Multiple Linear Regression

Assumptions & Special Predictors

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09.23.19

STA 210

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Announcements

- Lab 04 **due tomorrow at 11:59p**
- HW 02 **due Wednesday, 9/25 at 11:59p**
- Team Feedback #1 **due Wednesday, 9/25 at 11:59p**
  - Please provide honest and constructive feedback. This team feedback will be graded for completion.

Today's agenda

- Math details of multiple linear regression
- Assumptions for multiple linear regression
- Special predictors

R packages

```
library(tidyverse)
library(knitr)
library(broom)
library(Sleuth3) # case 1202 dataset
library(cowplot) # use plot_grid function
```

Starting wages data

Explanatory

- **`Educ`:** years of Education
- **`Exper`:** months of previous work Experience (before hire at bank)
- **`Female`:** 1 if female, 0 if male
- **`Senior`:** months worked at bank since hire
- **`Age`:** Age in months

Response

- **`Bsal`:** annual salary at time of hire

Starting wages

```
glimpse(wages)
```

```
## Observations: 93
## Variables: 6
## $ Bsal   <int> 5040, 6300, 6000, 6000, 6000, 6840, 8100, 6000, 6000, 690…
## $ Senior <int> 96, 82, 67, 97, 66, 92, 66, 82, 88, 75, 89, 91, 66, 86, 9…
## $ Age    <int> 329, 357, 315, 354, 351, 374, 369, 363, 555, 416, 481, 33…
## $ Educ   <int> 15, 15, 15, 12, 12, 15, 16, 12, 12, 15, 12, 15, 15, 15, 1…
## $ Exper  <dbl> 14.0, 72.0, 35.5, 24.0, 56.0, 41.5, 54.5, 32.0, 252.0, 13…
## $ Female <fct> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, …
```

Regression model

```
bsal_model <- lm(Bsal ~ Senior + Age + Educ + Exper + Female,
                 data=wages)
kable(tidy(bsal_model,conf.int=TRUE),format="html",digits=3)
```

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	6277.893	652.271	9.625	0.000	4981.434	7574.353
Senior	-22.582	5.296	-4.264	0.000	-33.108	-12.056
Age	0.631	0.721	0.876	0.384	-0.801	2.063
Educ	92.306	24.864	3.713	0.000	42.887	141.725
Exper	0.501	1.055	0.474	0.636	-1.597	2.598
Female1	-767.913	128.970	-5.954	0.000	-1024.255	-511.571

Math Details

Regression Model

- The multiple linear regression model assumes

$$y|x_1, x_2, \ldots, x_p \sim N(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p, \sigma^2)$$

- For a given observation $(x_{i1}, x_{i2}, \ldots, x_{ip}, y_i)$, we can rewrite the previous statement as

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + \epsilon_i \qquad \epsilon_i \sim N(0, \sigma^2)$$

Estimating σ^2

- For a given observation $(x_{i1}, x_{i2}, \dots, x_{ip}, y_i)$ the residual is

$$e_i = y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \dots + \hat{\beta}_p x_{ip})$$

- The estimated value of the regression variance , σ^2, is

$$\hat{\sigma}^2 = \frac{RSS}{n-p-1} = \frac{\sum_{i=1}^n e_i^2}{n-p-1}$$

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Estimating Coefficients

- One way to estimate the coefficients is by taking partial derivatives of the formula

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n}[y_i - (\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} \cdots + \beta_p x_{ip})]^2$$

- This produces messy formulas, so instead we can use matrix notation for multiple linear regression and estimate the coefficients using rules from linear algebra.
  - For more details, see Section 1.2 of the textbook and the supplemental notes Matrix Notation for Multiple Linear Regression
  - **Note:** You are <u>not</u> required to know matrix notation for MLR in this class

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Assumptions

Assumptions

Inference on the regression coefficients and predictions are reliable only when the regression assumptions are reasonably satisfied:

1. **Linearity:** Response variable has a linear relationship with the predictor variables in the model

2. **Constant Variance:** The regression variance is the same for all set of predictor variables $(x_1, \ldots, x_p)$

3. **Normality:** For a given set of predictors $(x_1, \ldots, x_p)$, the response, y, follows a Normal distribution around its mean

4. **Independence:** All observations are independent

Scatterplots

- Look at a scatterplot of the response variable vs. each of the predictor variables in the exploratory data analysis before calculating the regression model
- This is a good way to check for obvious departures from linearity
  - Could be an indication that a higher order term or transformation is needed

Residual Plots

- **Plot the residuals vs. the predicted values**
  - Can expose issues such at outliers or non-constant variance
  - Should have no systematic pattern
- **Plot the residuals vs. each of the predictors**
  - Can expose issues between the response and a predictor variable that didn't show in the exploratory data analysis
  - Use box plots to plot residuals versus categorical predictor variables
  - Should have no systematic pattern
- **Plot a histogram and QQ-plot of the residuals**
  - Check normality

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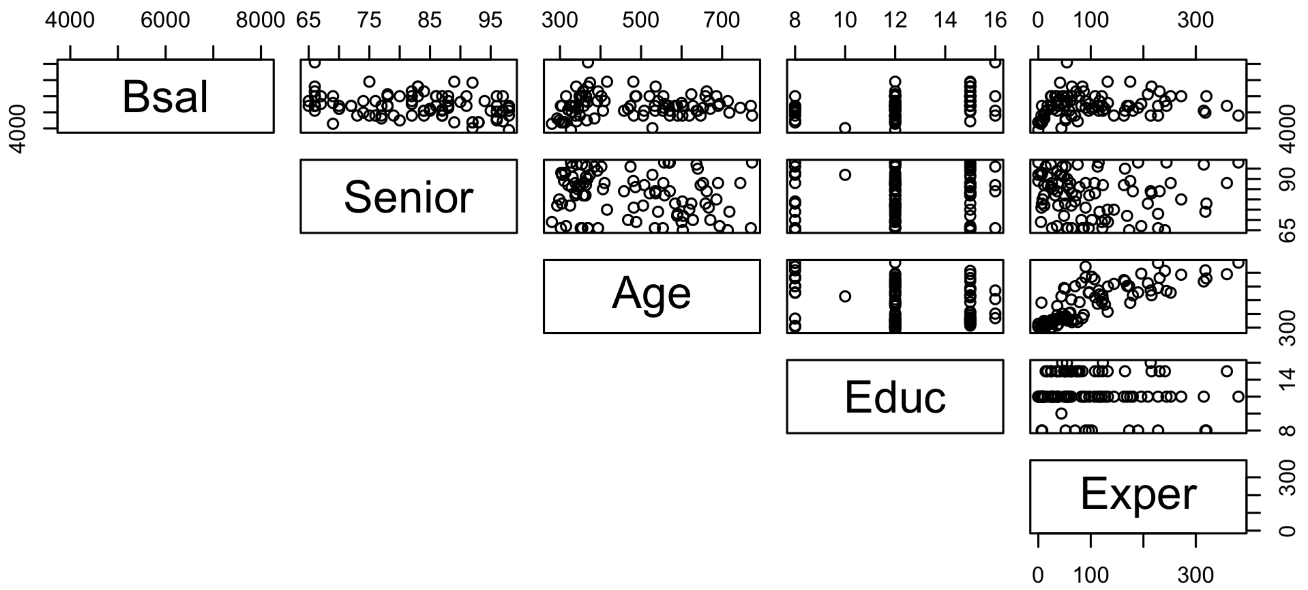
Scatterplots

```r
pairs(Bsal ~ Senior + Age + Educ + Exper, data = wages,
      lower.panel = NULL)
```



- Only include a 4 - 5 variables in a single pairs plot; otherwise, the scatterplots are too small to be readable

