

# Multiple Linear Regression

## Model Diagnostics

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# Announcements

- HW 03 due TODAY at 11:59p
- HW 04 due Thursday, October 10 at 11:59p
- Thursday's lab: Help Hours
- Looking ahead:
  - Exam 1 on Mon, Oct 14 in class
  - Practice exam on Sakai
  - Can bring 1 page of notes
  - Exam review on Oct 9

# R packages

```
library(tidyverse)  
library(knitr)  
library(broom)  
library(cowplot) # use plot_grid function
```

# Nested F Test

# Restaurant tips

What affects the amount customers tip at a restaurant?

- Response:

- **Tip**: amount of the tip

- Predictors:

- **Party**: number of people in the party
  - **Meal**: time of day (Lunch, Dinner, Late Night)
  - **Age**: age category of person paying the bill (Yadult, Middle, SenCit)

# Is Meal a significant predictor of tips?

term	estimate	std.error	statistic	p.value
(Intercept)	1.254	0.394	3.182	0.002
Party	1.808	0.121	14.909	0.000
AgeSenCit	0.390	0.394	0.990	0.324
AgeYadult	-0.505	0.412	-1.227	0.222
MealLate Night	-1.632	0.407	-4.013	0.000
MealLunch	-0.612	0.402	-1.523	0.130

# Tips data: Nested F Test

$$H_0 : \beta_{latenight} = \beta_{lunch} = 0$$

$$H_a : \text{at least one } \beta_j \text{ is not equal to } 0$$

```
reduced <- lm(Tip ~ Party + Age, data = tips)
```

```
full <- lm(Tip ~ Party + Age + Meal, data = tips)
```

```
kable(anova(reduced, full), format="markdown", digits = 3)
```

Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
165	686.444	NA	NA	NA	NA
163	622.979	2	63.465	8.303	0

At least one coefficient associated with **Meal** is not zero. Therefore, **Meal** is a significant predictor of **Tips**.



Why is it not good practice to use the individual p-values to determine a categorical variable with  $k > 2$  levels) is significant?

*Hint:* What does it actually mean if none of the  $k - 1$  p-values are significant?

# Practice with Interactions

term	estimate	std.error	statistic	p.value
(Intercept)	1.2764989	0.4910882	2.5993270	0.0102086
Party	1.7947980	0.1715003	10.4652753	0.0000000
AgeSenCit	0.4007889	0.3969295	1.0097230	0.3141431
AgeYadult	-0.4701634	0.4197146	-1.1201978	0.2642977
MealLate Night	-1.8454674	0.7089728	-2.6030159	0.0101039
MealLunch	-0.4608832	0.8651044	-0.5327487	0.5949421
Party:MealLate Night	0.1108600	0.2846584	0.3894491	0.6974586
Party:MealLunch	-0.0500822	0.2825586	-0.1772455	0.8595384

1. What is the baseline level for Meal?
2. How do we expect the mean tips to change when Meal == "Late Night", holding Age and Party constant?
3. How does the slope of Party change when Meal == "Late Night", holding Age and Party constant?

# Nested F test for interactions

Are there any significant interaction effects with Party in the model?

```
reduced <- lm(Tip ~ Party + Age + Meal, data = tips)
```

```
full <- lm(Tip ~ Party + Age + Meal + Age* Party + Meal * Party,  
           data = tips)
```

```
kable(anova(reduced, full ), format="markdown", digits = 3)
```

Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
163	622.979	NA	NA	NA	NA
159	615.380	4	7.6	0.491	0.742

## Final model for now

We conclude that there are no significant interactions with Party in the model. Therefore, we will use the original model that only included main effects.

term	estimate	std.error	statistic	p.value
(Intercept)	1.254	0.394	3.182	0.002
Party	1.808	0.121	14.909	0.000
AgeSenCit	0.390	0.394	0.990	0.324
AgeYadult	-0.505	0.412	-1.227	0.222
MealLate Night	-1.632	0.407	-4.013	0.000
MealLunch	-0.612	0.402	-1.523	0.130

