Logistic Regression

STA 198: Introduction to Health Data Science

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July 15, 2020

The following material was used by Yue Jiang during a live lecture.

Without the accompanying oral comments, the text is incomplete as a record of the presentation.

Models for binary outcomes

Suppose we have a binary outcome (e.g., $Y = 1$ if a condition is satisfied and $Y = 0$ if not) and predictors on a variety of scales.

If the predictors are discrete and the binary outcomes are independent, we can use the Bernoulli distribution for individual 0-1 data or the binomial distribution for grouped data that are counts of successes in each group.

Models for binary outcomes

Let's suppose we want to model $P(Y = 1)$.

One strategy might be to simply fit a linear regression model to the probabilities:

$$P(Y = 1)_i = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_p x_{pi} + \epsilon_i.$$

Primary biliary cirrhosis

The Mayo Clinic conducted a trial for primary biliary cirrhosis, comparing the drug D-penicillamine vs. placebo. Patients were followed for a specified duration, and their status at the end of follow-up (whether they died) was recorded.

Researchers are interested in predicting whether a patient died based on the following variables:

- ascites: whether the patient had ascites (1 = yes, 0 = no)
- bilirubin: serum bilirubin in mg/dL
- stage: histologic stage of disease (ordinal categorical variable with stages 1, 2, 3, and 4)

What can go wrong?

Suppose we fit the following model:

$$\begin{aligned} \text{P}(\text{Y} = 1)_i = \beta_0 + \beta_1(\text{ascites})_i + \beta_2(\text{bilirubin})_i + \beta_3(\text{stage} = 2)_i + \\ \beta_4(\text{stage} = 3)_i + \beta_5(\text{stage} = 4)_i + \epsilon_i \end{aligned}$$

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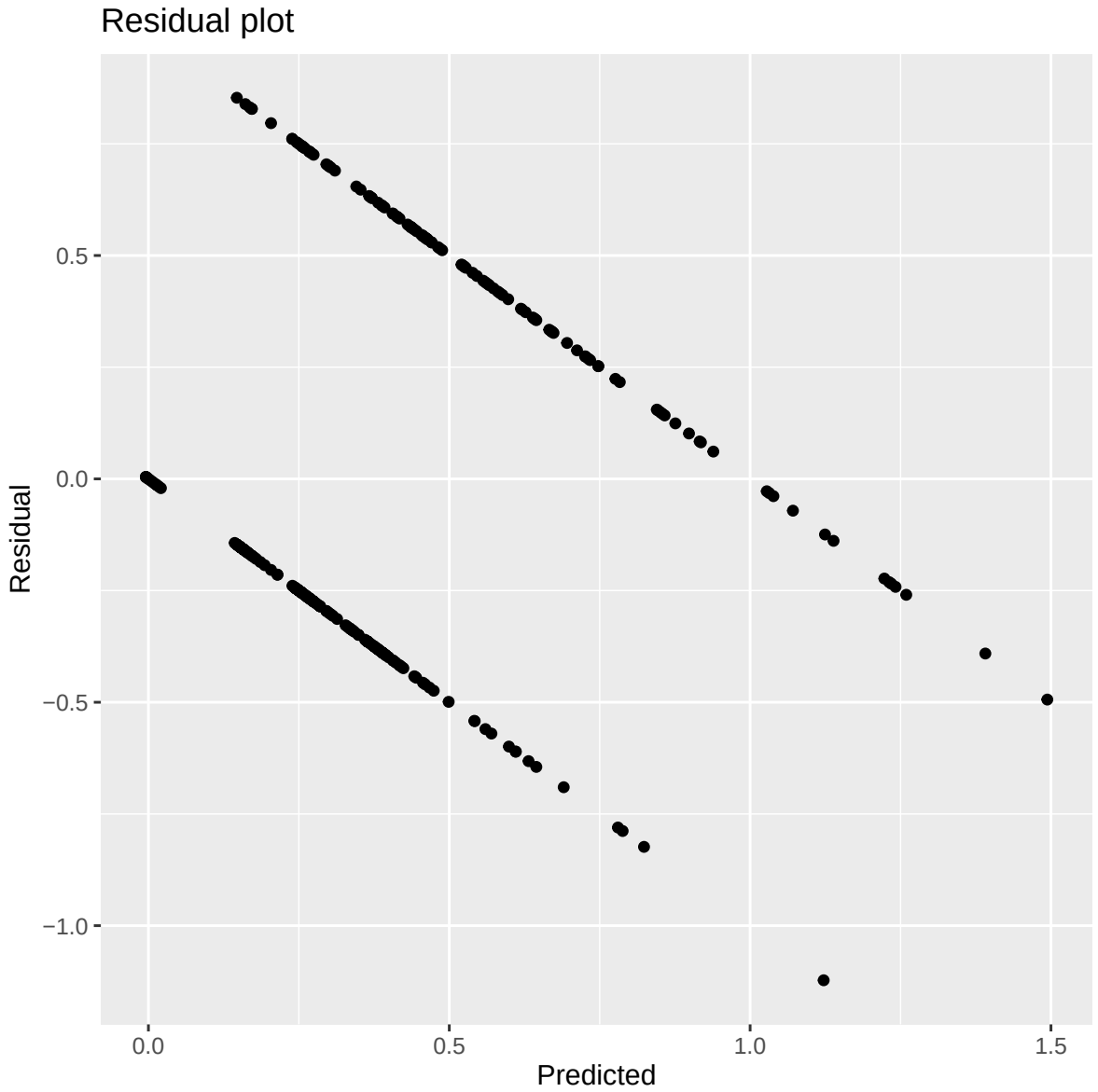

What can go wrong?

