# Multiple Linear Regression

## Model Assessment

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09.30.19

STA 210

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# Announcements

- Lab 05 **due Tuesday at 11:59p**
- HW 03 **due Wednesday at 11:59p**
- Reading 06 for Wednesday

# R packages

```
library(tidyverse)
library(knitr)
library(broom)
library(Sleuth3) # ex0824 data
library(cowplot) # use plot_grid function
```

# Log Transformations

# Respiratory Rate vs. Age

- A high respiratory rate can potentially indicate a respiratory infection in children. In order to determine what indicates a "high" rate, we first want to understand the relationship between a child's age and their respiratory rate.
- The data contain the respiratory rate for 618 children ages 15 days to 3 years.
- **Variables**:
  - **`Age`**: age in months
  - **`Rate`**: respiratory rate (breaths per minute)

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# Log transformation on $y$

```
log_model <- lm(log_rate ~ Age, data = respiratory)
  kable(tidy(log_model, conf.int = TRUE), format = "markdown", dig
```

| term | estimate | std.error | statistic | p.value | conf.low | conf.high |
|---|---|---|---|---|---|---|
| (Intercept) | 3.845 | 0.013 | 304.500 | 0 | 3.82 | 3.870 |
| Age | -0.019 | 0.001 | -25.839 | 0 | -0.02 | -0.018 |

$$\log \hat{rate} = 3.845 - 0.019 \times \text{Age}$$

- **Slope:** For every one month increase in Age, we expect the median respiratory rate to be multiplied by a factor of $\exp\{-0.019\} = 1.019$ breaths per minute.

- **Intercept:** The expected respiratory rate for a child who is 0 months old (a newborn) is $\exp\{3.845\} = 46.76$ breaths per minute.

# Confidence interval for $\beta_j$

- The confidence interval for the coefficient of $x$ describing its relationship with $\log(y)$ is

$$\hat{\beta}_j \pm t^* SE(\hat{\beta}_j)$$

- The confidence interval for the coefficient of $x$ describing its relationship with $y$ is

$$\exp\left\{\hat{\beta}_j \pm t^* SE(\hat{\beta}_j)\right\}$$

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# Coefficient of `Age`

| term | estimate | std.error | statistic | p.value | conf.low | conf.high |
|---|---|---|---|---|---|---|
| (Intercept) | 3.845 | 0.013 | 304.500 | 0 | 3.82 | 3.870 |
| Age | -0.019 | 0.001 | -25.839 | 0 | -0.02 | -0.018 |

The 95% confidence interval for the coefficient of Age in terms of `Rate`:

$$[\exp\{-0.02\}, \exp\{-0.018\}] = [0.981, 0.982]$$

**Interpretation:** We are 95% confident that for each additional month in age, we can expect the median respiratory rate to be multiplied by a factor of 0.981 to 0.982.

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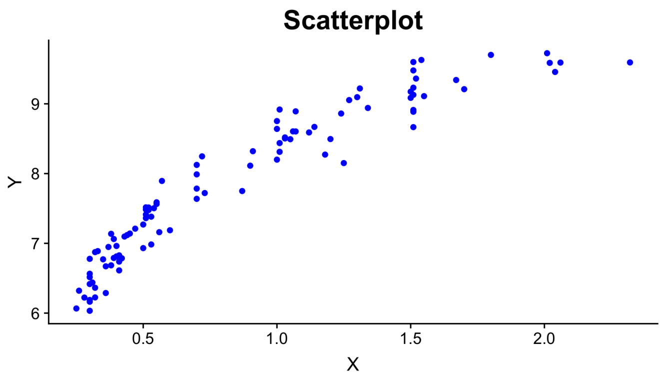
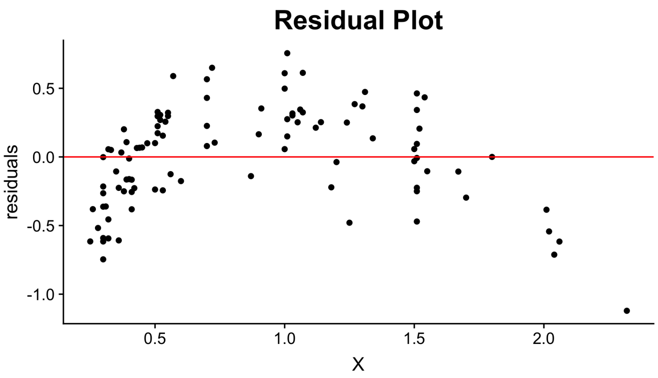
# Log Transformation on $x$




