Welcome to STA 101!

Statistics is a confrontation with uncertainty.

Statistics confronts uncertainty by quantifying it.

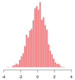
Data analysis

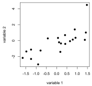
Transforming messy, incomplete, imperfect data into knowledge:

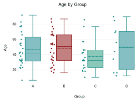
subject	variable_1	variable_2
1	-1.65692830	-2.16524631
2	-0.90396488	-2.97993045
3	1.37141732	0.09720280
4	-0.43176527	0.27970110
5	0.40649190	0.69143221
6	1.47092198	4.47233461
7	-0.78625051	-1.24276055
8	0.64835135	-0.06749005
9	0.06363568	0.33517580

`ggplot` $\Longrightarrow$

`|>` $\Longrightarrow$







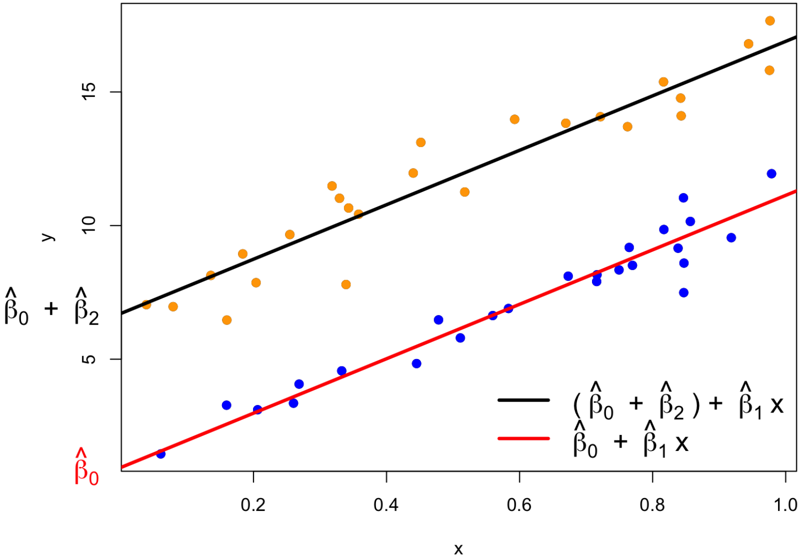
	mean	median	std
Variable 1	0.795	-0.292	0.200
Variable 2	0.343	0.616	0.834
Variable 3	1.587	0.508	2.501

Statistical inference

Quantifying our uncertainty about that knowledge:

- **Question**: What's the number?
- **Answer**: best-guess $\pm$ margin-of-error

One model, but two lines?



One model, but two lines?

We fit a model with a numerical predictor (x_1) and a categorical predictor (x_2) with two level:

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2.$$

The categorical predictor works like this:

$$x_2 = \begin{cases} 0 & \text{if book is hardback} \\ 1 & \text{if book is paperback.} \end{cases}$$

This essentially nests two models:

$$\hat{y} = \begin{cases} \hat{\beta}_0 + \hat{\beta}_1 x_1 & \text{if book is hardback} \\ \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 & \text{if book is paperback.} \end{cases}$$

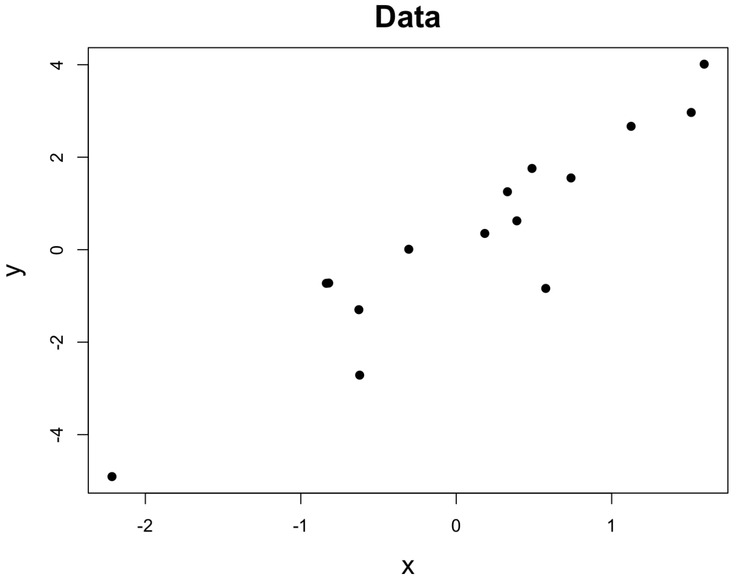
The intercept shifts.

What are models good for?

