Analysis of Variance

(ANOVA)

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STA 210

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Announcements

- Lab 03 - **due Tuesday, 9/17 at 11:59p**
- HW 01 - **due Wednesday, 9/18 at 11:59p**
- Use Piazza for questions instead of email
  - access it through Sakai
  - feel free to reply if you know the answer to question
  - let me know if you're not on Piazza

Check in

- Any questions from last class?

Today's Agenda

- Analysis of Variance to compare group means
- Multiple comparisons

Packages and Data

```r
library(tidyverse)
library(broom)
library(knitr)
```

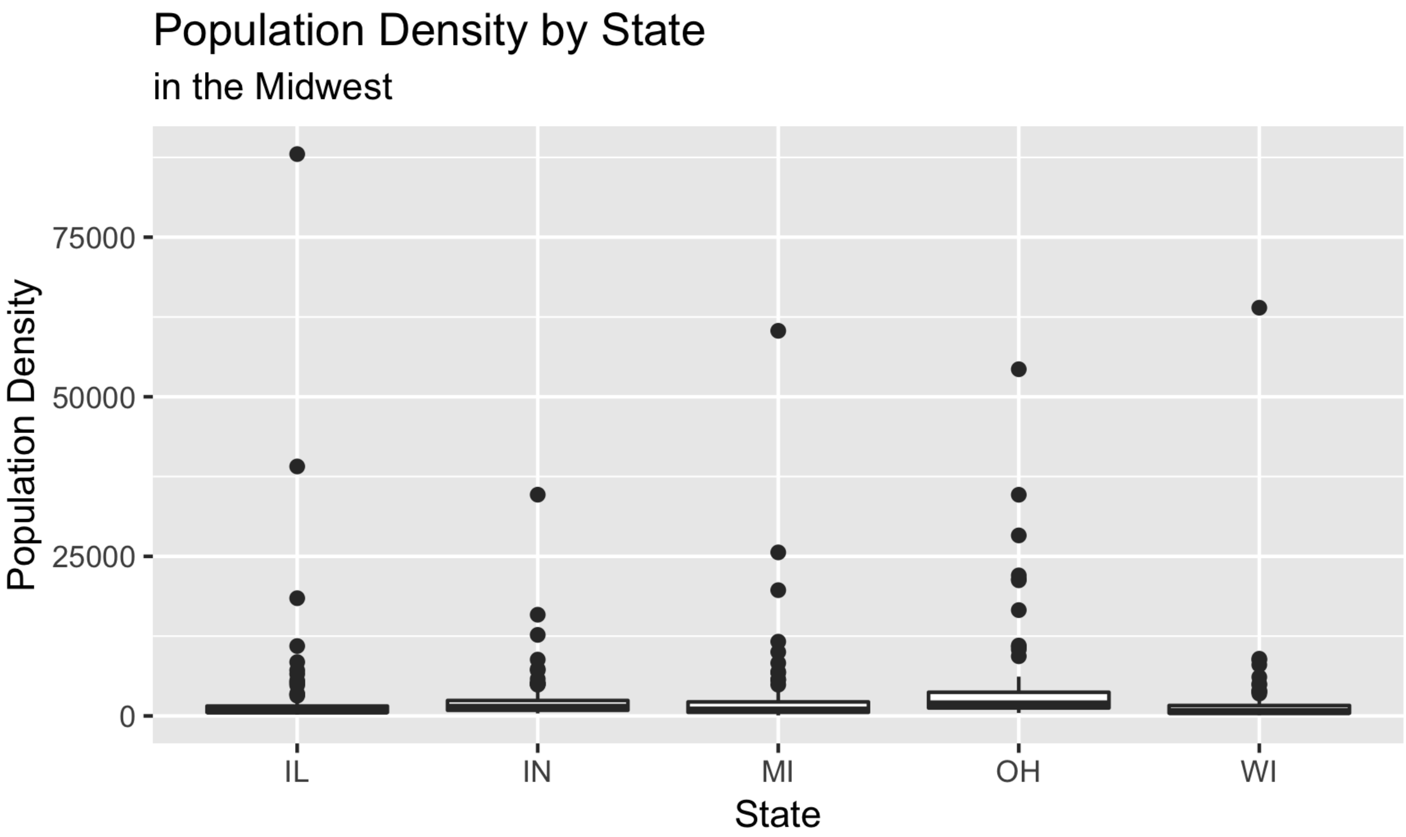
Population densities in the Midwest

- Data is in the `midwest` dataset in the ggplot2 package
- The data contains demographic information for all counties in each of the states in the Midwest: Illinois (IL), Indiana (IN), Michigan (MI), Ohio (OH), and Wisconsin (WI)
    - We will focus on the population density, **`popdensity`**

```
glimpse(midwest)
```