- Try a transformation on $X$ if the scatterplot shows some curvature but the variance is constant for all values of $X$

# Model with Transformation on $x$

$$y = \beta_0 + \beta_1 \log(x)$$

- **Intercept:** When $\log(x) = 0$, $(x = 1)$, $y$ is expected to be $\beta_0$ (i.e. the mean of $y$ is $\beta_0$)
- **Slope:** When $x$ is multiplied by a factor of $\mathbf{C}$, $y$ is expected to change by $\boldsymbol{\beta_1}\log(\mathbf{C})$ units, i.e. the mean of $y$ changes by $\boldsymbol{\beta_1}\log(\mathbf{C})$
    - *Example*: when $x$ is multiplied by a factor of 2, $y$ is expected to change by $\boldsymbol{\beta_1}\log(\mathbf{2})$ units

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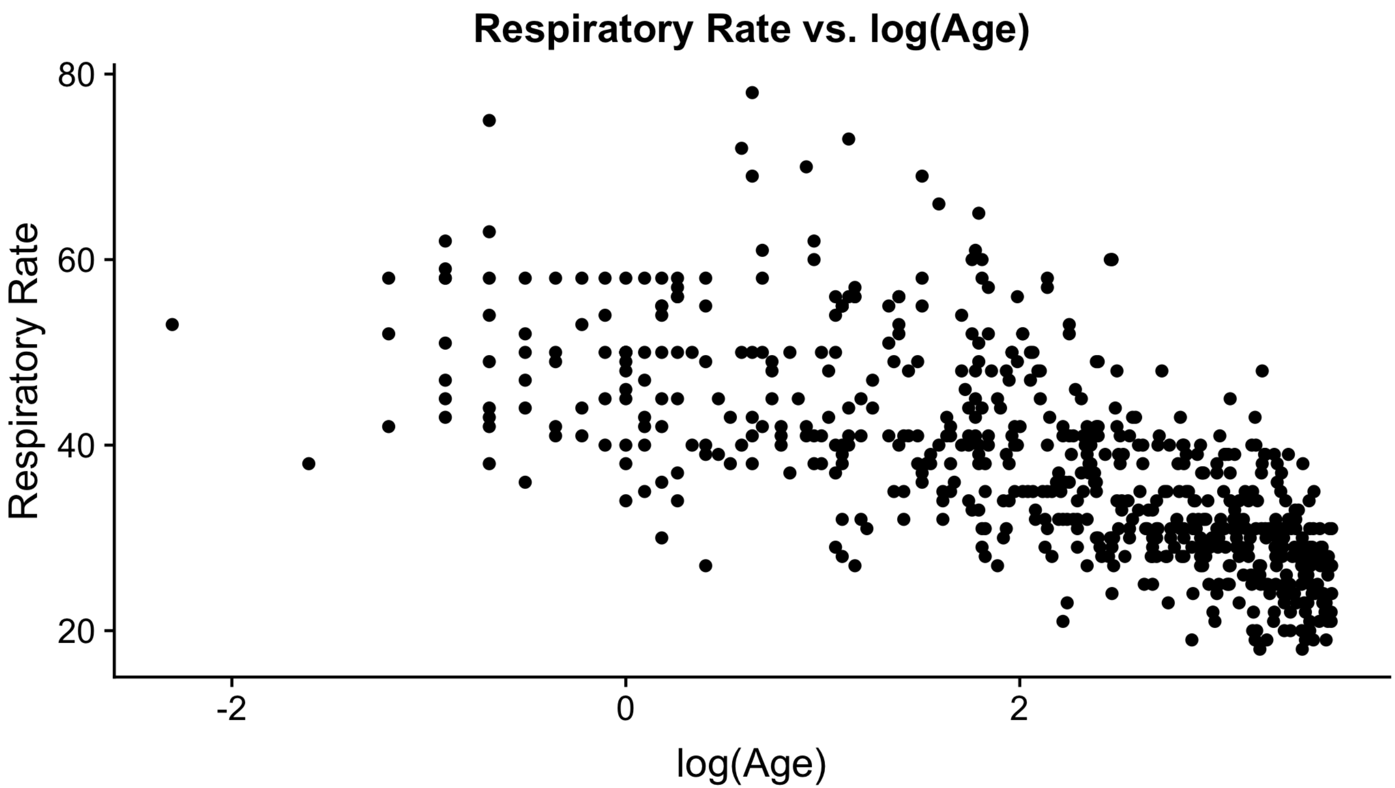
# Rate vs. log(Age)