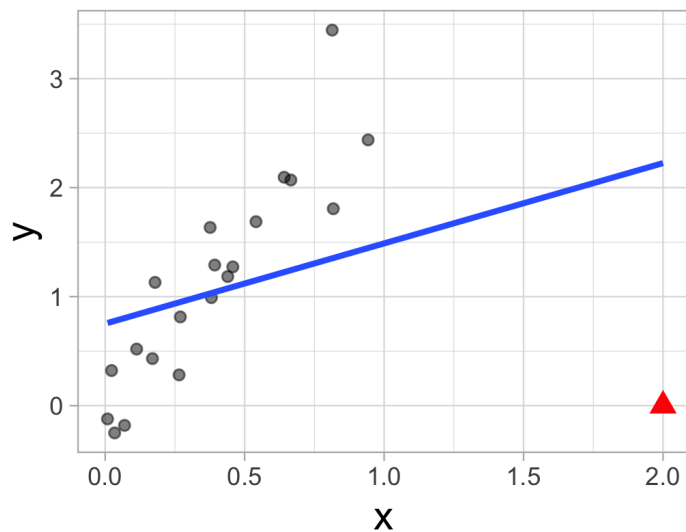
# Model Diagnostics

# Influential and Leverage Points

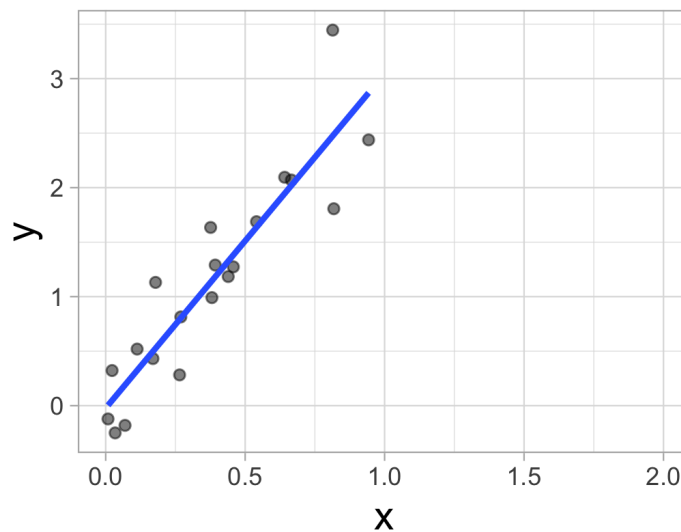
# Influential Observations

An observation is **influential** if removing it substantially changes the coefficients of the regression model

With Influential Point



Without Influential Point



# Influential Observations

- In addition to the coefficients, influential observations can have a large impact on the standard errors
- Occasionally these observations can be identified in the scatterplot
  - This is often not the case - especially when dealing with multivariate data
- We will use measures to quantify an individual observation's influence on the regression model
  - **leverage, standardized residuals, and Cook's distance**



# Leverage

- **Leverage:** measure of the distance between an observation's explanatory variable values and the average explanatory variables for the whole data set
- An observation has high leverage if its combination of values for the explanatory variables is very far from the typical combinations in the data
  - It is potentially an influential point, i.e. may have a large impact on the coefficient estimates and standard errors
- **Note:** Identifying points with high leverage has nothing to do with the values of the response variables

# Calculating Leverage

- **Simple Regression:** leverage of the  $i^{th}$  observation is

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{j=1}^n (x_j - \bar{x})^2}$$

- **Multiple Regression:** leverage of the  $i^{th}$  observation is the  $i^{th}$  diagonal of

$$\mathbf{H} = \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T$$

# High Leverage

- Values of leverage are between  $\frac{1}{n}$  and 1 for each observation
- The average leverage for all observations in the data set is  $\frac{(p+1)}{n}$
- Determination of **high leverage** is relatively arbitrary
  - one threshold is  $\frac{2(p+1)}{n}$
- Observations with high leverage tend to have small residuals

# High Leverage

- Questions to check if you identify points with high leverage:
  - Are they a result of data entry errors?
  - Are they in the scope for the individuals for which you want to make predictions?
  - Are they impacting the estimates of the model coefficients, especially for interactions?
- Just because a point has high leverage does not necessarily mean it will have a substantial impact on the regression. Therefore you should check other measures.