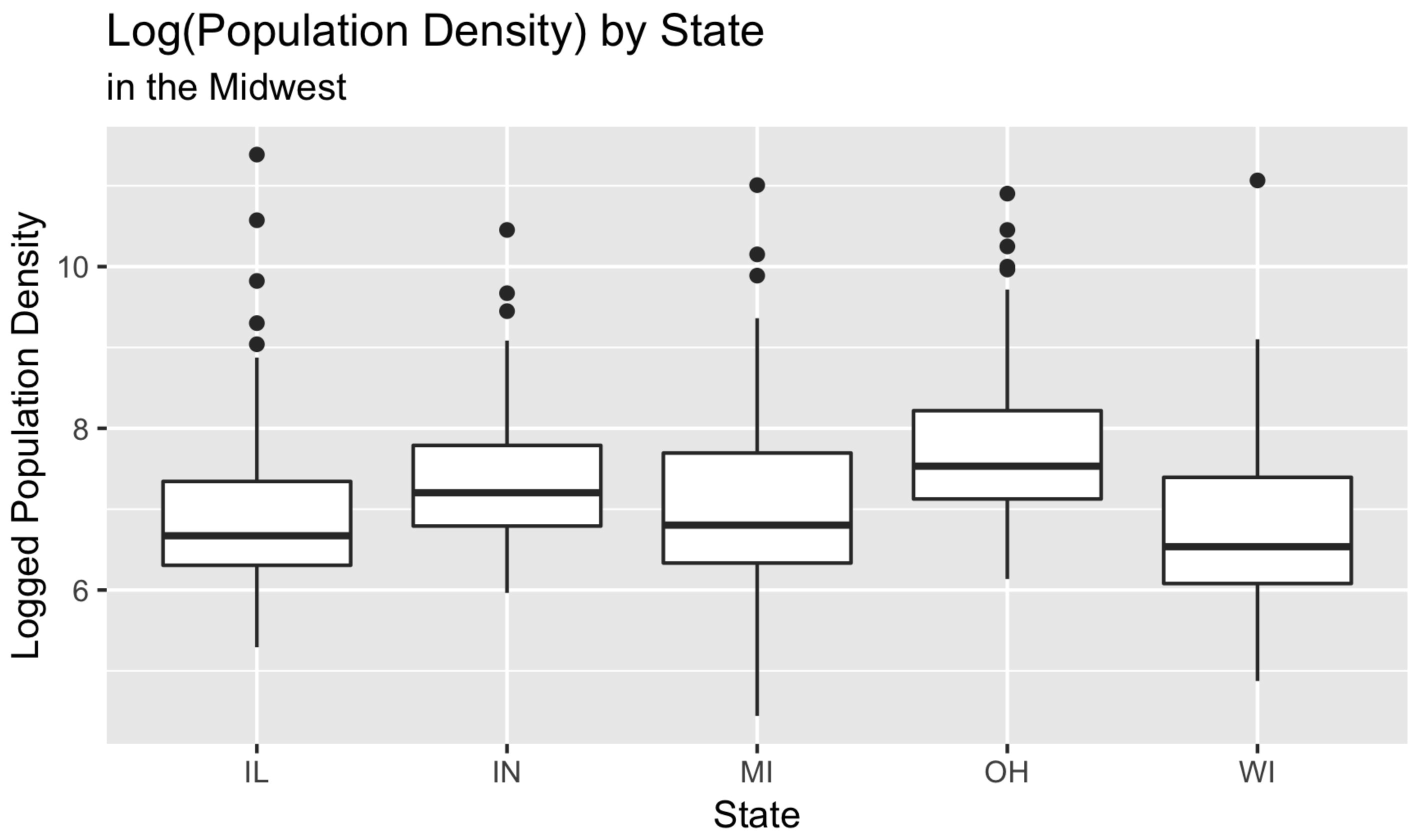
```
## Observations: 437
## Variables: 28
## $ PID                  <int> 561, 562, 563, 564, 565, 566, 567, 568, 569.
## $ county               <chr> "ADAMS", "ALEXANDER", "BOND", "BOONE", "BRO.
## $ state                <chr> "IL", "IL", "IL", "IL", "IL", "IL", "IL", ".
## $ area                 <dbl> 0.052, 0.014, 0.022, 0.017, 0.018, 0.050, 0.
## $ poptotal             <int> 66090, 10626, 14991, 30806, 5836, 35688, 53.
## $ popdensity           <dbl> 1270.9615, 759.0000, 681.4091, 1812.1176, 3.
## $ popwhite             <int> 63917, 7054, 14477, 29344, 5264, 35157, 529.
## $ popblack             <int> 1702, 3496, 429, 127, 547, 50, 1, 111, 16, .
## $ popamerindian        <int> 98, 19, 35, 46, 14, 65, 8, 30, 8, 331, 51, .
```

