Multiple Linear Regression

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09.18.19

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Announcements

- HW 01 **due TODAY at 11:59p**
- Reading 03 for Monday
- HW 02 due Wednesday, 9/25 at 11:59p

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Today's Agenda

- Introducing multiple linear regression

R Packages used in the notes

```
library(tidyverse)
library(knitr)
library(broom)
library(Sleuth3) # case 1202 dataset
library(cowplot) # use plot_grid function
```

Multiple Linear Regression

Example: Starting Wages

- In the 1970s Harris Trust and Savings Bank was sued for discrimination on the basis of gender.
- The defense presented an analysis of the salaries for skilled, entry-level clerical employees as evidence.
- **Question**: Did female employees receive lower starting salaries on average than male employees with similar experience and qualifications?

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Data

```
glimpse(wages)
```

```
## Observations: 93
## Variables: 6
## $ Bsal   <int> 5040, 6300, 6000, 6000, 6000, 6840, 8100, 6000, 6000, 690…
## $ Senior <int> 96, 82, 67, 97, 66, 92, 66, 82, 88, 75, 89, 91, 66, 86, 9…
## $ Age    <int> 329, 357, 315, 354, 351, 374, 369, 363, 555, 416, 481, 33…
## $ Educ   <int> 15, 15, 15, 12, 12, 15, 16, 12, 12, 15, 12, 15, 15, 15, 1…
## $ Exper  <dbl> 14.0, 72.0, 35.5, 24.0, 56.0, 41.5, 54.5, 32.0, 252.0, 13…
## $ Female <fct> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, …
```

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Variables

Explanatory

- **`Educ`**: years of education
- **`Exper`**: months of previous work experience (before hire at bank)
- **`Female`**: 1 if female, 0 if male
- **`Senior`**: months worked at bank since hire
- **`Age`**: age in months

Response

- **`Bsal`**: annual salary at time of hire

Salary comparison

- **Question:** Did female employees receive lower starting salaries on average than male employees with similar experience and qualifications?

Using ANOVA

$$H_0 : \mu_F = \mu_M$$
$$H_a : \mu_F \neq \mu_M$$

term	df	sumsq	meansq	statistic	p.value
Female	1	14045183	14045183.2	39.597	0
Residuals	91	32278107	354704.5	NA	NA

- What's your conclusion?
- What is a disadvantage to using this method to answer the question?

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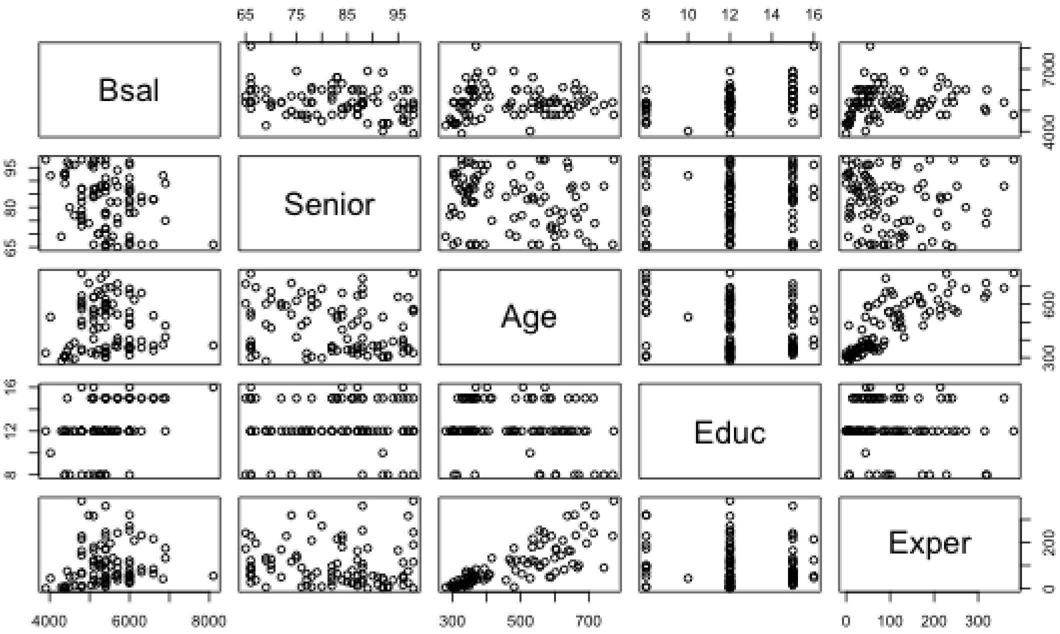
Salary vs. Other Variables

```
pairs(Bsal ~ Senior + Age + Educ + Exper, data=wages)
```