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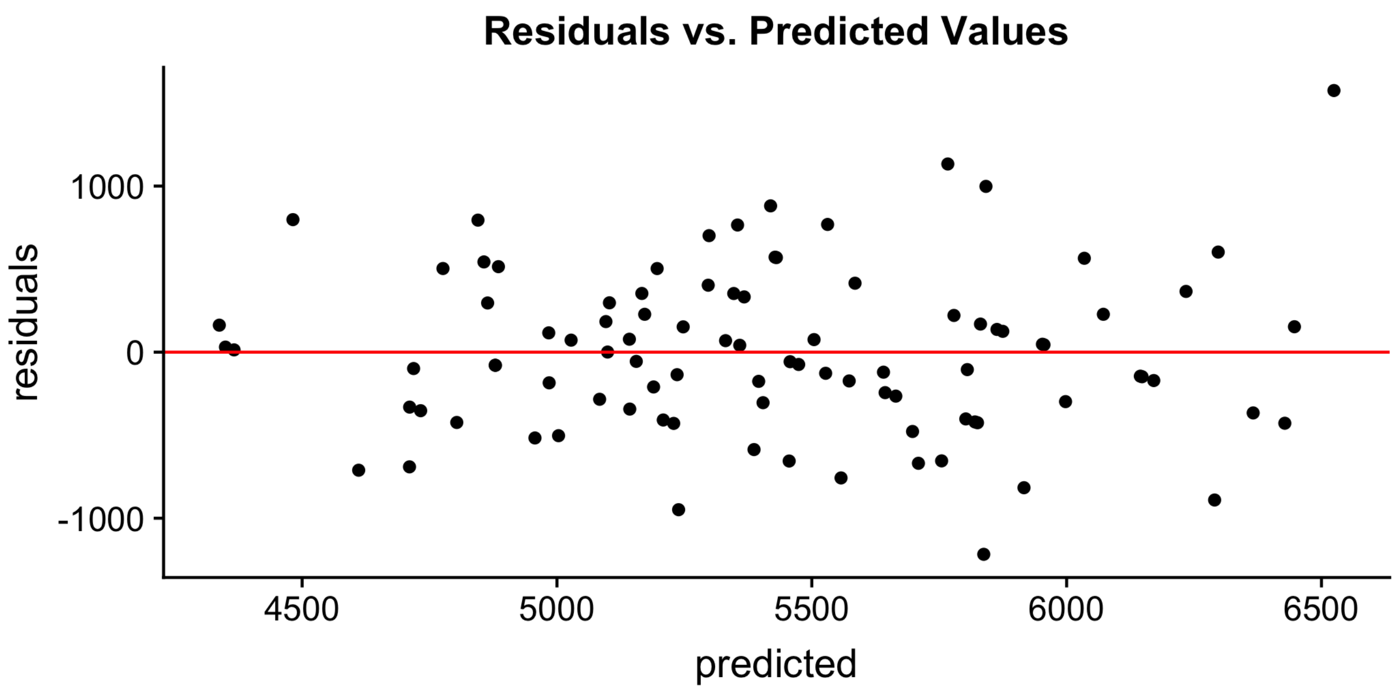
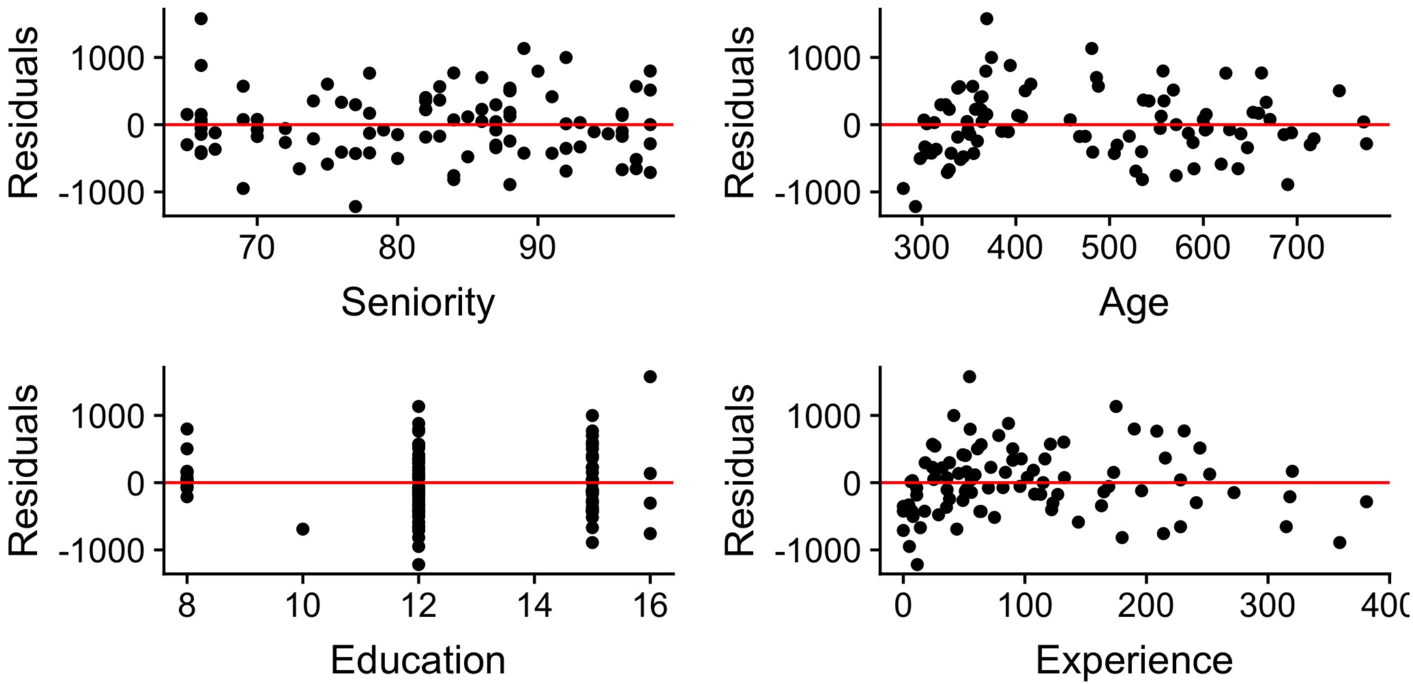
Residuals vs. Predicted Values

```
wages <- wages %>%
  mutate(predicted = predict.lm(bsal_model), residuals = resid(bsa
ggplot(data=wages,aes(x=predicted, y=residuals)) +
  geom_point() +
  geom_hline(yintercept=0,color="red") +
  labs(title="Residuals vs. Predicted Values")
```