Exploratory Data Analysis

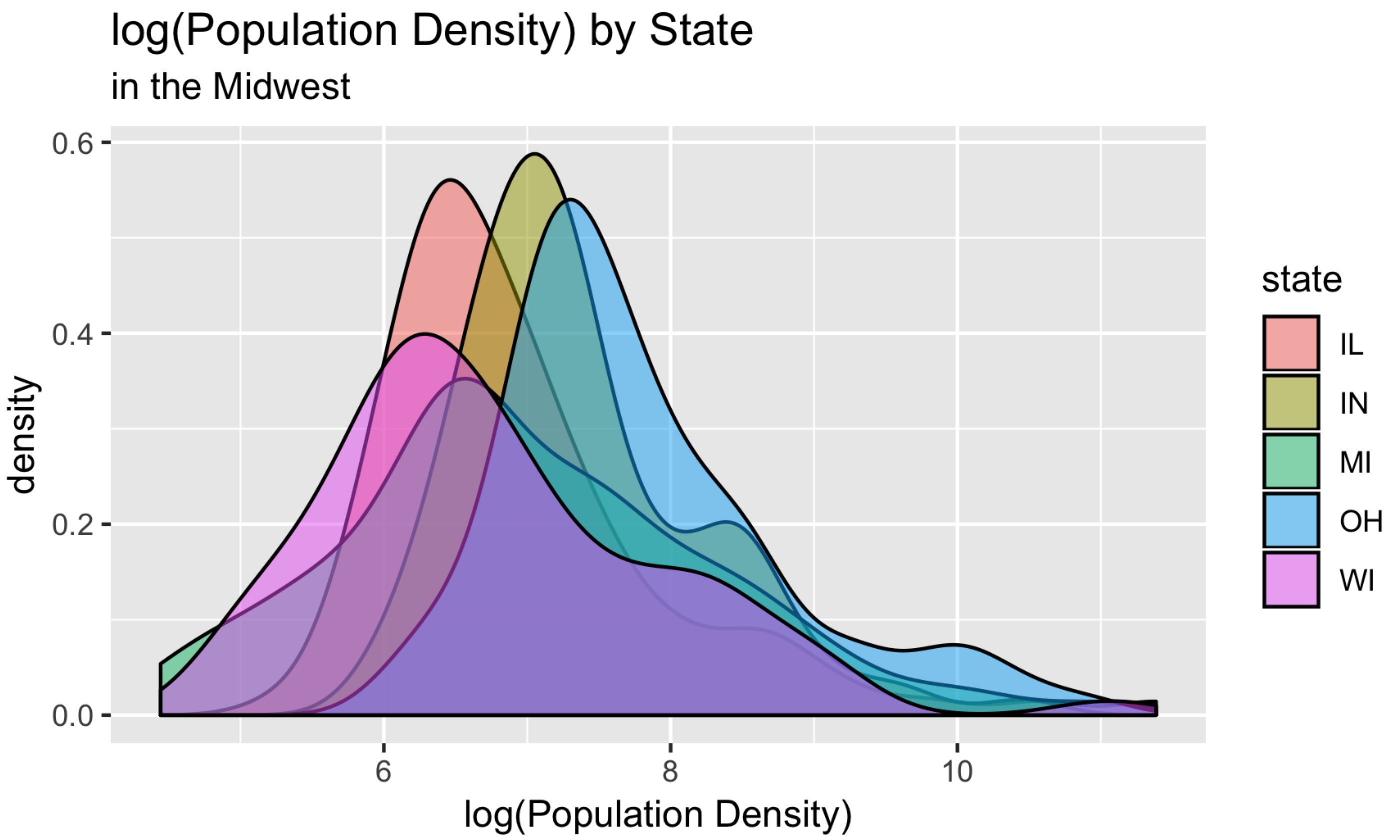


The distributions are very skewed by outliers, so let's look at the log of population density (more on log transformations in a few weeks)

```
midwest <- midwest %>% mutate(log_popdensity = log(popdensity))
```



```r
ggplot(data = midwest, aes(x = log_popdensity, fill = state)) +
  geom_density(alpha = 0.5) +
  labs(title = "log(Population Density) by State",
       subtitle = "in the Midwest",
       x = "log(Population Density)",
       color = "State")
```



Exploratory Data Analysis

```
midwest %>%
  group_by(state) %>%
  summarise(mean = mean(log_popdensity), var = var(log_popdensity)
```

```
## # A tibble: 5 x 3
##   state  mean   var
##   <chr> <dbl> <dbl>
## 1 IL     6.97 1.07
## 2 IN     7.37 0.719
## 3 MI     7.00 1.70
## 4 OH     7.79 0.982
## 5 WI     6.77 1.38
```

Using ANOVA to compare group means

So far, we have used a *quantitative* predictor variable to understand the variation in a quantitative response variable.

Now, we will use a *categorical (qualitative)* predictor variable to understand the variation in a quantitative response variable.

Notation

- K is number of mutually exclusive groups. We index the groups as $i = 1, \dots, K$.
- n_i is number of observations in group i
- $n = n_1 + n_2 + \cdots + n_K$ is the total number of observations in the data
- y_{ij} is the j^{th} observation in group i, for all i, j
- μ_i is the population mean for group i, for $i = 1, \dots, K$

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Motivating ANOVA

- **Question:** Is there a significant relationship between the predictor variable x and the response variable y?
- In other words, is the mean value of the response equal for all groups?

Model structure:

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij}$$

- μ is the overall mean,
- α_i is how much the mean for group i deviates from μ