# Standardized & Studentized Residuals

- What is the best way to identify outliers (points that don't fit the pattern from the regression line)?
- Look for points that have large residuals
- We want a common scale, so we can more easily identify "large" residuals
- We will look at each residual divided by its standard error

# Standardized Residuals

$$std.res_i = \frac{e_i}{\hat{\sigma}\sqrt{1 - h_i}}$$

- The standard error of a residual,  $\hat{\sigma}\sqrt{1 - h_i}$  depends on the value of the predictor variables
- Residuals for observations that are high leverage have smaller variance than residuals for observations that are low leverage
  - This is because the regression line tries to fit high leverage observations as closely as possible

# Standardized Residuals

- Values with very large standardized residuals are outliers, since they don't fit the pattern determined by the regression model
- Observations with standardized residuals with magnitude  $> 2$  should be more closely examined
- Observations with large standardized residuals are outliers but may not have an impact on the regression line
- **Good Practice:** Make residual plots with standardized residuals
  - It is easier to identify outliers and check for constant variance assumption

# Motivating Cook's Distance

- If a observation has a large impact on the estimated regression coefficients, when we drop that observation...
  - The estimated coefficients should change
  - The predicted  $\hat{Y}$  value for that observation should change
- One way to determine each observation's impact could be to delete it, rerun the regression, compare the predicted  $\hat{Y}$  values from the new and original models
  - This could be very time consuming
- Instead, we can use **Cook's Distance** which gives a measure of the change in the predicted  $\hat{Y}$  value when an observation is dropped



# Cook's Distance

- **Cook's Distance:** Measure of an observation's overall impact, i.e. the effect removing the observation has on the estimated coefficients
- For the  $i^{th}$  observation, we can calculate Cook's Distance as

$$D_i = \frac{1}{p} (std. res_i)^2 \left( \frac{h_i}{1 - h_i} \right)$$

- *Note:* Cook's distance,  $D_i$ , incorporates both the residual and the leverage for each observation
- An observation with large  $D_i$  is said to have a strong influence on the predicted values

# Using these measures

- Standardized residuals, leverage, and Cook's Distance should all be examined together
- Examine plots of the measures to identify observations that may have an impact on your regression model
- Some thresholds for flagging potentially influential observations:
  - **Leverage:**  $h_i > \frac{2(p+1)}{n}$  (some software uses  $2p/n$ )
  - **Standardized Residuals:**  $|std.res_i| > 2$
  - **Cook's Distance:**  $D_i > 1$

# What to do with outliers/influential observations?

- It is **OK** to drop an observation based on the **predictor variables** if...
  - It is meaningful to drop the observation given the context of the problem
  - You intended to build a model on a smaller range of the predictor variables. Mention this in the write up of the results and be careful to avoid extrapolation when making predictions
- It is **not OK** to drop an observation based on the response variable
  - These are legitimate observations and should be in the model
- You can try transformations or increasing the sample size by collecting more data
- In either instance, you can try building the model with and without the outliers/influential observations

# Model diagnostics in R

- Use the **augment** function in the broom package to output the model diagnostics (along with the predicted values and residuals)
- Output from augment :
  - response and predictor variables in the model
  - `.fitted`: predicted values
  - `.se.fit`: standard errors of predicted values
  - `.resid`: residuals
  - **.hat**: leverage
  - `.sigma`: estimate of residual standard deviation when corresponding observation is dropped from model
  - **.cooks**: Cook's distance
  - **.std.resid**: standardized residuals

# Example: Restaurant tips

What affects the amount customers tip at a restaurant?