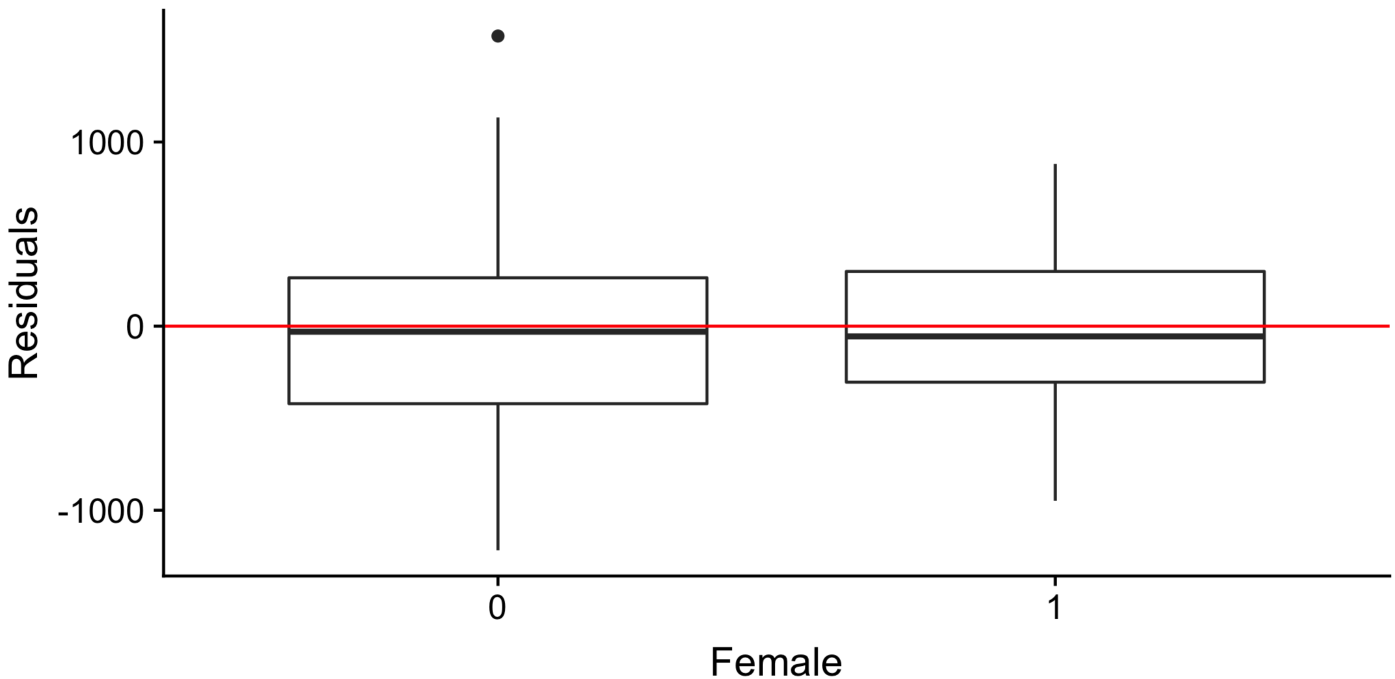


Residuals vs. Predictors

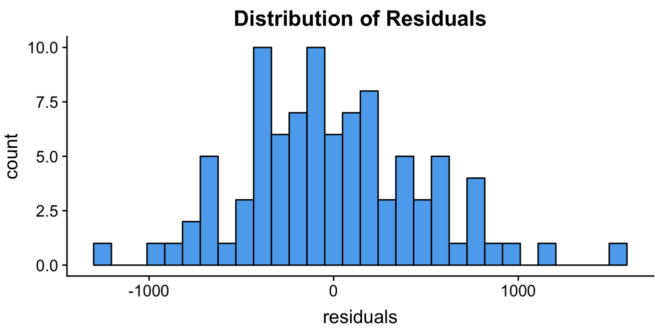


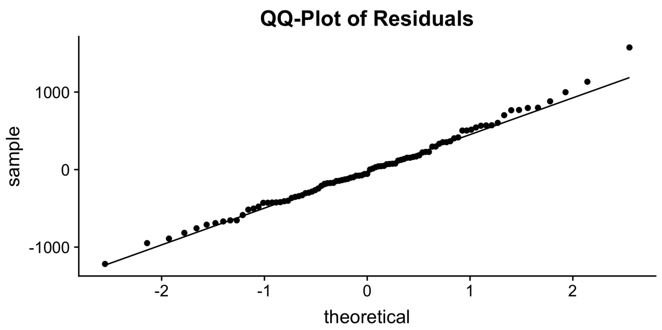
Residuals vs. Predictors

```
ggplot(data=wages,aes(x=Female,y=residuals)) +
  geom_boxplot() +
  geom_hline(yintercept=0,color="red") +
  labs(x = "Female",
       y="Residuals")
```



Normality of Residuals





Special Predictors

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Interpreting the Intercept

term	estimate	std.error	statistic	p.value
(Intercept)	6277.893	652.271	9.625	0.000
Senior	-22.582	5.296	-4.264	0.000
Age	0.631	0.721	0.876	0.384
Educ	92.306	24.864	3.713	0.000
Exper	0.501	1.055	0.474	0.636
Female1	-767.913	128.970	-5.954	0.000

- Interpret the intercept.
- Is this interpretation meaningful? Why or why not?