# Rate vs. Age

| term | estimate | std.error | statistic | p.value | conf.low | conf.high |
| --- | --- | --- | --- | --- | --- | --- |
| (Intercept) | 50.134533 | 0.6319775 | 79.32961 | 0 | 48.893441 | 51.375625 |
| log.age | -5.982434 | 0.2626097 | -22.78070 | 0 | -6.498153 | -5.466715 |

1. Write the equation for the model of $y$ regressed on $\log(x)$.
2. Interpret the intercept in the context of the problem.
3. Interpret the slope in terms of how the mean respiratory rate changes when a child's age doubles.
4. Suppose a doctor has a patient who is currently 3 years old. Will this model provide a reliable prediction of the child's respiratory rate when her age doubles? Why or why not?

See Log Transformations in Linear Regression for more details about interpreting regression models with log-transformed variables.

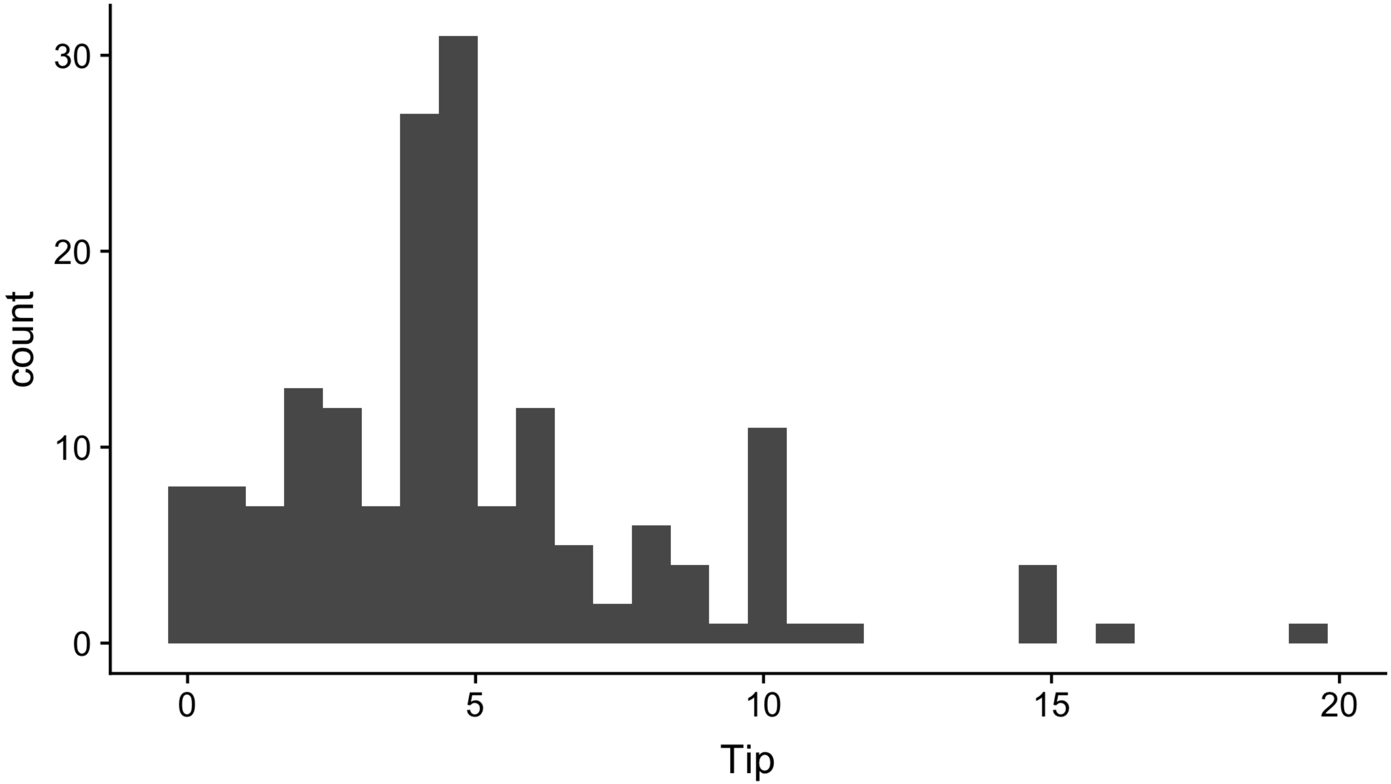
# Model Assessment & Selection

# Restaurant tips

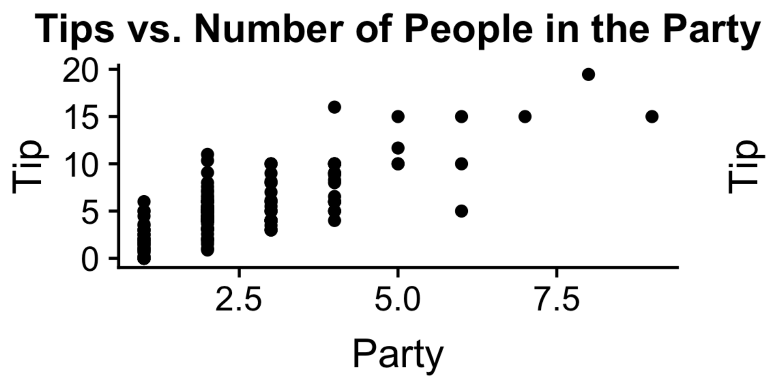
What affects the amount customers tip at a restaurant?

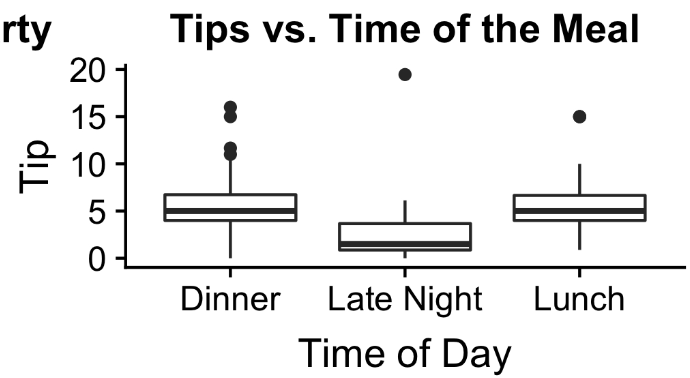
- **Response:**
  - `Tip`: amount of the tip
- **Predictors:**
  - `Party`: number of people in the party
  - `Meal`: time of day (Lunch, Dinner, Late Night)
  - `Age`: age category of person paying the bill (Yadult, Middle, SenCit)