- **Response:**
  - **Tip:** amount of the tip
- **Predictors:**
  - **Party:** number of people in the party
  - **Meal:** time of day (Lunch, Dinner, Late Night)
  - **Age:** age category of person paying the bill (Yadult, Middle, SenCit)

```
tips <- read_csv("data/tip-data.csv") %>%  
  filter(!is.na(Party))
```

# Example: Tips

```
model1 <- lm(Tip ~ Party + Meal + Age , data = tips)
kable(tidy(model1),format="html",digits=3)
```

term	estimate	std.error	statistic	p.value
(Intercept)	1.254	0.394	3.182	0.002
Party	1.808	0.121	14.909	0.000
MealLate Night	-1.632	0.407	-4.013	0.000
MealLunch	-0.612	0.402	-1.523	0.130
AgeSenCit	0.390	0.394	0.990	0.324
AgeYadult	-0.505	0.412	-1.227	0.222

# Using `augment` function

- Use the `augment` function to add predicted values and model diagnostics to data
  - Add the observation number for diagnostic plots

```
tips_output <- augment(model1) %>%  
  mutate(obs_num = row_number())
```

# Augmented data

```
glimpse(tips_output)
```

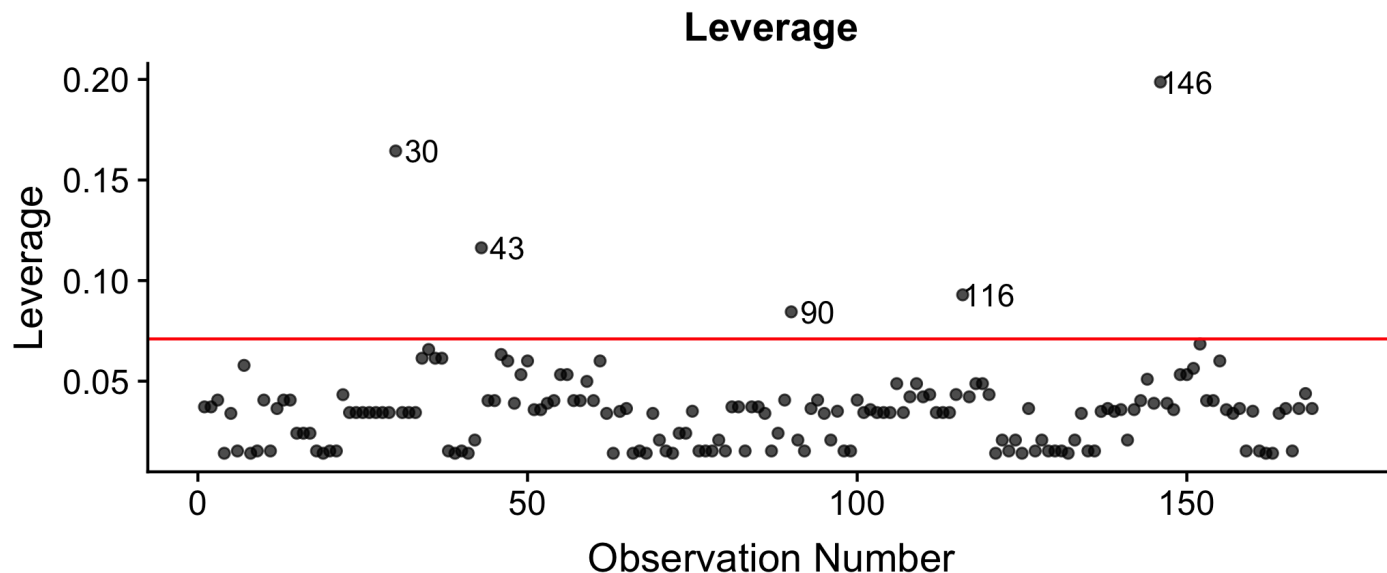
```
## Observations: 169
## Variables: 12
## $ Tip          <dbl> 2.99, 2.00, 5.00, 4.00, 10.34, 4.85, 5.00, 4.00, 5.00,
## $ Party        <dbl> 1, 1, 1, 3, 2, 2, 4, 3, 2, 1, 2, 2, 1, 1, 1, 1, 2,
## $ Meal         <chr> "Dinner", "Dinner", "Dinner", "Dinner", "Dinner", "Di
## $ Age          <chr> "Yadult", "Yadult", "SenCit", "Middle", "SenCit", "Mi
## $ .fitted      <dbl> 2.5562830, 2.5562830, 3.4515838, 6.6766419, 5.2591209,
## $ .se.fit      <dbl> 0.3771863, 0.3771863, 0.3939434, 0.2327069, 0.3604347,
## $ .resid       <dbl> 0.43371698, -0.55628302, 1.54841620, -2.67664190, 5.0,
## $ .hat         <dbl> 0.03722423, 0.03722423, 0.04060519, 0.01416878, 0.033,
## $ .sigma       <dbl> 1.960700, 1.960502, 1.957071, 1.949536, 1.918486, 1.9,
## $ .cooksd      <dbl> 3.294208e-04, 5.419132e-04, 4.612379e-03, 4.554814e-0,
## $ .std.resid   <dbl> 0.226100143, -0.289994804, 0.808622859, -1.378941928,
## $ obs_num     <int> 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16,
```