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Mean-Centered Variables

- To have a meaningful interpretation of the intercept, use **mean-centered** predictor variables in the model (quantitative predictors only)

- A mean-centered variable is calculated by subtracting the mean from each value of the variable, i.e.

$$x_{ip} - \bar{x}_{.p}$$

- Now the intercept is interpreted as the expected value of the response at the mean value of all quantitative predictors

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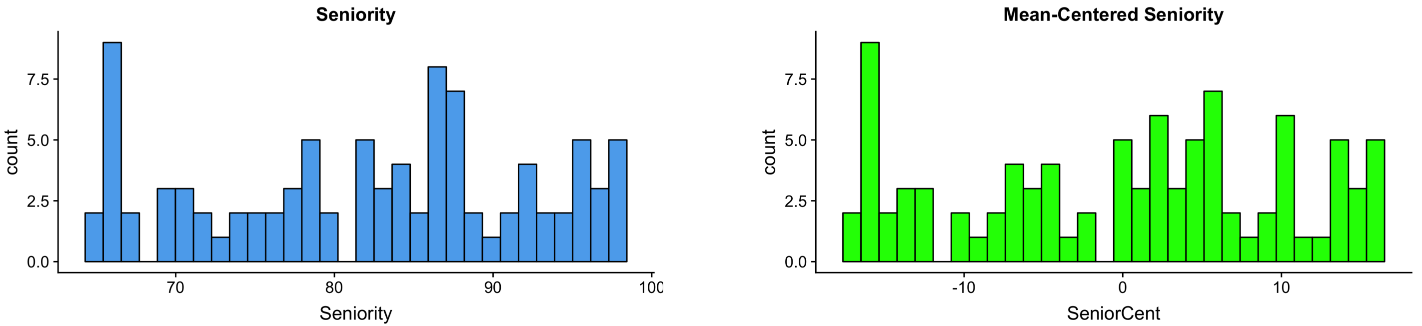
Salary: Mean-Centered Variables

```
wages <- wages %>%
  mutate(SeniorCent = Senior - mean(Senior),
         AgeCent = Age-mean(Age),
         EducCent = Educ - mean(Educ),
         ExperCent = Exper - mean(Exper))
```



