# Logistic Regression

## Odds + probabilities

Prof. Maria Tackett

STA 210

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# Topics

- Logistic regression for binary response variable
- Relationship between odds and probabilities
- Use logistic regression model to calculate predicted odds and probabilities

# Types of response variables

**Quantitative response variable**:

- Sales price of a house in Levittown, NY
- **Model**: Expected sales price given the number of bedrooms, lot size, etc.

**Categorical response variable**:

- High risk of coronary heart disease
- **Model**: Probability an adult is high risk of heart disease given their age, total cholesterol, etc.

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# Models for categorical response variables

## Logistic Regression

2 Outcomes

1: Yes, 0: No

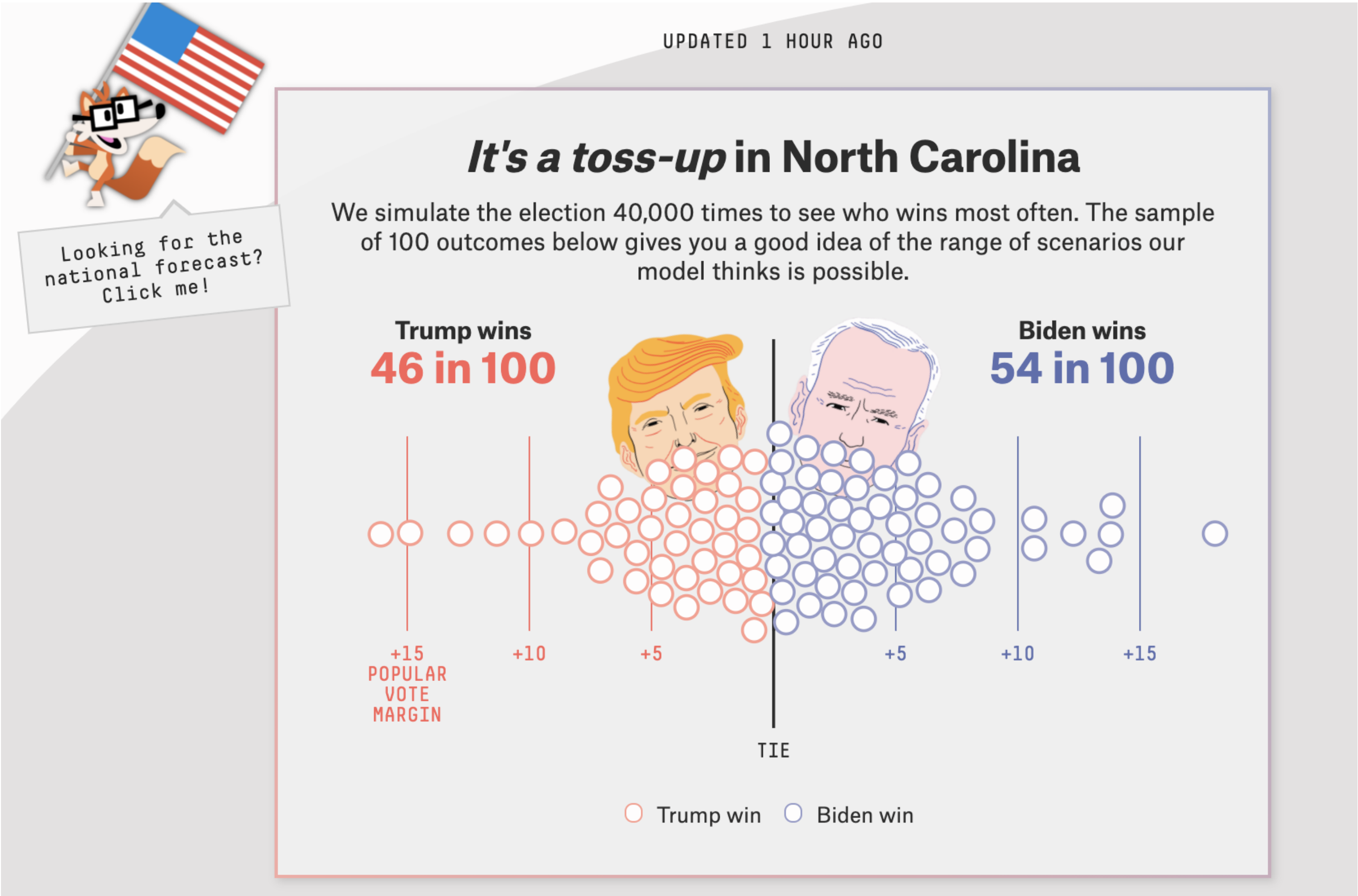
## Multinomial Logistic Regression

3+ Outcomes

1: Democrat, 2: Republican, 3: Independent

**Let's focus on logistic regression models for now.**

# FiveThirtyEight 2020 election forcasts




[FiveThirtyEight Election Forcasts]