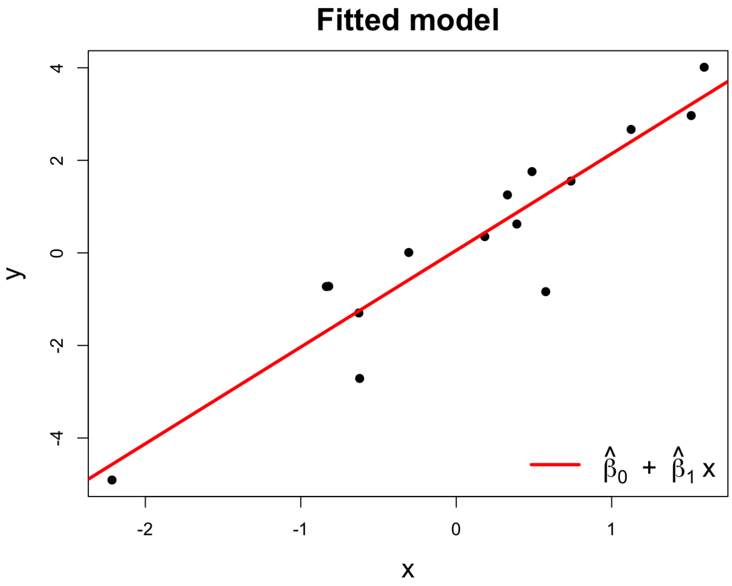
"All models are wrong but some are useful."

- We would love to use models to draw causal conclusions from messy, non-experimental data, but this is very very difficult;
- A slightly easier task that some models are very good at is *prediction*...

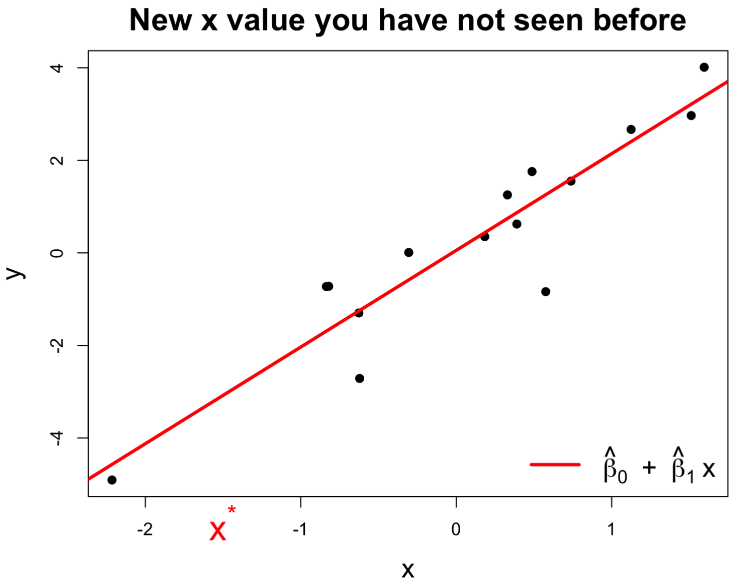
Prediction



Prediction

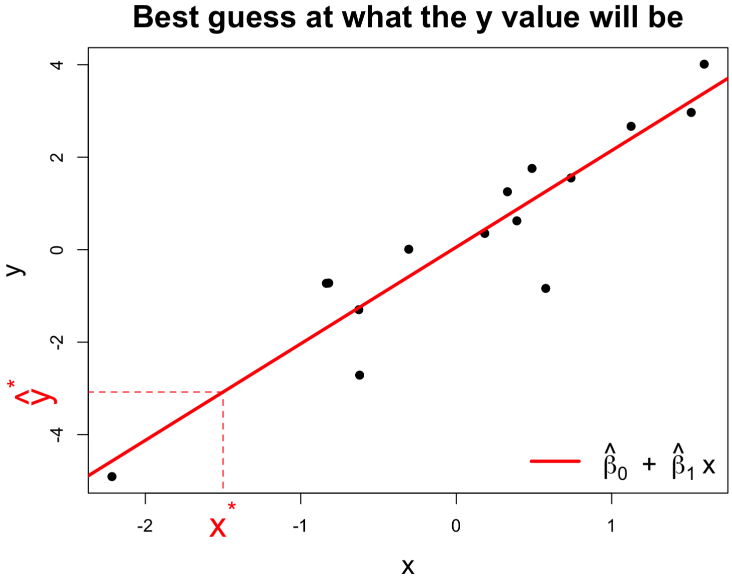


Prediction



New x value you have not seen before
y
x
4
2
0
-2
-4
-2
-1
0
1
x*
$\hat{\beta}_0 + \hat{\beta}_1 x$

Prediction



Best guess at what the y value will be
4
2
0
-2
-4
y
$\hat{y}^*$
-2
x^*
-1
0
1
x
$\hat{\beta}_0 + \hat{\beta}_1 x$

Prediction, summary

Collect data:

$$(x_1,\ y_1), (x_2,\ y_2), (x_3,\ y_3), ..., (x_n,\ y_n)$$

Fit a model:

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

Prediction: Someone hands you an $x^\star$ you've never seen before. What's your best guess at what y is going to be?

$$\hat{y}^\star = \hat{\beta}_0 + \hat{\beta}_1 x^\star.$$

Historical Data + Model = Predictions you can use to aid decision making in an uncertain world.