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Multiple Regression Model

- We will calculate a multiple linear regression model with the following form:

$$Bsal = \beta_0 + \beta_1 \text{Senior} + \beta_2 \text{Age} + \beta_3 \text{Educ} + \beta_4 \text{Exper} + \beta_5 \text{Female}$$

- Similar to simple linear regression, this model assumes that at each combination of the predictor variables, the values `Bsal` follow a Normal distribution

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Regression Model

- Recall: The simple linear regression model assumes

$$y|x \sim N(\beta_0 + \beta_1 x, \sigma^2)$$

- Similarly: The multiple linear regression model assumes

$$y|x_1, x_2, \ldots, x_p \sim N(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p, \sigma^2)$$

- For a given observation $(x_{i1}, x_{i2} \ldots, x_{iP}, y_i)$

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + \epsilon_i \quad \epsilon_i \sim N(0, \sigma^2)$$

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Regression Model

- At any combination of $x's$, the true mean value of y is

$$\mu_y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p$$

- We will use multiple linear regression to estimate the mean y for any combination of $x's$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \cdots + \hat{\beta}_p x_p$$

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Regression Output

```
bsal_model <- lm(Bsal ~ Senior + Age + Educ + Exper + Female,
                 data=wages)
kable(tidy(bsal_model),format="html",digits=3)
```

term	estimate	std.error	statistic	p.value
(Intercept)	6277.893	652.271	9.625	0.000
Senior	-22.582	5.296	-4.264	0.000
Age	0.631	0.721	0.876	0.384
Educ	92.306	24.864	3.713	0.000
Exper	0.501	1.055	0.474	0.636
Female1	-767.913	128.970	-5.954	0.000

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Interpreting $\hat{\beta}_j$

- An estimated coefficient $\hat{\beta}_j$ is the amount y is expected to change when x_j increases by one unit **holding the values all other predictor variables constant**
- *Example:* The estimated coefficient for `Educ` is 92.31. This means for each additional year of education an employee has, we expect starting salary to increase by about $92.31, holding all other predictor variables constant.

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Hypothesis Tests for $\hat{\beta}_j$

- We want to test whether a particular coefficient has a value of 0 in the population, given all other variables in the model:

$$
\begin{aligned}
H_0 &: \beta_j = 0 \\
H_a &: \beta_j \neq 0
\end{aligned}
$$

- The test statistic reported in R is the following:

$$\text{test statistic} = t = \frac{\hat{\beta}_j - 0}{SE(\hat{\beta}_j)}$$

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Salary

term	estimate	std.error	statistic	p.value
(Intercept)	6277.893	652.271	9.625	0.000
Senior	-22.582	5.296	-4.264	0.000
Age	0.631	0.721	0.876	0.384
Educ	92.306	24.864	3.713	0.000
Exper	0.501	1.055	0.474	0.636
Female1	-767.913	128.970	-5.954	0.000

Given the other variables in the model, are the following significant predictors of salary at time of hire (`Bsal`)?

- Education (`Educ`)
- Experience (`Exper`)

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Confidence Interval for β_j

The C confidence interval for β_j

$$\hat{\beta}_j \pm t^* SE(\hat{\beta}_j)$$

where t^* follows a t distribution with with $(n-p-1)$ degrees of freedom

- **General Interpretation**: We are C confident that the interval LB to UB contains the population coefficient of x_j. Therefore, for every one unit increase in x_j, we expect y to change LB to UB units, holding all else constant.

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CI for **Educ**

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	6277.893	652.271	9.625	0.000	4981.434	7574.353
Senior	-22.582	5.296	-4.264	0.000	-33.108	-12.056
Age	0.631	0.721	0.876	0.384	-0.801	2.063
Educ	92.306	24.864	3.713	0.000	42.887	141.725
Exper	0.501	1.055	0.474	0.636	-1.597	2.598
Female1	-767.913	128.970	-5.954	0.000	-1024.255	-511.571

Interpret the 95% confidence interval for the coefficient of Educ.