Salary: Mean-Centered Variables

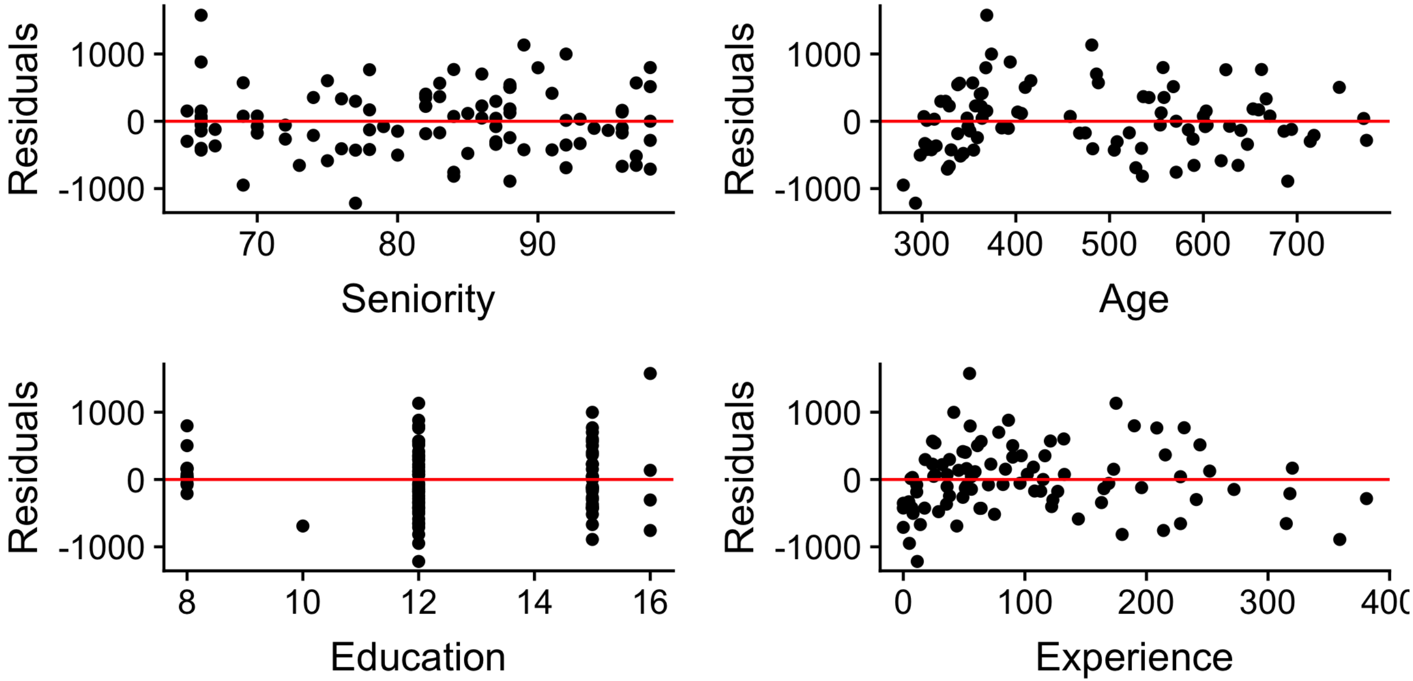
Application Exercise: Fit the regression model using the mean-centered variables.

- How did the model stay the same?
- How did the model change?

Quadratic Terms

- Sometimes the response variable may have a quadratic relationship with one or more predictor variables
    - You can see this in a plot of the residuals vs. a predictor variable
    - Include quadratic terms in the model to capture the relationship
- **Good Practice:** Also include all lower order terms even if they are not significant.
    - This helps with interpretation
- You can show quadratic relationships by plotting the predicted mean response for different values of the predictors variable
- Note: The same ideas apply for higher-order polynomial terms

Below are plots of the residuals versus each quantitative predictor variable.



Which variables (if any) appear to have a quadratic relationship with `Bsal`?

Indicator (dummy) variables

- Suppose there is a categorical variable with k levels (categories)
- Make k indicator variables (also known as dummy variables)
- Use $k-1$ of the indicator variables in the model
  - Can't uniquely estimate all k variables at once if the intercept is in the model
- Level that doesn't have a variable in the model is called the **baseline**
- Coefficients interpreted as the change in the mean of the response over the baseline

Indicator variables when $k = 2$

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	5924.007	99.659	59.443	0.000	5725.925	6122.090
Female1	-767.913	128.970	-5.954	0.000	-1024.255	-511.571
SeniorCent	-22.582	5.296	-4.264	0.000	-33.108	-12.056
AgeCent	0.631	0.721	0.876	0.384	-0.801	2.063
EducCent	92.306	24.864	3.713	0.000	42.887	141.725
ExperCent	0.501	1.055	0.474	0.636	-1.597	2.598

- Write the model equation used to predict the starting salary for males.
- Write the model equation used to predict the starting salary for males.