- ϵ_{ij} is the amount y_{ij} deviates from the group mean

- Note that the mean response for group i is $\mu_i = \mu + \alpha_i$.

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Motivating ANOVA

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij}$$

- **Assumption:** ϵ_{ij} follows a Normal distribution with mean 0 and constant variance σ^2

$$\epsilon_{ij} \sim N(0, \sigma^2)$$

- This is the same as

$$y_{ij} \sim N(\mu_i, \sigma^2)$$

Hypotheses

- **Question of interest** Is there a significant difference in the means across the K groups?

- To answer this question, we will test the following hypotheses:

$$
\begin{aligned}
&H_0 : \mu_1 = \mu_2 = \cdots = \mu_K \\
&H_a : \text{At least one } \mu_i \text{ is not equal to the others}
\end{aligned}
$$

- **How to think about it:** If the sample means are "far apart", " there is evidence against H_0

- We will calculate a test statistic to quantify "far apart" in the context of the data

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Analysis of Variance (ANOVA)

- **Main Idea:** Decompose the total variation in the data into the variation between groups and the variation within each group

$$\sum_{i=1}^{K}\sum_{j=1}^{n_i}(y_{ij}-\bar{y})^2 = \sum_{i=1}^{K} n_i(\bar{y}_i-\bar{y})^2 + \sum_{i=1}^{K}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_i)^2$$

- If the variation between groups is significantly greater than the variation within each group, then there is evidence against the null hypothesis.

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ANOVA table for comparing means

	Sum of Squares	DF	Mean Square	F-Stat	p-value
Between (Model)	$\sum_{i=1}^{K} n_i(\bar{y}_i - \bar{y})^2$	$K - 1$	$SSB/(K-1)$	MSB/MSW	$P(F > \text{F-Stat})$
Within (Residual)	$\sum_{i=1}^{K}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_i)^2$	$n - K$	$SSW/(n-K)$		
Total	$\sum_{i=1}^{K}\sum_{j=1}^{n_i}(y_{ij} - \bar{y})^2$	$n - 1$	$SST/(n-1)$		