Notes about CI and Hypothesis Tests

- If the sample size is large enough, the test will likely result in rejecting $H_0 : \beta_j = 0$ even x_j has a very small effect on y
  - Consider the practical significance of the result not just the statistical significance
  - Use the confidence interval to draw conclusions instead of p-values
- If the sample size is small, there may not be enough evidence to reject $H_0 : \beta_j = 0$
  - When you fail to reject the null hypothesis, **DON'T** immediately conclude that the variable has no association with the response.
  - There may be a linear association that is just not strong enough to detect given your data, or there may be a non-linear association.

Prediction

- We calculate predictions the same as with simple linear regression
- **Example:** Suppose we want to predict the starting wages for a female who is 28 years old with 12 years of education, 11 months seniority and 2 years of prior experience.

$$\begin{aligned}\widehat{bsal} = & 6277.893 - 22.582 \times \text{Senior} + 0.631 \times \text{Age} \\ & + 92.306 \times \text{Educ} + 0.501 \times \text{Exper} - 767.913 \times \text{Female}\end{aligned}$$

```
6277.893 - 22.582 * 11 + 0.631 * 28 + 92.306 * 12 + 0.501 * 24 - 7
```

```
## [1] 6398.942
```

Prediction

- Just like with simple linear regression, we can use the **`predict.lm()`** function in R to calculate the appropriate intervals for our predicted values

- Suppose we want to predict the starting wages for a female who is 28 years old with 12 years of education, 11 months seniority and 2 years of prior experience.

```
x0 <- data.frame(Senior= 11, Age = 28, Educ = 12, Exper = 24, Fema
predict.lm(bsal_model, x0, interval = "prediction")
```

```
##        fit      lwr      upr
## 1 6398.93 4967.054 7830.805
```

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Prediction

Suppose we want to predict the mean age for the subset of all females who are 28 years old with 12 years of education, 11 months of seniority and 2 years of prior experience.

- How will the predicted value change?
- How will the interval change?

```
x0 <- data.frame(Senior= 11, Age = 28, Educ = 12, Exper = 24, Fema
predict.lm(bsal_model, x0, interval = "confidence")
```

```
##        fit      lwr      upr
## 1 6398.93 5383.844 7414.016
```

Cautions

- **Do not extrapolate!** Because there are multiple explanatory variables, you can extrapolation in many ways
- The multiple regression model only shows **association, not causality**
    - To prove causality, you must have a carefully designed experiment or carefully account for confounding variables in an observational study

Assumptions

Assumptions

The confidence intervals and hypothesis tests are reliable only when the regression assumptions are reasonably satisfied

1. **Linearity:** Response variable has a linear relationship with the explanatory variables in the model
2. **Constant Variance:** The regression variance is the same for all set of predictor variables $(x_1, \dots, x_p)$
3. **Normality:** For a given $(x_1, \dots, x_p)$, the distribution of y around its mean is Normal
4. **Independence:** All observations are independent

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Scatterplots

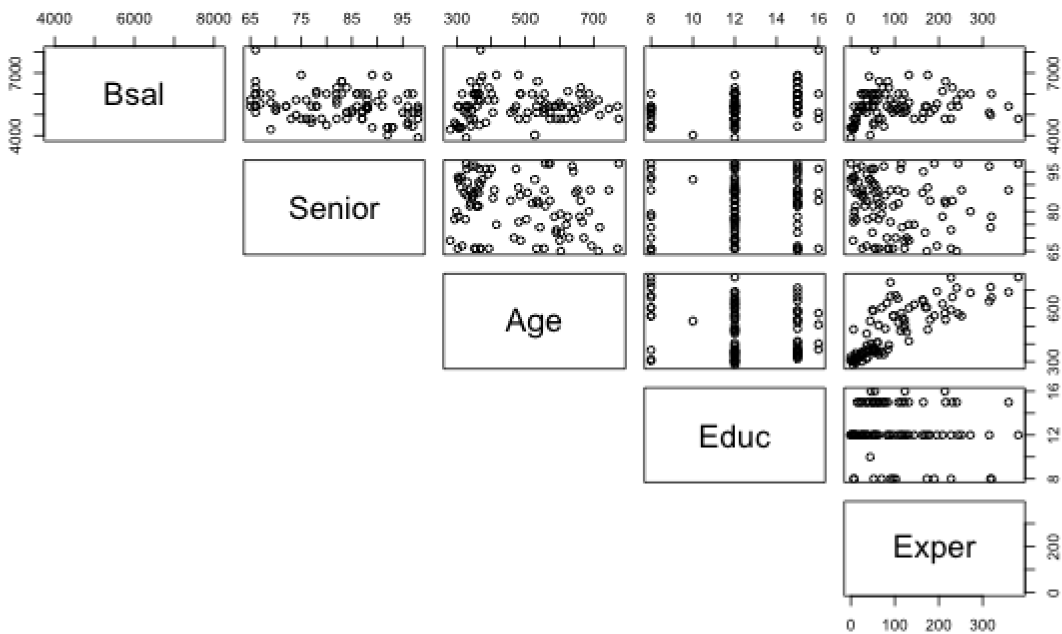
- Look at a scatterplot of the response variable vs. each of the predictor variables in the exploratory data analysis before calculating the regression model
- This is a good way to check for obvious departures from linearity
  - Could be an indication that a higher order term or transformation is needed (will discuss this next class)