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# FiveThirtyEight NBA finals predictions

**Friday, Oct. 2** FINALS

| Game 2 • FINAL | RAPTOR SPREAD | WIN PROB. | SCORE |
|---|---|---|---|
| Heat | | 43% | 114 |
| Lakers 2-0 | -2 | 57% | ✓ 124 |

**Wednesday, Sept. 30** FINALS

| Game 1 • FINAL | RAPTOR SPREAD | WIN PROB. | SCORE |
|---|---|---|---|
| Heat | -5 | 68% | 98 |
| Lakers 1-0 | | 32% | ✓ 116 |

[2019-20 NBA Predictions]

# Do teenagers get 7+ hours of sleep?

Students in grades 9 - 12 surveyed about health risk behaviors including whether they usually get 7 or more hours of sleep.

**`Sleep7`**

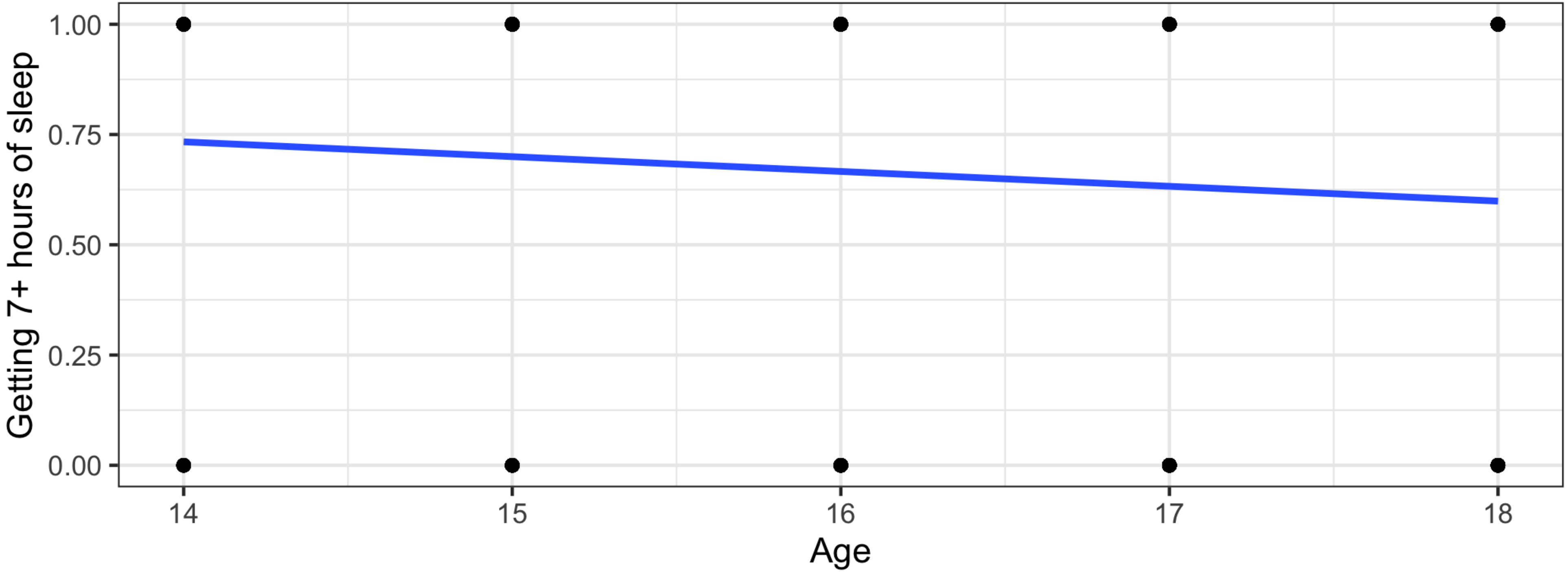
1: yes

0: no

| Age | Sleep7 |
|---|---|
| 16 | 1 |
| 17 | 0 |
| 18 | 0 |
| 17 | 1 |
| 15 | 0 |
| 17 | 0 |
| 17 | 1 |
| 16 | 1 |
| 16 | 1 |
| 18 | 0 |