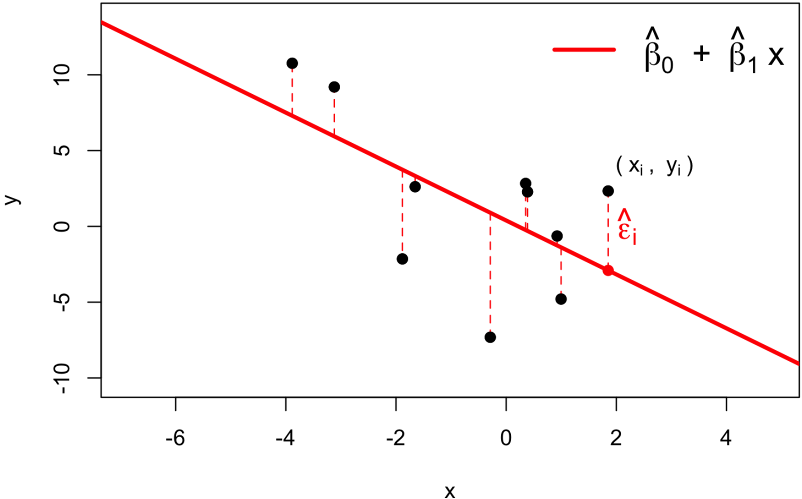
How well does a linear model fit the data?

Question: can we *quantify* this?

Recall the residuals

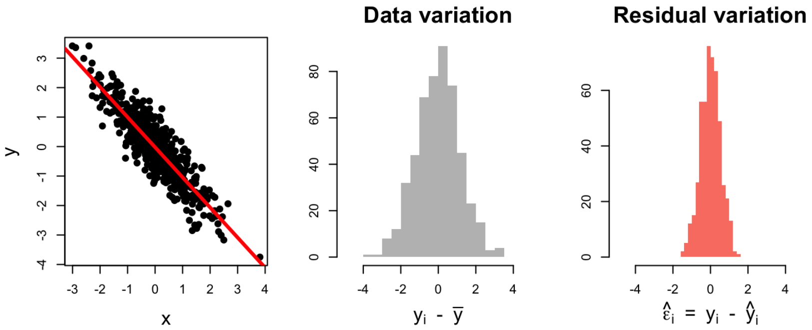
$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i \qquad \text{(fitted model)}$$

$$\hat{\varepsilon}_i = y_i - \hat{y}_i \qquad \text{(residuals)}$$



How variable are the residuals compared to the data?

Idea: Fit a model, and compare histogram of $\{y_1, y_2, ..., y_n\}$ to histogram of $\{\hat{\varepsilon}_1, \hat{\varepsilon}_2, ..., \hat{\varepsilon}_n\}$:



- Variation in y_i is what we seek to "explain" with a model;
- Variation in $\hat{\varepsilon}_i$ is the leftover that our model does not explain;
- If there's not a lot of leftover, we did pretty well.

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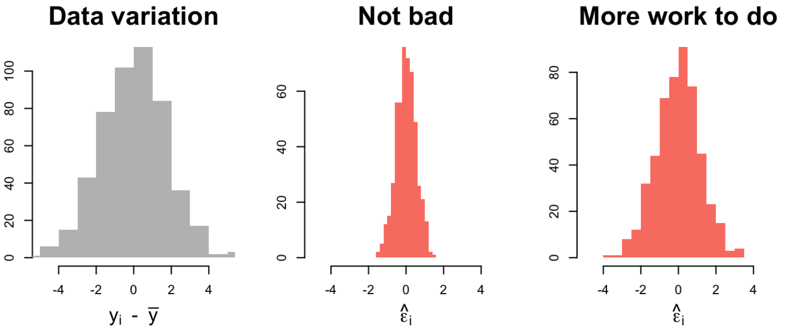
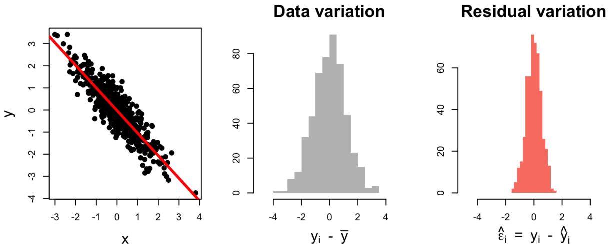


How do we quantify this comparison?