Residual Plots

- **Plot the residuals vs. the predicted values**
  - Can expose issues such at outliers or nonconstant variance
- **Plot the residuals vs. each of the predictors**
  - Can expose issues between the response and a predictor variable that didn't show in the exploratory data analysis
  - Use boxplots to plot residuals versus categorical predictor variables
- Residual plots should show no systematic pattern
- Plot a histogram and QQ-plot of the residuals to check Normality

Scatterplots

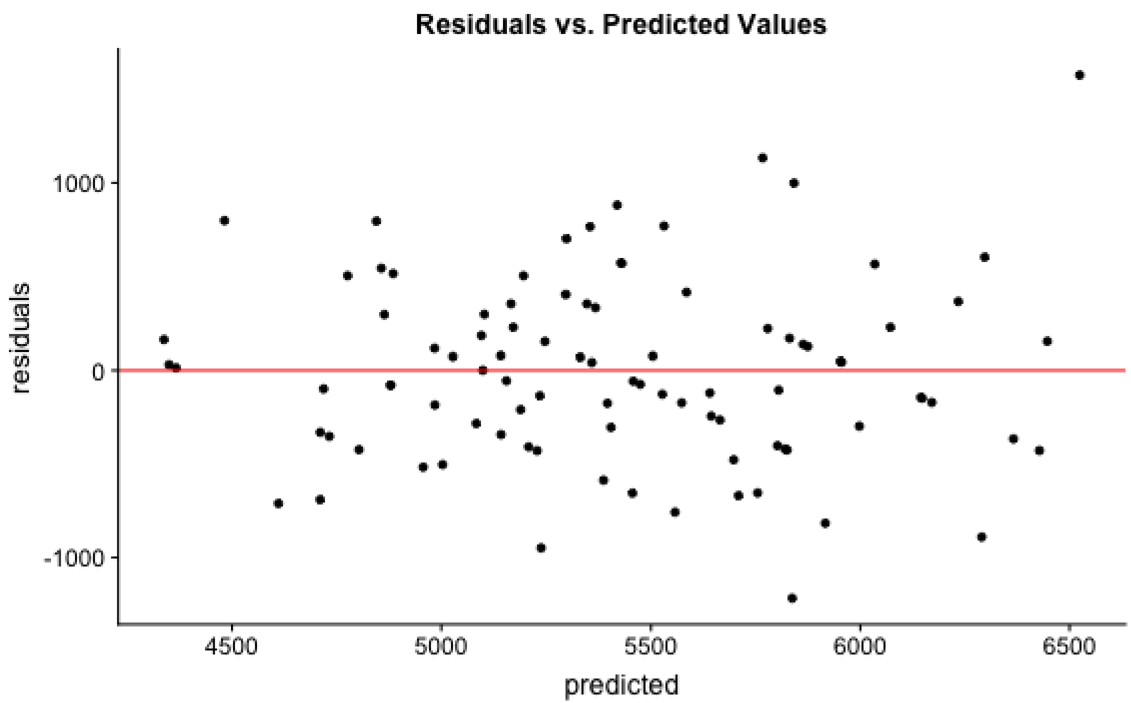
```
pairs(Bsal ~ Senior + Age + Educ + Exper, data = wages, lower.pane
```