Indicator variables when $k > 2$

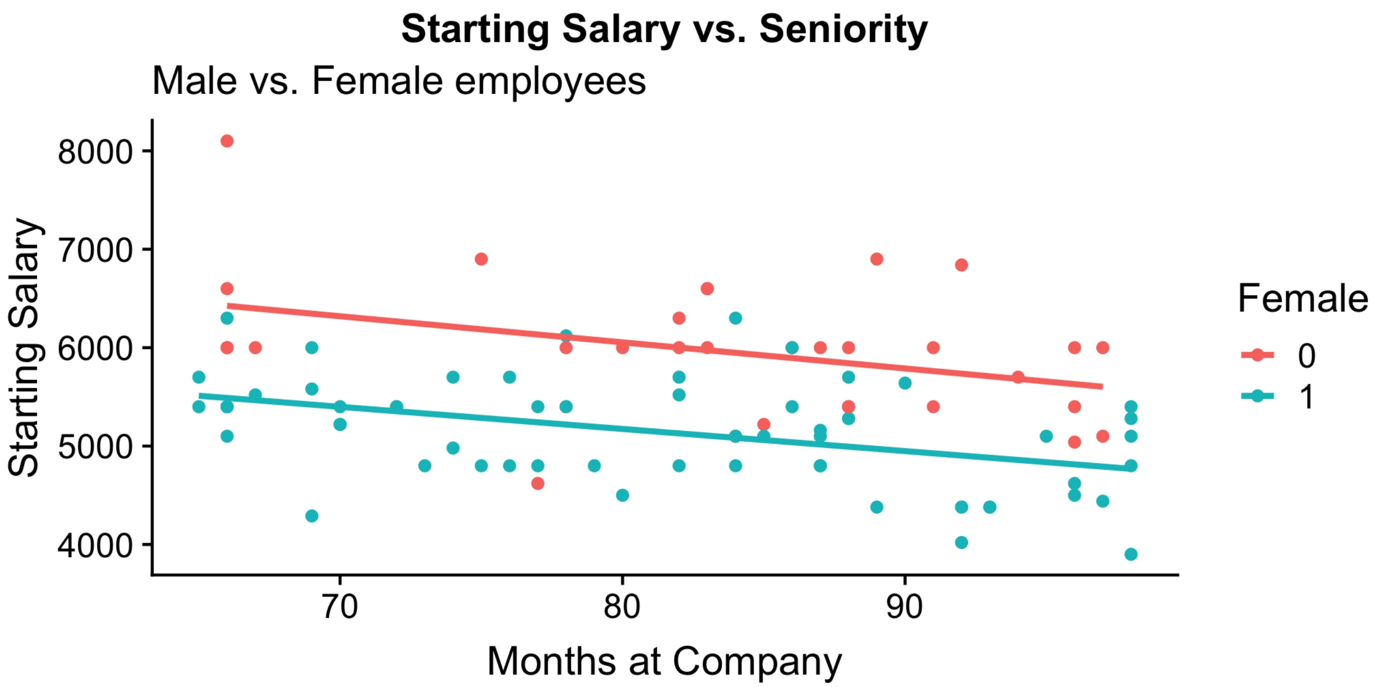
Application Exercise: Fit a regression model with Education treated as a categorical variable.

- What is the baseline for Education?
- Interpret the coefficient for `EducCat16`.
- What is your conclusion from the p-value of `EducCat16`?
- Write the model equation for those with 8 years of education.
- Write the model equation for those with 16 years of education.

Interaction Terms

- **Case:** Relationship of the predictor variable with the response depends on the value of another predictor variable
  - This is an **interaction effect**
- Create a new interaction variable that is one predictor variable times the other in the interaction
- **Good Practice:** When including an interaction term, also include the associated **main effects** (each predictor variable on its own) even if they are not statistically significant

Interaction effects



Do you think there is a significant interaction effect between `Female` and `Senior`? Why or why not?

Before next class

- Review Reading 03 on special predictors
- Reading 04 on transformations