The probability, p_i must be in the interval [0, 1], but there is nothing in the model that enforces this constraint.

With this model, you could be estimating probabilities that are negative or that are greater than 1!

From probabilities to log-odds

Suppose the probability of an event is p

Then the odds that the event occurs is $\frac{p}{1-p}$

Taking the (natural) log of the odds, we have the logit of p: the log-odds:

$$logit(p) = \log\left(\frac{p}{1-p}\right).$$

Note that although p is constrained to lie between 0 and 1, the logit of p is unconstrained - it can be anything from $-\infty$ to ∞

Logistic regression model

Let's create a model for the logit of p:

$$logit(p_i) = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_p x_{pi}$$

This is a linear model for a transformation of the outcome of interest, and is also equivalent to

$$p_i = \frac{exp(\beta_0 + \beta_1 x_{1i} + \cdots + \beta_p x_{pi})}{1 + exp(\beta_0 + \beta_1 x_{1i} + \cdots + \beta_p x_{pi})}.$$

The expression on the right is called a *logistic function* and cannot yield a value that is negative or a value that is > 1. Fitting a model of this form is known as *logistic regression*.

Logistic regression

$$\text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_p x_{pi}$$

Negative logits represent probabilities less than one-half, and positive logits represent probabilities above one-half.

Interpreting parameters in logistic regression

Typically we interpret *functions* of parameters in logistic regression rather than the parameters themselves. For the simple model

$$\log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 x_1,$$

we note that the probability that $Y = 1$ when $X = 0$ is

$$\frac{\exp(\beta_0)}{1+\exp(\beta_0)}.$$

Interpreting binary predictors

Suppose that X is a binary (0/1) variable (e.g., $X = 1$ for males and 0 for non-males). In this case, we interpret $\exp(\beta_1)$ as the odds ratio (OR) of the response for the two possible levels of X.

How does this translate to dummy variables for categorical predictors?

Interpreting continuous predictors

The log odds of response for $X = 1$ is given by $\beta_0 + \beta_1$, and the log odds of response for $X = 0$ is β_0. So the odds ratio of response comparing $X = 1$ to $X = 0$ is given by $\frac{\exp(\beta_0+\beta_1)}{\exp(\beta_0)} = \exp(\beta_1)$.

In a *multiple logistic regression* model with more than one predictor, this OR is interpreted conditionally on values of other variables (i.e., controlling for them).

Back to the PBC data

Fitting a logistic regression model, we obtain

	Est.	SE	p-value
(Intercept)	-3.14	1.05	0.003
ascites	2.87	1.07	0.007
bilirubin	0.31	0.06	< 0.001
stage $= 2$	1.25	1.10	0.253
stage $= 3$	1.72	1.07	0.109
stage $= 4$	2.17	1.08	0.044

Remember, this is for the linear effect on the log-odds (the logit).
How might we interpret these coefficients as odds ratios?

Back to the PBC data

Remember, we are interested in the probability that a patient died during follow-up (a "success"). We are predicting the log-odds of this event happening.

- The $\hat{\beta}$ corresponding to ascites was 2.87. Thus, the odds ratio for dying is $\exp(2.87) \approx 17.6$. That is, patients with ascites have 17.6 times the odds of dying compared to patients that do not, holding all other variables constant.
- The $\hat{\beta}$ corresponding to bilirubin was 0.31. Thus, the odds ratio for dying for a patient with 1 additional mg/dL serum bilirubin compared to another is $\exp(0.31) \approx 1.36$, holding all other variables constant.
- The baseline stage was 1. The $\hat{\beta}$ corresponding to stage 3 was 1.72. Thus, patients in stage 3 have approximately 5.58 times the odds of dying compared to stage 1 patients, holding all other variables constant.

Predicted probabilities

There is a one-to-one relationship between p and logit(p). So, if we predict logit(p), we can "back-transform" to get back to a predicted probability.

Predicted probabilities

Suppose a patient does not have ascites, has a bilirubin level of 5 mg/dL, and is a stage 2 patient.

Their predicted log-odds are

$$-3.14 + 0.31 \times 5 + 1.25 = -0.34$$

Thus, the predicted probability of dying for this individual is

$$\frac{\exp(-0.34)}{1 + \exp(-0.34)} = 0.42.$$

Hypothesis tests in logistic regression

Generally, we wish to know whether the OR $= 1$ or equivalently, whether the logit of p (a β coefficient) $= 0$.

To test $H_0 : \beta_j = 0$, we can compare the ratio of a parameter estimate to its standard error using the standard normal distribution (reason we use Z instead of t is a bit technical).

Confidence intervals in logistic regression

Confidence intervals for the effects on the logit scale,

$$\hat{\beta}_j \pm z^{\star}_{1-\alpha/2} \times \widehat{SE}(\hat{\beta}_j),$$

are typically translated into confidence intervals for ORs by exponentiating the lower and upper confidence limits:

$$\left(\exp\left(\hat{\beta}_j - z^{\star}_{1-\alpha/2} \times \widehat{SE}(\hat{\beta}_j)\right), \exp\left(\hat{\beta}_j + z^{\star}_{1-\alpha/2} \times \widehat{SE}(\hat{\beta}_j)\right)\right).$$