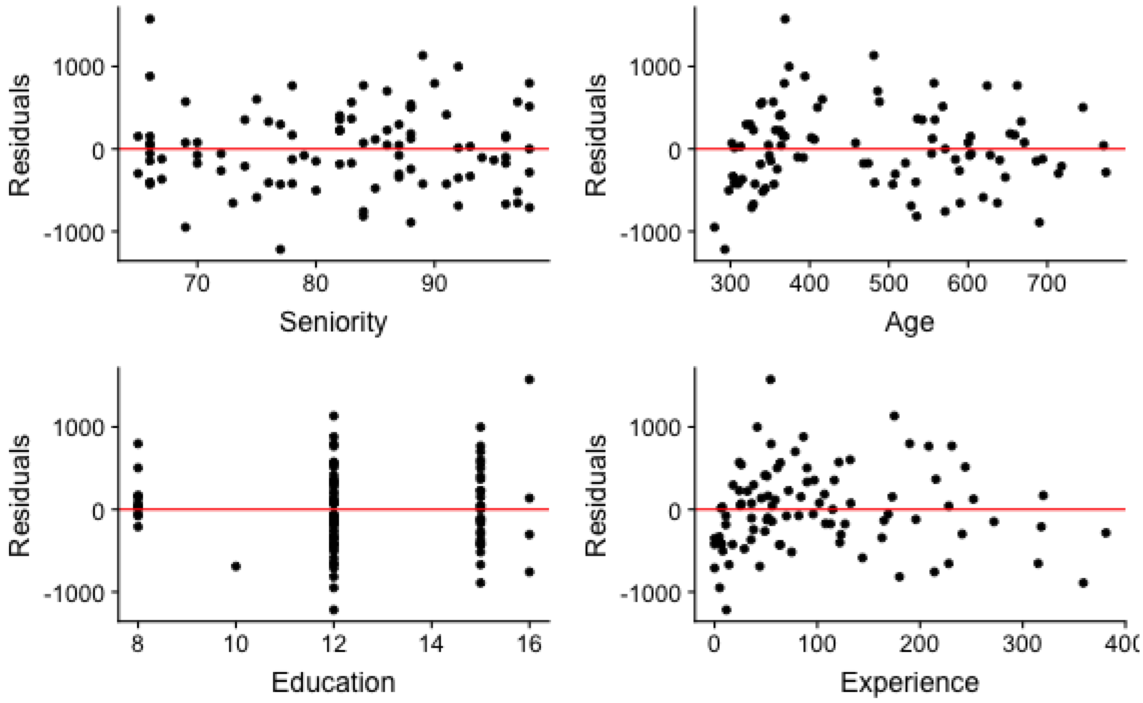
- Only include a few variables in a single pairs plot; otherwise, the scatterplots are too small to be readable.

Residuals vs. Predicted Values

```
wages <- wages %>%
  mutate(predicted = predict.lm(bsal_model), residuals = resid(bsa
ggplot(data=wages,aes(x=predicted, y=residuals)) +
  geom_point() +
  geom_hline(yintercept=0,color="red") +
  labs(title="Residuals vs. Predicted Values")
```