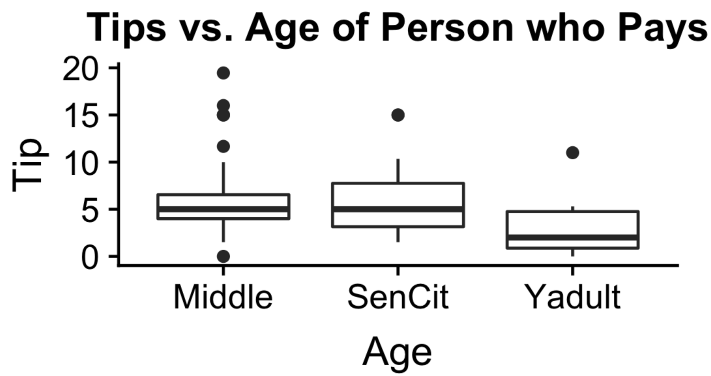
# Examining the Response Variable



# Response vs. Predictors



# ANOVA table for regression

We can use the Analysis of Variance (ANOVA) table to decompose the variability in our response variable

| | Sum of Squares | DF | Mean Square | F-Stat | p-value |
|---|---|---|---|---|---|
| Regression (Model) | $\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$ | $p$ | $\frac{MSS}{p}$ | $\frac{MMS}{RMS}$ | $P(F > \text{F-Stat})$ |
| Residual | $\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$ | $n-p-1$ | $\frac{RSS}{n-p-1}$ | | |
| Total | $\sum_{i=1}^{n}(y_i - \bar{y})^2$ | $n-1$ | $\frac{TSS}{n-1}$ | | |

The estimate of the regression variance, $\hat{\sigma}^2 = RMS$

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# $R^2$

- **Recall**: $R^2$ is the proportion of the variation in the response variable explained by the regression model
- $R^2$ will always increase as we add more variables to the model
    - If we add enough variables, we can always achieve $R^2 = 100\%$
- If we only use $R^2$ to choose a best fit model, we will be prone to choose the model with the most predictor variables

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# Restaurant tips: model

```
model1 <- lm(Tip ~ Party + Meal + Age , data = tips)
kable(tidy(model1),format="html",digits=3)
```

| term | estimate | std.error | statistic | p.value |
|---|---|---|---|---|
| (Intercept) | 1.254 | 0.394 | 3.182 | 0.002 |
| Party | 1.808 | 0.121 | 14.909 | 0.000 |
| MealLate Night | -1.632 | 0.407 | -4.013 | 0.000 |
| MealLunch | -0.612 | 0.402 | -1.523 | 0.130 |
| AgeSenCit | 0.390 | 0.394 | 0.990 | 0.324 |
| AgeYadult | -0.505 | 0.412 | -1.227 | 0.222 |

**Is this the best model to explain variation in Tips?**

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# Adjusted $R^2$

- **Adjusted $R^2$**: a version of $R^2$ that penalizes for unnecessary predictor variables
- Similar to $R^2$, it measures the proportion of variation in the response that is explained by the regression model
- Differs from $R^2$ by using the mean squares rather than sums of squares and therefore adjusting for the number of predictor variables

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# $R^2$ and Adjusted $R^2$

$$R^2 = \frac{\text{Total Sum of Squares} - \text{Residual Sum of Squares}}{\text{Total Sum of Squares}}$$

$$Adj.R^2 = \frac{\text{Total Mean Square} - \text{Residual Mean Square}}{\text{Total Mean Square}}$$

- $Adj.R^2$ can be used as a quick assessment to compare the fit of multiple models; however, it should not be the only assessment!
- Use $R^2$ when describing the relationship between the response and predictor variables

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# Restaurant tips: ANOVA

- R output

```
kable(anova(model1), format = "markdown", digits = 3)
```

| | Df | Sum Sq | Mean Sq | F value | Pr(>F) |
|---|---|---|---|---|---|
| Party | 1 | 1188.636 | 1188.636 | 311.002 | 0.000 |
| Meal | 2 | 88.460 | 44.230 | 11.573 | 0.000 |
| Age | 2 | 13.032 | 6.516 | 1.705 | 0.185 |
| Residuals | 163 | 622.979 | 3.822 | NA | NA |

- ANOVA table

| | Sum of Squares | DF | Mean Square | F-Stat | p-value |
|---|---|---|---|---|---|
| Regression (Model) | 1290.12829 | 5 | 258.025658 | 67.5113618 | 0 |
| Residual | 622.97932 | 163 | 3.821959 | | |
| Total | 1913.10761 | 168 | | | |