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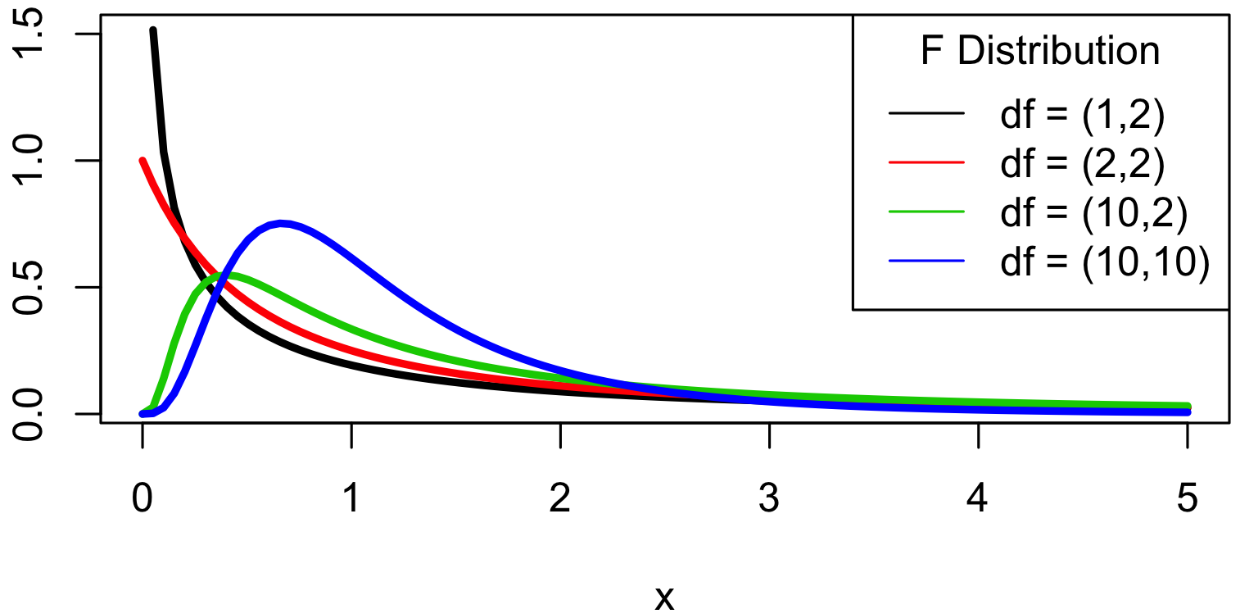
F-Distribution

The ANOVA test statistic follows an F distribution



Total Variation

- Total variation = variation between and within groups

$$SST = \sum_{i=1}^{K} \sum_{j=1}^{n_i} (y_{ij} - \bar{y})^2$$

- Degrees of freedom

$$DFT = n - 1$$

- Estimate of the variance across all observations:

$$\frac{SST}{DFT} = \frac{\sum_{i=1}^{K} \sum_{j=1}^{n_i} (y_{ij} - \bar{y})^2}{n - 1} = s_y^2$$

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Between Variation (Model)

- Variation in the group means

$$SSB = \sum_{i=1}^{K} n_i(\bar{y}_i - \bar{y})^2$$

- **Degrees of freedom**

$$DFB = K - 1$$

- **Mean Squares Between**

$$MSB = \frac{SSB}{DFB} = \frac{\sum_{i=1}^{K} n_i(\bar{y}_i - \bar{y})^2}{K - 1}$$

- MSB is an estimate of the variance of the μ_i's

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Within Variation (Residual)

- Variation within each group

$$SSW = \sum_{i=1}^{K} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_k)^2$$

- **Degrees of freedom**

$$DFW = n - K$$

- **Mean Squares Within**

$$MSW = \frac{SSW}{DFW} = \frac{\sum_{i=1}^{K} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2}{n - K}$$

- MSW is the estimate of σ^2, the variance within each group

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Population densities in the Midwest

```
pop_anova <- aov(log(popdensity) ~ state, data = midwest)
tidy(pop_anova) %>% kable(format = "markdown", digits = 3)
```

term	df	sumsq	meansq	statistic	p.value
state	4	55.682	13.921	12.13	0
Residuals	432	495.770	1.148	NA	NA

- How many observations (counties) are in the data?
- What is $\hat{\sigma}^2$, the estimated variance within each group?
- State the null and alternative hypothesis for this test. What is your conclusion?