# Leverage

```
leverage_threshold <- 2*(5+1)/nrow(tips)
```

```
ggplot(data = tips_output, aes(x = obs_num, y = .hat)) +  
  geom_point(alpha = 0.7) +  
  geom_hline(yintercept = leverage_threshold, color = "red") +  
  labs(x = "Observation Number", y = "Leverage", title = "Leverage") +  
  geom_text(aes(label = ifelse(.hat > leverage_threshold, as.character(obs_num), "")))
```



# Points with high leverage

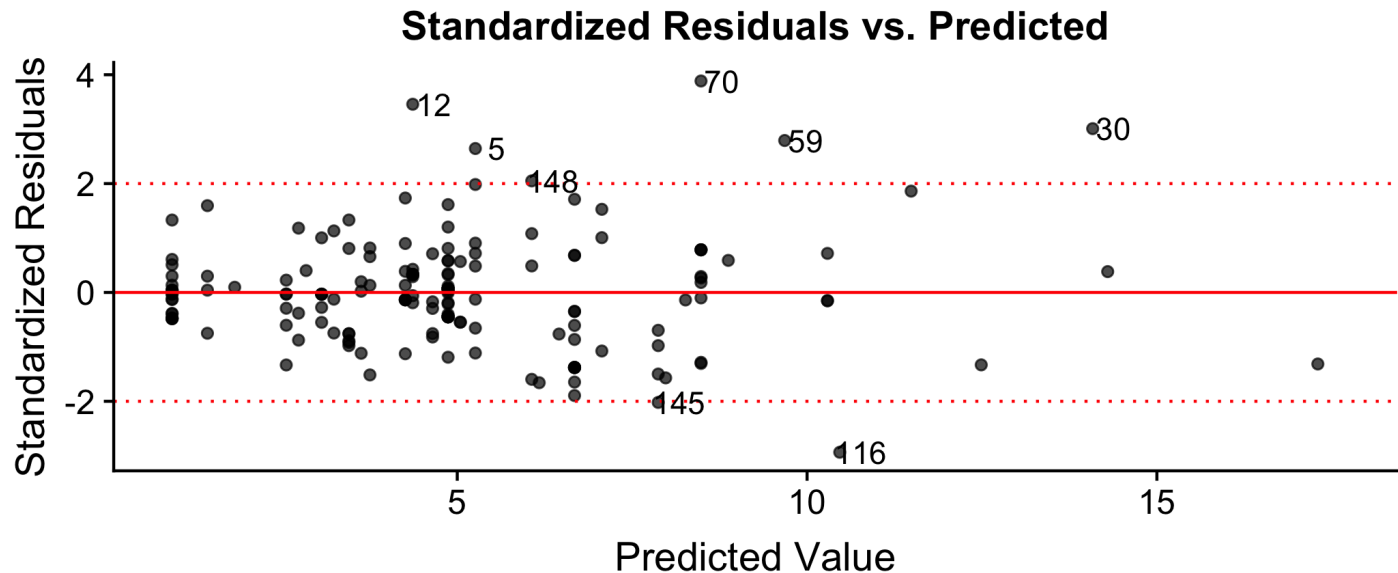
```
tips_output %>% filter(.hat > leverage_threshold) %>%  
  select(Party, Meal, Age)
```

```
## # A tibble: 5 x 3  
##   Party Meal      Age  
##   <dbl> <chr>    <chr>  
## 1     8 Late Night Middle  
## 2     7 Dinner    SenCit  
## 3     6 Dinner    SenCit  
## 4     6 Late Night Middle  
## 5     9 Lunch     SenCit
```