Summarize variation with a measure of *spread* and compare:

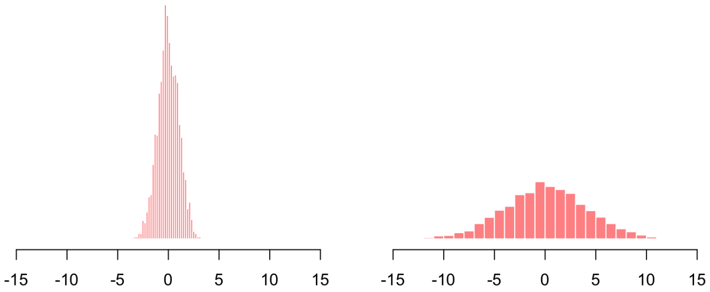
$$\begin{aligned}
\text{fit quality} &= \text{proportion of variation explained} \\
&= \frac{\text{explained variation}}{\text{total data variation}} \\
&= \frac{\text{total data variation - unexplained variation}}{\text{total data variation}} \\
&= \frac{\text{spread of } y_i - \text{spread of } \hat{\varepsilon}_i}{\text{spread of } y_i}.
\end{aligned}$$

How do we measure spread?

With the **variance**:

the average squared distance from the mean.

"how far are the data, typically, from their center?"



Left: data are typically close to the center (low variance)

Right: data are typically farther from the center (higher variance).

How do we measure spread?

With the **variance**:

the average squared distance from the mean.

"how far are the data, typically, from their center?"

Recall that the mean is the same thing as the average:

$$\bar{y} = \frac{y_1 + y_2 + y_3 + \ldots + y_n}{n} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

So in formulas...

$$\begin{aligned} \mathrm{var}(y_i) &= \frac{(y_1 - \bar{y})^2 + (y_2 - \bar{y})^2 + \ldots + (y_n - \bar{y})^2}{n} \\ &= \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 \\ &= \mathrm{sd}(y_i)^2. \end{aligned}$$

And similarly for $\mathrm{var}(\hat{\varepsilon}_i)$

Back to measuring fit

Before:

$$\text{fit quality} = \text{proportion of variation explained by the model}$$

$$= \frac{\text{spread of } y_i - \text{spread of } \hat{\varepsilon}_i}{\text{spread of } y_i}.$$

Now:

$$R^2 = \frac{\text{var}(y_i) - \text{var}(\hat{\varepsilon}_i)}{\text{var}(y_i)}$$

$$= 1 - \frac{\text{var}(\hat{\varepsilon}_i)}{\text{var}(y_i)}.$$

R^2 is called the **coefficient of determination**.

Facts about R^2

- $\text{var}(y_i)$ is always bigger than $\text{var}(\hat{\varepsilon}_i)$, so R^2 is a number between zero and one;
- If $\text{var}(\hat{\varepsilon}_i) = 0$, then the model explained everything. The fit is perfect (poifect!), and $R^2 = 1$;
- If $\text{var}(\hat{\varepsilon}_i) = \text{var}(y_i)$, then the model explained absolutely nothing and $R^2 = 0$;
- Most of the time we are somewhere in between, and we can use R^2 to quantify the quality of a model's fit and *rank* competing models.

Adjusted R^2

Possible use of R^2:

- decide which covariates to include in a big multiple regression. The set of covariates that delivers the highest R^2 is the winner;

Problem:

- R^2 has a nasty mathematical property that it *always goes up* every time you add *any* covariate to the model, even if that covariate is silly and useless;

Goal:

- We want a measure of fit that will not give all variables a participation trophy just for showing up, but actually rewards honest-to-goodness improvements in fit;

Solution:

- Adjusted R^2.

Variable selection: backward elimination

Start with the *full model* (the model that includes all potential predictor variables). Variables are eliminated one-at-a-time from the model until we cannot improve the model any further.

Procedure:

1. Start with a model that has all predictors we consider and compute the adjusted R^2.
2. Next fit every possible model with 1 fewer predictor.
3. Compare adjusted R^2s to select the best model (highest adjusted R^2) with 1 fewer predictor.
4. Repeat steps 2 and 3 until adjusted R^2 no longer increases.

Variable selection: forward stepwise

Forward stepwise regression is the reverse of the backward elimination technique. Instead, of eliminating variables one-at-a-time, we add variables one-at-a-time until we cannot find any variables that improve the model any further.

Procedure:

1. Start with a model that has no predictors.
2. Next fit every possible model with 1 additional predictor and calculate adjusted R^2 of each model.
3. Compare adjusted R^2 values to select the best model (highest adjusted R^2) with 1 additional predictor.
4. Repeat steps 2 and 3 until adjusted R^2 no longer increases.