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Assumptions for ANOVA

- **Normality:** $y_{ij} \sim N(\mu_i, \sigma^2)$
- **Equal (Constant) Variance:** The population distribution for each group has a common variance, σ^2
- **Independence:** The observations are independent from one another
    - This applies to observation within and between groups
- We can typically check these assumptions in the exploratory data analysis

Robustness to Assumptions

- **Normality:** $y_{ij} \sim N(\mu_i, \sigma^2)$
  - ANOVA relatively robust to departures from Normality.
  - Concern when there are strongly skewed distributions with different sample sizes (especially if sample sizes are small, < 10 in each group)
- **Independence:** There is independence within and across groups
  - If this doesn't hold, should use methods that account for correlated errors

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Robustness to Assumptions

- **Equal (Constant) Variance:** The population distribution for each group has a common variance, σ^2
  - Critical assumption, since the pooled (combined) variance is important for ANOVA
  - General rule: If the sample sizes within each group are approximately equal, the results of the F-test are valid if the largest variance is no more than 4 times the small variance (i.e. the largest standard deviation is no more than 2 times the smallest standard deviation)

Multiple Comparisons

After ANOVA: Individual Group Means

- Suppose you conduct an ANOVA and conclude that at least one group mean has a different mean response value.
- The next question you want to answer is **which group?**
- One way to answer this question is to compare the estimated means for each group, accounting for the random variability we'd naturally expect
- Since we've assumed the variance is the same for all groups, we can use a pooled standard error with $n - K$ degrees of freedom to calculate the confidence

$$\bar{y}_i \pm t^* \times \frac{s_P}{\sqrt{n_i}}$$

where s_P is the pooled standard error

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After ANOVA: Difference in Means

- We can also estimate the difference in two means, $\mu_1 - \mu_2$ for each pair of groups

$$(\bar{y}_1 - \bar{y}_2) \pm t^* \times s_P \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

where s_P is the pooled standard error

- If we have K groups, we will make $\binom{K}{2} = K(K-1)/2$ such comparisons
  - Ex: If we have 6 groups, we'll make $\binom{6}{2} = 6(6-1)/2 = 15$ comparisons

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Multiple Comparisons

- When making multiple comparisons, there is a higher chance that a Type I error will occur, e.g. conclude that there is a significant difference between two groups even when there is not

- **At a Minimum**: When calculating multiple confidence intervals or conducting multiple hypothesis tests to compare means, you should clearly state how many CIs and/or tests you computed.

- **Good practice**: Account for the number of comparisons being made in the analysis
  - We will discuss one method: **Bonferroni correction**

Confidence levels

- **Individual confidence level:** success rate of a procedure for calculating a <u>single</u> confidence interval
- **Familywise confidence level:** success rate of a procedure for calculating a <u>family</u> of confidence intervals
  - "success": all intervals in the family capture their parameters
- **Issue:** There is an increased chance of making at least one error when calculating multiple confidence intervals
  - The same is true when conducting multiple hypothesis tests

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Bonferroni correction

- **Goal:** Achieve at least $100(1-\alpha)\%$ familywise confidence level for C confidence intervals
  - Where α is the significance level for the corresponding two-sided hypothesis test
- Calculate each of the k confidence intervals at a $100(1-\frac{\alpha}{C})\%$ confidence level
  - When there are K groups, there are $C = \frac{K(K-1)}{2}$ pairs of means that can be compared
- Notes:
  - The exact familywise confidence level is not easily predictable. This partially depends on the level of dependence between the intervals.
  - Bonferroni correction is sometimes too conservative, i.e don't reject H_0 as much as you should

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Population Density in the Midwest

- There are 5 groups (states) in the `midwest` data, so we will do $\binom{5}{2} = 10$ comparisons.

- If we want a familywise confidence level of 95%, then we should use a $(1 - 0.05/10) \times 100 = 99.5$ % confidence level for each pairwise comparison

Pairwise CI

```
library(pairwiseCI)
pairwiseCI(log_popdensity ~ state, data = midwest, method = "Paran
  kable(format = "markdown")
```

estimate	lower	upper	comparison
0.4089452	0.0212811	0.7966093	IN-IL
0.0315392	-0.4563571	0.5194355	MI-IL
0.8237068	0.4049660	1.2424476	OH-IL
-0.1959042	-0.6744822	0.2826737	WI-IL
-0.3774060	-0.8457153	0.0909032	MI-IN
0.4147616	0.0245751	0.8049481	OH-IN
-0.6048494	-1.0546829	-0.1550160	WI-IN
0.7921676	0.2903355	1.2939997	OH-MI
-0.2274434	-0.7987309	0.3438440	WI-MI
-1.0196110	-1.5070486	-0.5321735	WI-OH