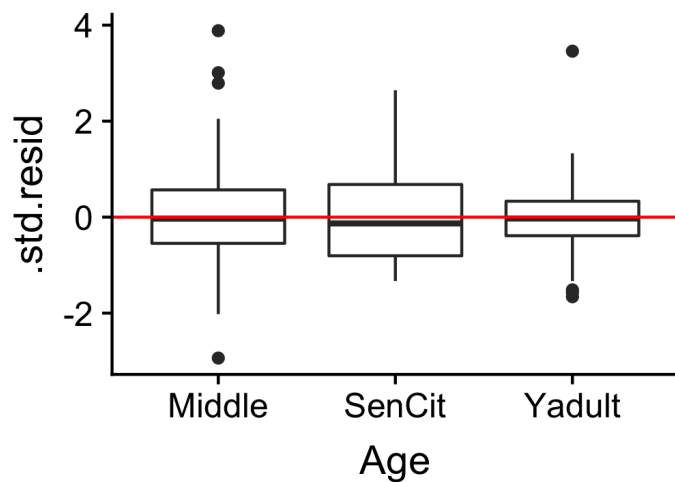
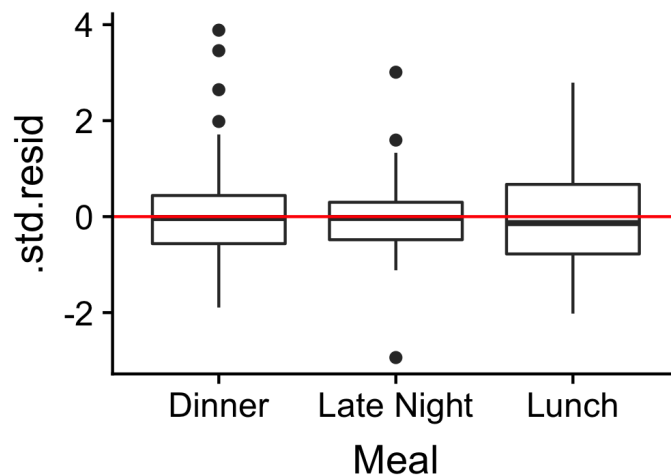
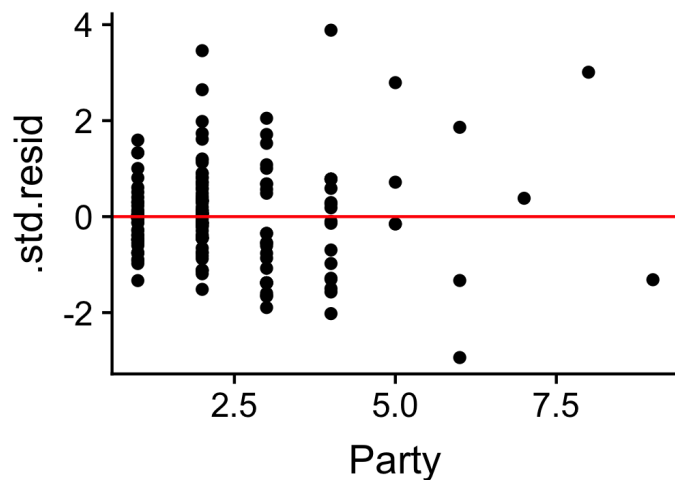
Why do you think these points have high leverage?