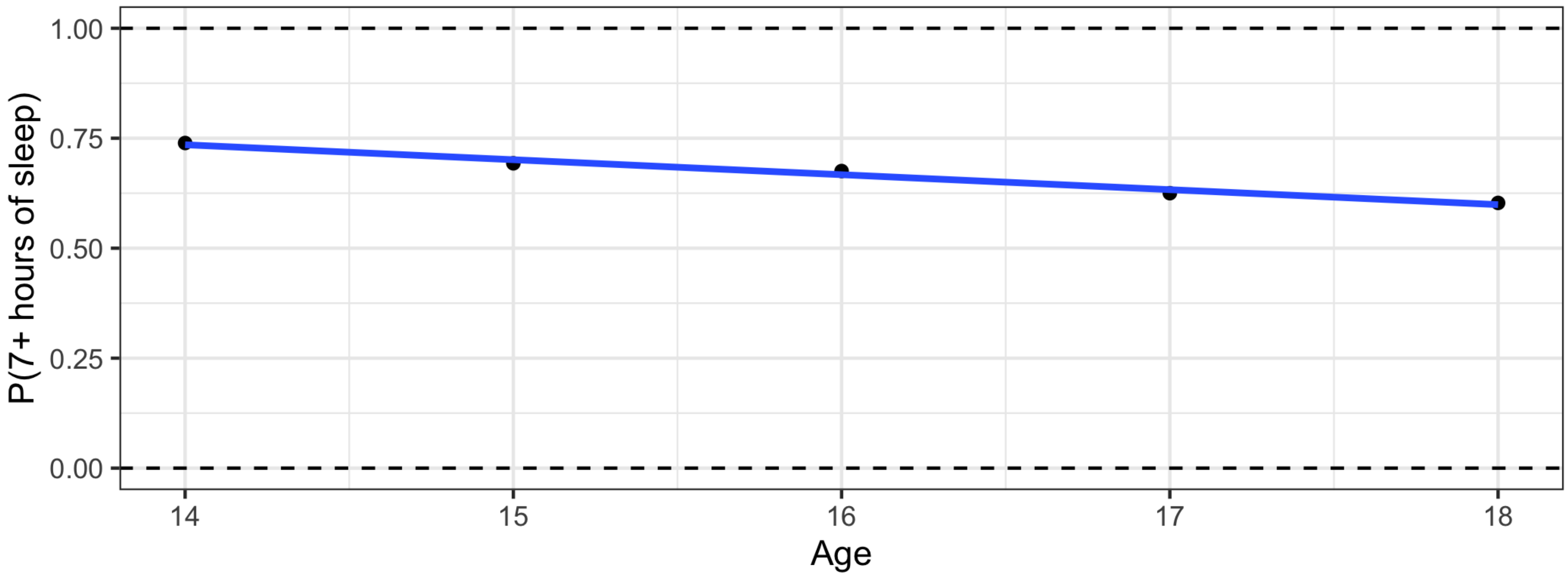
# Let's fit a linear regression model

**Response**: $Y$ = 1: yes, 0: no



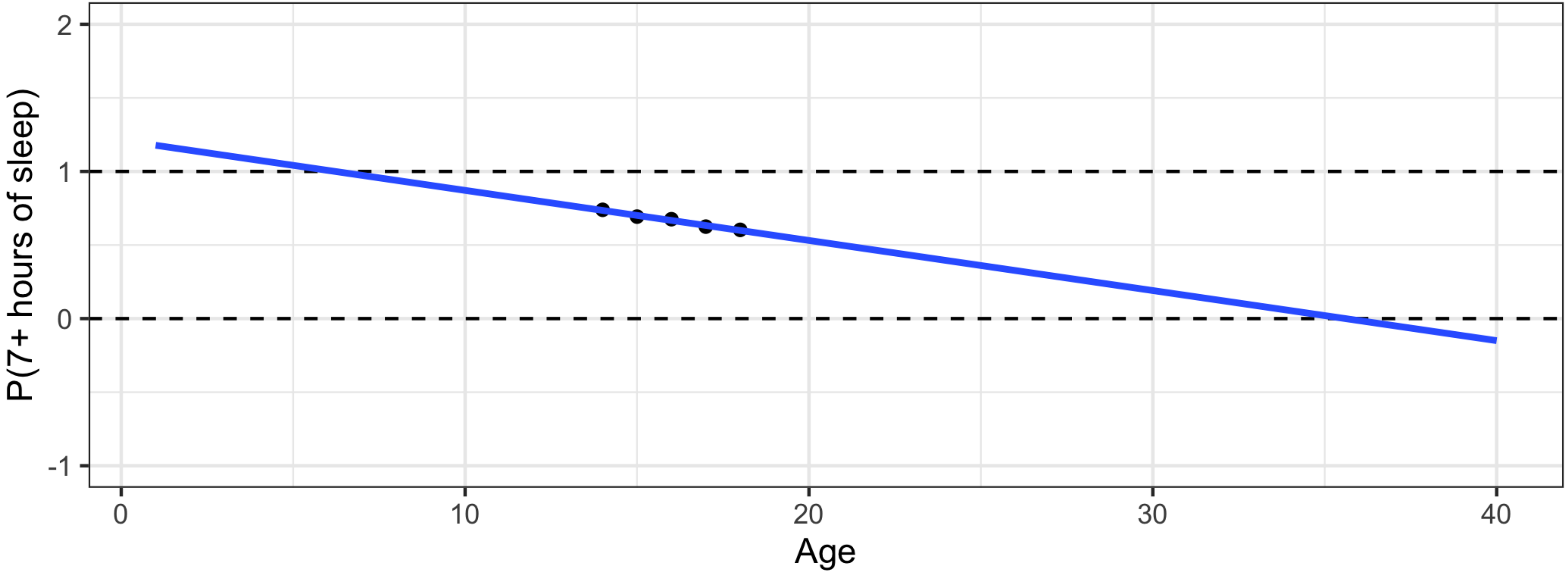