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# Calculating $R^2$ and Adj $R^2$

| | Sum of Squares | DF | Mean Square | F-Stat | p-value |
|---|---|---|---|---|---|
| Regression (Model) | 1290.12829 | 5 | 258.025658 | 67.5113618 | 0 |
| Residual | 622.97932 | 163 | 3.821959 | | |
| Total | 1913.10761 | 168 | | | |

```
#r-squared
mss <- 1290.12829
rss <- 622.97932
tss <- mss + rss
(r_sq <- (tss - rss)/tss)
```

```
## [1] 0.6743626
```

```
#adj r-squared
rms <- 3.821959
tms <- tss/(nrow(tips)-1)
(adj_r_sq <- (tms - rms)/tms)
```

```
## [1] 0.6643738
```

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# Restaurant tips: $R^2$ and Adj. $R^2$

```
glance(model1)
```

```
## # A tibble: 1 x 11
##   r.squared adj.r.squared sigma statistic  p.value    df logLik   AIC
##       <dbl>         <dbl> <dbl>     <dbl>    <dbl> <int>  <dbl> <dbl> <d
## 1     0.674         0.664  1.95      67.5 6.14e-38     6  -350.  714.  7
## # … with 2 more variables: deviance <dbl>, df.residual <int>
```

- Close values of $R^2$ and Adjusted $R^2$ indicate that the variables in the model are significant in understanding variation in the response, i.e. that there aren't a lot of unnecessary variables in the model

# ANOVA F Test

- Using the ANOVA table, we can test whether any variable in the model is a significant predictor of the response. We conduct this test using the following hypotheses:

$$\begin{aligned} H_0 &: \beta_1 = \beta_2 = \cdots = \beta_p = 0 \\ H_a &: \text{at least one } \beta_j \text{ is not equal to 0} \end{aligned}$$

- The statistic for this test is the $F$ test statistic in the ANOVA table
- We calculate the p-value using an $F$ distribution with $p$ and $(n-p-1)$ degrees of freedom

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# ANOVA F Test in R

```
model0 <- lm(Tip ~ 1, data = tips)
```

```
model1 <- lm(Tip ~ Party + Meal + Age , data = tips)
```

```
kable(anova(model0, model1), format="markdown", digits = 3)
```

| Res.Df | RSS | Df | Sum of Sq | F | Pr(>F) |
|---:|---:|---:|---:|---:|---:|
| 168 | 1913.108 | NA | NA | NA | NA |
| 163 | 622.979 | 5 | 1290.128 | 67.511 | 0 |

At least one coefficient is non-zero, i.e. at least one predictor in the model is significant

# Testing subset of coefficients

- Sometimes we want to test whether a subset of coefficients are all equal to 0
- This is often the case when we want test
  - whether a categorical variable with $k$ levels is a significant predictor of the response
  - whether the interaction between a categorical and quantitative variable is significant
- To do so, we will use the **Nested F Test**

# Nested F Test

- Suppose we have a full and reduced model:

$$\begin{aligned} \text{Full} &: y = \beta_0 + \beta_1 x_1 + \dots + \beta_q x_q + \beta_{q+1} x_{q+1} + \dots \beta_p x_p \\ \text{Red} &: y = \beta_0 + \beta_1 x_1 + \dots + \beta_q x_q \end{aligned}$$

- We want to test whether any of the variables $x_{q+1}, x_{q+2}, \ldots, x_p$ are significant predictors. To do so, we will test the hypothesis:

$$\begin{aligned} &H_0 : \beta_{q+1} = \beta_{q+2} = \dots = \beta_p = 0 \\ &H_a : \text{at least one } \beta_j \text{ is not equal to } 0 \end{aligned}$$

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# Nested F Test

- The test statistic for this test is

$$F = \frac{(RSS_{reduced} - RSS_{full}) \Big/ (p_{full} - p_{reduced})}{RSS_{full} \Big/ (n - p_{full} - 1)}$$

- Calculate the p-value using the F distribution with $(p_{full} - p_{reduced})$ and $(n - p_{full} - 1)$ degrees of freedom