# Standardized Residuals vs. Predicted

```
ggplot(data = tips_output, aes(x = .fitted, y = .std.resid)) +  
  geom_point(alpha = 0.7) +  
  geom_hline(yintercept = 0, color = "red") +  
  geom_hline(yintercept = -2, color = "red", linetype = "dotted") +  
  geom_hline(yintercept = 2, color = "red", linetype = "dotted") +  
  labs(x = "Predicted Value", y = "Standardized Residuals", title = "Standardized Residuals vs. Predicted") +  
  geom_text(aes(label = ifelse(abs(.std.resid) > 2, as.character(obs_id), "")))
```



# Standardized residuals vs. predictors



# Points with large magnitude std.res.

```
tips_output %>% filter(abs(.std.resid) > 2) %>%  
  select(Party, Meal, Age, Tip)
```

```
## # A tibble: 8 x 4  
##   Party Meal      Age      Tip  
##   <dbl> <chr>    <chr> <dbl>  
## 1     2 Dinner SenCit 10.3  
## 2     2 Dinner Yadult 11  
## 3     8 Late Night Middle 19.5  
## 4     5 Lunch   Middle 15  
## 5     4 Dinner   Middle 16  
## 6     6 Late Night Middle 5  
## 7     4 Lunch   Middle 4  
## 8     3 Lunch   Middle 10
```

- Why do you think these points have standardized residuals with large magnitude?
- What other variables could you examine?

# Why we want to find outliers

Estimate of regression standard deviation,  $\hat{\sigma}$ , using all observations

```
glance(model1)$sigma
```

```
## [1] 1.954983
```

Estimate of  $\hat{\sigma}$  without points with large magnitude standardized residuals

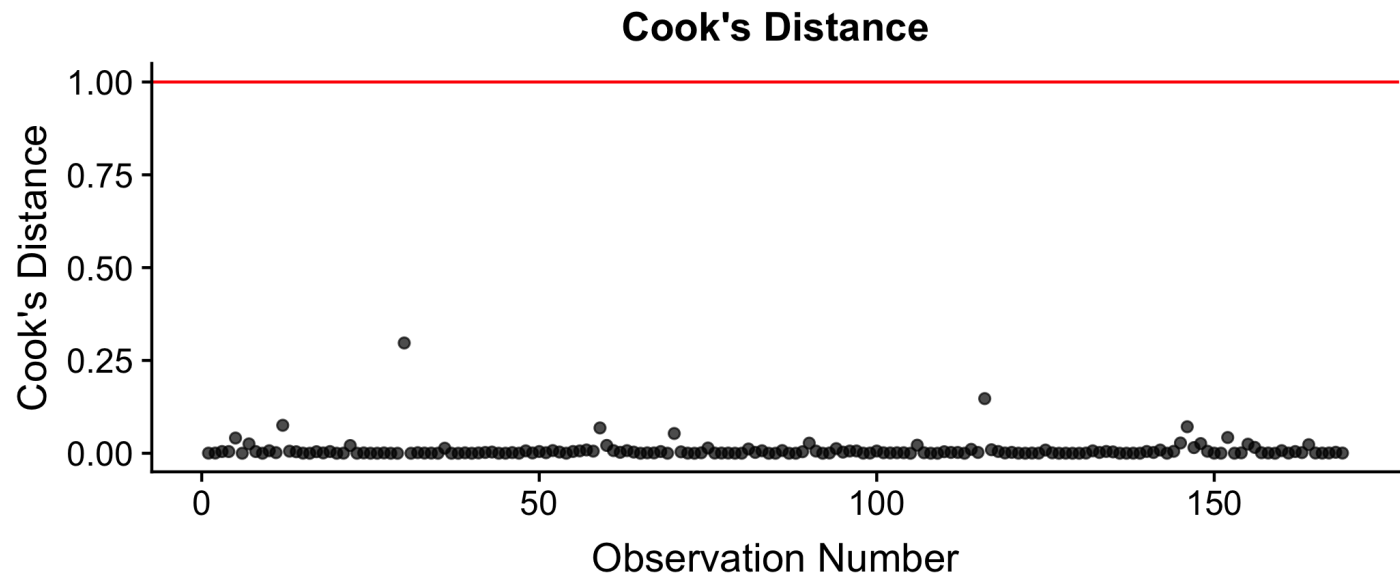
```
tips_output %>%  
  filter(abs(.std.resid) <= 2) %>%  
  summarise(sigma_est = sqrt(sum(.resid^2)/(n() - 5 - 1)))
```

```
## # A tibble: 1 x 1  
##   sigma_est  
##   <dbl>  
## 1      1.56
```