# Let's use proportions

**Response**: Probability of getting 7+ hours of sleep



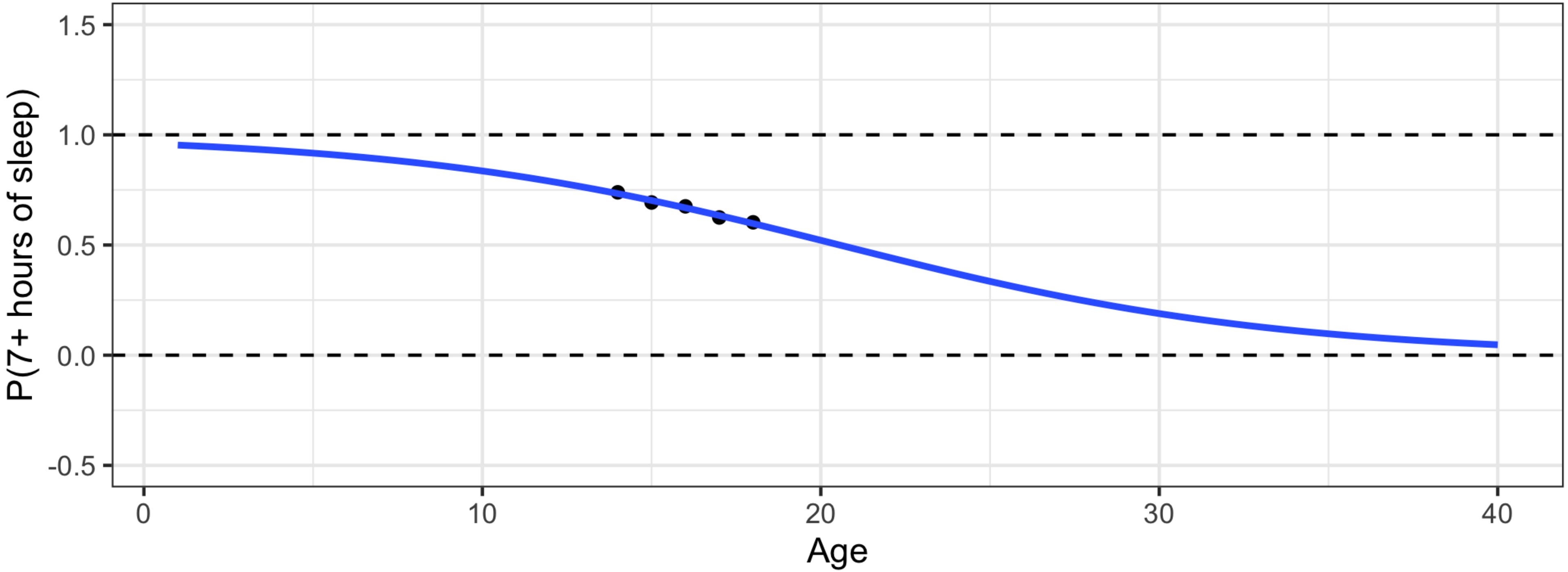
# What happens if we zoom out?