Residuals vs. Predictors



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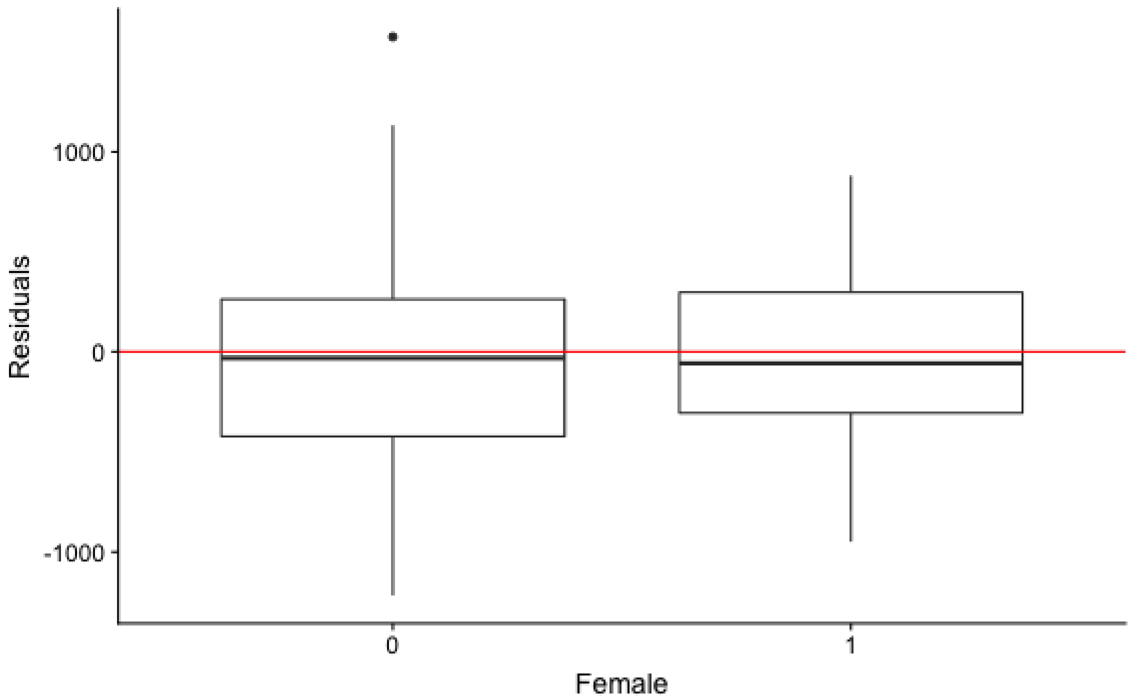
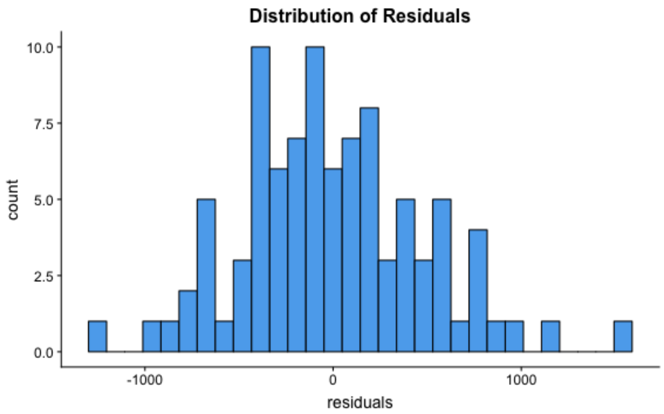
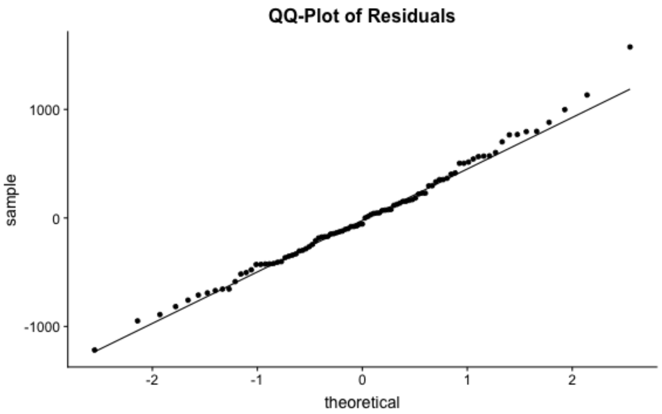
Residuals vs. Predictors

```
ggplot(data=wages,aes(x=Female,y=residuals)) +
  geom_boxplot() +
  geom_hline(yintercept=0,color="red") +
  labs(x = "Female",
       y="Residuals")
```



Normality of Residuals





Math Foundation

Regression Model

- The multiple linear regression model assumes

$$y|x_1, \dots, x_p \sim N(\beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p, \sigma^2)$$

- For a given observation $(x_{i1}, \dots, x_{ip}, y_i)$, we can rewrite the previous statement as

$$y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \epsilon_i \qquad \epsilon_i \sim N(0, \sigma^2)$$

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Estimating σ^2

- For a given observation $(x_{i1}, \dots, x_{ip}, y_i)$ the residual is

$$e_i = y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \dots + \hat{\beta}_p x_{ip})$$

- The estimated regression variance is

.alert[

$$\hat{\sigma}^2 = \frac{RSS}{n-p-1} = \frac{\sum\limits_{i=1}^{n} [y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \dots + \hat{\beta}_p x_{ip})]^2}{n-p-1}$$

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Calculating $\hat{\sigma}^2$

Salary: Estimating $\hat{\sigma}^2$

```
(glance(bsal_model)$sigma)^2
```

```
## [1] 258156
```

```
kable(tidy(aov(bsal_model)),format="html",digits=3)
```

term	df	sumsq	meansq	statistic	p.value
Senior	1	3784914.70	3784914.70	14.661	0.000
Age	1	17010.44	17010.44	0.066	0.798
Educ	1	8814046.86	8814046.86	34.142	0.000
Exper	1	2095479.05	2095479.05	8.117	0.005
Female	1	9152264.30	9152264.30	35.452	0.000
Residuals	87	22459574.96	258156.03	NA	NA