Recall that we use  $\hat{\sigma}$  to calculate the standard errors for all confidence intervals and p-values, so outliers can affect conclusions drawn from model

# Cook's Distance

```
ggplot(data = tips_output, aes(x = obs_num, y = .cooks_d)) +  
  geom_point(alpha = 0.7) +  
  geom_hline(yintercept=1,color = "red")+  
  labs(x= "Observation Number",y = "Cook's Distance",title = "Cook's Distance") +  
  geom_text(aes(label = ifelse(.hat>1,as.character(obs_num),"")))
```





# Multicollinearity

# Why multicollinearity is a problem

- We can't include two variables that have a perfect linear association with each other
- If we did so, we could not pick a unique best fit model

# Why multicollinearity is a problem

- Ex. Suppose the true population regression equation is  $\mu\{Y|X\} = 3 + 4X$
- Suppose we try estimating that regression model using the variables  $X$  and  $Z = X/10$

$$\begin{aligned}\hat{\mu}\{Y|X\} &= \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 \frac{X}{10} \\ &= \hat{\beta}_0 + \left( \hat{\beta}_1 + \frac{\hat{\beta}_2}{10} \right) X\end{aligned}$$

- We can set  $\hat{\beta}_1$  and  $\hat{\beta}_2$  to any two numbers such that  $\hat{\beta}_1 + \frac{\hat{\beta}_2}{10} = 4$ 
  - The data then is unable to choose the "best" combination of  $\hat{\beta}_1$  and  $\hat{\beta}_2$

# Why multicollinearity is a problem

- When we have almost perfect collinearities (i.e. highly correlated explanatory variables), the standard errors for our regression coefficients inflate
- In other words, we lose precision in our estimates of the regression coefficients

# Detecting Multicollinearity

Multicollinearity may occur when...

- There are very high correlations ( $r > 0.9$ ) among two or more explanatory variables, especially for smaller sample sizes
- One (or more) explanatory variables is an almost perfect linear combination of the others
- Include quadratic terms without first mean-centering the variables before squaring
- Including interactions with two or more continuous variables

# Detecting Multicollinearity

- Look at a correlation matrix of the explanatory variables, including all dummy variables
  - Look out for values close to 1 or -1
- If you think one explanatory variable is an almost perfect linear combination of other explanatory variables, you can run a regression of that explanatory variable vs. the others and see if  $R^2$  is close to 1

# Detecting Multicollinearity (VIF)

- **Variance Inflation Factor (VIF)**: Measure of multicollinearity

$$VIF(\hat{\beta}_j) = \frac{1}{1 - R_{X_j|X_{-j}}^2}$$

where  $R_X^2$  is the proportion of variation  $X$  that is explained by the linear combination of the other explanatory variables in the model.

- Typically  $VIF > 10$  indicates concerning multicollinearity
- Use the **vif()** function in the rms package to calculate VIF

# Tips VIF

- Calculate VIF using the **vif** function in the rms package

```
library(rms)
tidy(vif(model1))
```

```
## # A tibble: 5 x 2
##   names          x
##   <chr>        <dbl>
## 1 Party        1.19
## 2 MealLate Night 1.25
## 3 MealLunch     1.09
## 4 AgeSenCit     1.10
## 5 AgeYadult     1.40
```



# Calculating VIF for Party

```
party_model <- lm(Party ~ Meal + Age, data=tips)
r.sq <- glance(party_model)$r.squared
(vif <- 1/(1-r.sq))
```

```
## [1] 1.193821
```

# Calculating VIF for MealLateNight

```
# create indicator variables for Meal
```

```
tips <- tips %>%  
  mutate(late_night = if_else(Meal=="Late Night",1,0),  
         lunch = if_else(Meal=="Lunch",1,0))
```

```
late_night_model <- lm(late_night ~ lunch + Party + Age, data=tips)  
r.sq <- glance(late_night_model)$r.squared  
(vif <- 1/(1-r.sq))
```

```
## [1] 1.250908
```

See the supplemental notes [Details on Model Diagnostics](#) for more details about standardized residuals, leverage points, and Cook's distance.