**Response**: Probability of getting 7+ hours of sleep




🛑 This model produces predictions outside of 0 and 1.

# Let's try another model




✅ This model (called a **logistic regression model**) only produces predictions between 0 and 1.

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# Different types of models

| Method | Response Type | Model |
|---|---|---|
| Linear Regression | Quantitative | $Y = \beta_0 + \beta_1\ X$ |
| Linear regression (transform Y) | Quantitative | $\log(Y) = \beta_0 + \beta_1\ X$ |
| Logistic regression | Binary | $\log\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1\ X$ |

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# Binary response variable

- $Y = 1: \text{ yes}, 0: \text{ no}$
- $\pi$: probability that $Y = 1$, i.e., $P(Y = 1)$
- $\frac{\pi}{1-\pi}$: odds that $Y = 1$
- $\log\left(\frac{\pi}{1-\pi}\right)$: log odds
- Go from $\pi$ to $\log\left(\frac{\pi}{1-\pi}\right)$ using the logit transformation

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# Odds

Suppose there is a **70% chance** it will rain tomorrow

- Probability it will rain is $\mathbf{p = 0.7}$
- Probability it won't rain is $\mathbf{1 - p = 0.3}$
- Odds it will rain are **7 to 3**, 7:3, $\mathbf{\frac{0.7}{0.3} \approx 2.33}$

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# Are teenagers getting enough sleep?

```
## # A tibble: 2 x 3
##   Sleep7     n     p
##    <int> <int> <dbl>
## 1      0   150 0.336
## 2      1   296 0.664
```

$$P(7\text{+ hours of sleep}) = P(Y = 1) = p = 0.664$$

$$P(< 7\text{ hours of sleep}) = P(Y = 0) = 1 - p = 0.336$$

$$P(\text{odds of 7+ hours of sleep}) = \frac{0.664}{0.336} = 1.976$$

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# From odds to probabilities

**odds**

$$\omega = \frac{\pi}{1-\pi}$$

**probability**

$$\pi = \frac{\omega}{1+\omega}$$

# Logistic model: from odds to probabilities

1️⃣ **Logistic model**: log odds = $\log\big(\frac{\pi}{1-\pi}\big) = \beta_0 + \beta_1~X$

2️⃣ **odds =** $\exp\big\{\log\big(\frac{\pi}{1-\pi}\big)\big\} = \frac{\pi}{1-\pi}$

Combining 1️⃣ and 2️⃣ with what we saw earlier

$$\text{probability} = \pi = \frac{\exp\{\beta_0 + \beta_1~X\}}{1 + \exp\{\beta_0 + \beta_1~X\}}$$

# Logistic regression model

Logit form:

$$\log\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1\ X$$

Probability form:

$$\pi = \frac{\exp\{\beta_0 + \beta_1\ X\}}{1 + \exp\{\beta_0 + \beta_1\ X\}}$$

# Risk of coronary heart disease

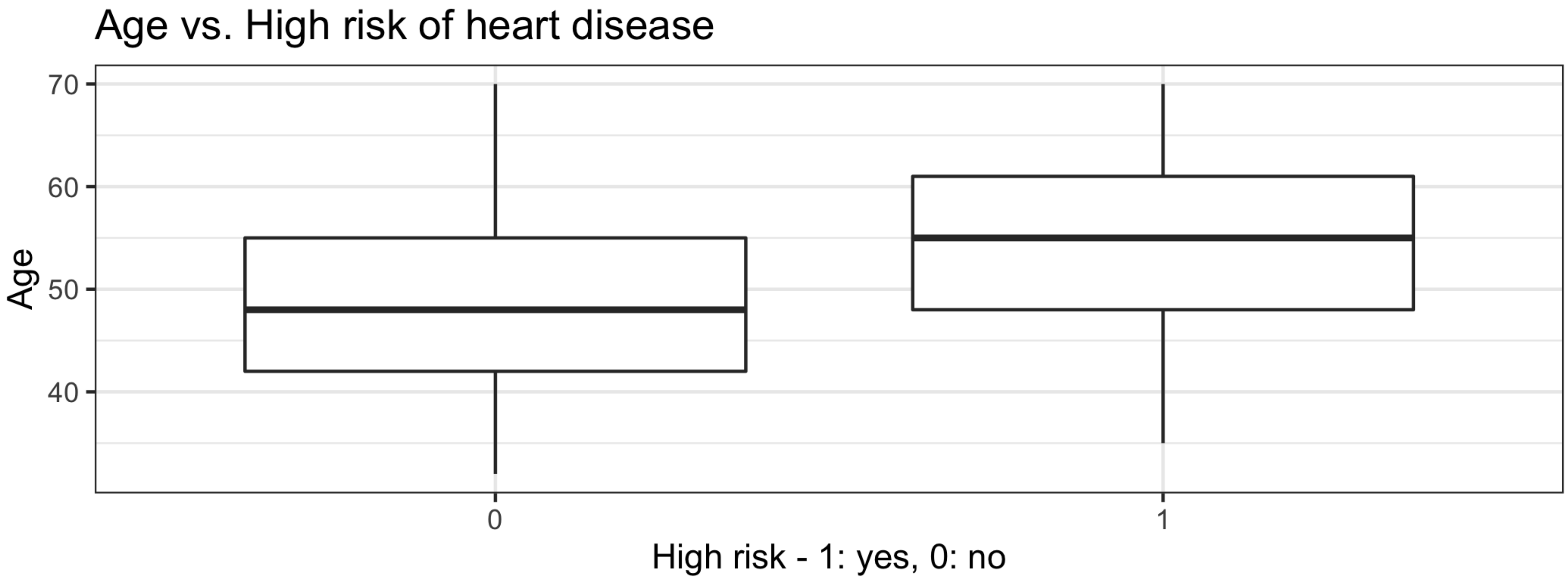
This dataset is from an ongoing cardiovascular study on residents of the town of Framingham, Massachusetts. We want to use **`age`** to predict if a randomly selected adult is high risk of having coronary heart disease in the next 10 years.

**`high_risk`**:

- 1: High risk of having heart disease in next 10 years
- 0: Not high risk of having heart disease in next 10 years

**`age`**: Age at exam time (in years)

# High risk vs. age



# Let's fit the model

```
high_risk_model <- glm(high_risk ~ age,
                       data = heart_data,
                       family = "binomial")
tidy(high_risk_model) %>%
  kable(digits = 3)
```

| term | estimate | std.error | statistic | p.value |
|---|---|---|---|---|
| (Intercept) | -5.561 | 0.284 | -19.599 | 0 |
| age | 0.075 | 0.005 | 14.178 | 0 |

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# Let's fit the model

| term | estimate | std.error | statistic | p.value |
|---|---|---|---|---|
| (Intercept) | -5.561 | 0.284 | -19.599 | 0 |
| age | 0.075 | 0.005 | 14.178 | 0 |

$$\log\Big(\frac{\hat{\pi}}{1-\hat{\pi}}\Big) = -5.561 + 0.075 \times \text{age}$$

where $\hat{\pi}$ is the predicted probability of being high risk

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# Predicted log odds

```
predict(high_risk_model)
```

```
##      1      2      3      4      5      6      7      8      9     10
## -2.650 -2.127 -1.978 -1.007 -2.127 -2.351 -0.858 -2.202 -1.679 -2.351
```

## For observation 1

$$\text{predicted odds} = \hat{\omega} = \frac{\hat{\pi}}{1-\hat{\pi}} = \exp\{-2.650\} = 0.071$$

# Predcited probabilities

```
predict(high_risk_model,
        type = "response")
```

```
##     1     2     3     4     5     6     7     8     9    10
## 0.066 0.106 0.122 0.267 0.106 0.087 0.298 0.100 0.157 0.087
```

$$\text{predicted probabilities} = \hat{\pi} = \frac{\exp\{-2.650\}}{1 + \exp\{-2.650\}} = 0.066$$

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# Recap

- Logistic regression for binary response variable
- Relationship between odds and probabilities
- Used logistic regression model to calculate predicted odds and probabilities