# Is `Meal` a significant predictor of tips?

| term | estimate | std.error | statistic | p.value |
|---|---|---|---|---|
| (Intercept) | 1.254 | 0.394 | 3.182 | 0.002 |
| Party | 1.808 | 0.121 | 14.909 | 0.000 |
| AgeSenCit | 0.390 | 0.394 | 0.990 | 0.324 |
| AgeYadult | -0.505 | 0.412 | -1.227 | 0.222 |
| MealLate Night | -1.632 | 0.407 | -4.013 | 0.000 |
| MealLunch | -0.612 | 0.402 | -1.523 | 0.130 |

# Tips data: Nested F Test

$$H_0: \beta_{latenight} = \beta_{lunch} = 0$$
$$H_a: \text{ at least one } \beta_j \text{ is not equal to } 0$$

```
reduced <- lm(Tip ~ Party + Age, data = tips)
```

```
full <- lm(Tip ~ Party + Age + Meal, data = tips)
```

```
kable(anova(reduced, full), format="markdown", digits = 3)
```

| Res.Df | RSS | Df | Sum of Sq | F | Pr(>F) |
|---:|---:|---:|---:|---:|---:|
| 165 | 686.444 | NA | NA | NA | NA |
| 163 | 622.979 | 2 | 63.465 | 8.303 | 0 |

**At least one coefficient associated with `Meal` is not zero. Therefore, `Meal` is a significant predictor of `Tips`.**

Why is it not good practice to use the individual p-values to determine a categorical variable with $k > 2$ levels) is significant?

*Hint*: What does it actually mean if none of the $k - 1$ p-values are significant?

# Practice with Interactions

| term | estimate | std.error | statistic | p.value |
|---|---|---|---|---|
| (Intercept) | 1.2764989 | 0.4910882 | 2.5993270 | 0.0102086 |
| Party | 1.7947980 | 0.1715003 | 10.4652753 | 0.0000000 |
| AgeSenCit | 0.4007889 | 0.3969295 | 1.0097230 | 0.3141431 |
| AgeYadult | -0.4701634 | 0.4197146 | -1.1201978 | 0.2642977 |
| MealLate Night | -1.8454674 | 0.7089728 | -2.6030159 | 0.0101039 |
| MealLunch | -0.4608832 | 0.8651044 | -0.5327487 | 0.5949421 |
| Party:MealLate Night | 0.1108600 | 0.2846584 | 0.3894491 | 0.6974586 |
| Party:MealLunch | -0.0500822 | 0.2825586 | -0.1772455 | 0.8595384 |

1. What is the baseline level for `Meal`?
2. How do we expect the mean tips to change when `Meal == "Late Night"`, holding Age and Party constant?
3. How does the slope of `Party` change when `Meal == "Late Night"`, holding Age and Party constant?

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# Nested F test for interactions

## Are there any significant interaction effects with Party in the model?

```
reduced <- lm(Tip ~ Party + Age + Meal, data = tips)
```

```
full <- lm(Tip ~ Party + Age + Meal + Age* Party + Meal * Party,
           data = tips)
```

```
kable(anova(reduced, full ), format="markdown", digits = 3)
```

| Res.Df | RSS | Df | Sum of Sq | F | Pr(>F) |
|---:|---:|---:|---:|---:|---:|
| 163 | 622.979 | NA | NA | NA | NA |
| 159 | 615.380 | 4 | 7.6 | 0.491 | 0.742 |

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# Final model for now

We conclude that there are no significant interactions with `Party` in the model. Therefore, we will use the original model that only included main effects.

| term | estimate | std.error | statistic | p.value |
|---|---|---|---|---|
| (Intercept) | 1.254 | 0.394 | 3.182 | 0.002 |
| Party | 1.808 | 0.121 | 14.909 | 0.000 |
| AgeSenCit | 0.390 | 0.394 | 0.990 | 0.324 |
| AgeYadult | -0.505 | 0.412 | -1.227 | 0.222 |
| MealLate Night | -1.632 | 0.407 | -4.013 | 0.000 |
| MealLunch | -0.612 | 0.402 | -1.523 | 0.130 |