Inference with mixed effects models

Data: flipped classrooms?

Data set has 375 rows (one per student), and the following variables:

- `professor`: which professor taught the class (1 -- 15)
- `style`: which teaching style the professor used (no flip, some flip, fully flipped)
- `score`: the student's score on the final exam

Inference with linear models

$$Score_i = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + \varepsilon_i$$

Research question: Is there a relationship between teaching style and student score?

> What are my null and alternative hypotheses, in terms of one or more model parameters?

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Research question: Is there a relationship between teaching style and student score?

$H_0 : \beta_1 = \beta_2 = 0$

$H_A : \text{at least one of } \beta_1, \beta_2 \neq 0$

> What test would I use to test these hypotheses?

F tests

$$Score_i = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + \varepsilon_i$$

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> What are my degrees of freedom for the F test?

F tests for mixed effects models

$$Score_{ij} = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + u_i + \varepsilon_{ij}$$

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$$H_0 : \beta_1 = \beta_2 = 0$$

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Test: We will use an F test again

- numerator df = number of parameters tested = 2
- denominator df = ??

What *are* degrees of freedom?

Suppose we want to estimate the mean μ of a distribution. We observe n observations $X_1, \ldots, X_n$, and calculate the sample mean

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

> Suppose we know the value of $\overline{X}$. How many of the values $X_1, \ldots, X_n$ are "free to vary" (i.e., can be any number they want)?

Example: simple linear regression

Observe $(X_1, Y_1), \ldots, (X_n, Y_n)$ and calculate

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i$$

> What are my degrees of freedom?

Denominator degrees of freedom for mixed models

$$Score_{ij} = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + u_i + \varepsilon_{ij}$$

$$H_0 : \beta_1 = \beta_2 = 0 \qquad H_A : \text{at least one of } \beta_1, \beta_2 \neq 0$$

Test: We will use an F test again

- numerator df = number of parameters tested = 2
- denominator df =

$$\text{number of independent observations} - \text{number of parameters}$$

> Are all observations independent?

Bounds on the denominator degrees of freedom

$$Score_{ij} = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + u_i + \varepsilon_{ij}$$

$$u_i \overset{iid}{\sim} N(0, \sigma_u^2) \qquad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$$

$$\#\,\text{groups} - p \;\leq\; \text{denominator df} \;\leq\; n - p$$

- $p =$ number of parameters in full model
- $n =$ total number of observations

Approximating the degrees of freedom

```
library(lme4)
library(lmerTest)
m1 <- lmer(score ~ style + (1|professor),
           data = teaching)
anova(m1)
```

```
## Type III Analysis of Variance Table with Satterthwaite's
##        Sum Sq Mean Sq NumDF DenDF F value   Pr(>F)
## style 65.437  32.718     2    12  7.6949 0.007072 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
```

- The `lmerTest` package approximates degrees of freedom with *Satterthwaite's method* (details are beyond the scope of this course)
- This allows us to calculate (approximate) p-values

Why degrees of freedom matter

- If we use the wrong degrees of freedom, we get the wrong p-value
- Often this means our p-value is smaller than it should be (we overestimate the strength of evidence)

Using the correct degrees of freedom:

```
pf(7.69, 2, 12, lower.tail=F)
```

```
## [1] 0.007087398
```

If we just did $n - p$ (wrong):

```
pf(7.69, 2, 372, lower.tail=F)
```

```
## [1] 0.0005339106
```

Class activity

https://sta214-f22.github.io/class_activities/ca_lecture_28.html

Class activity

> There are 1561 rentals in 43 neighborhoods. What are the lower and upper bounds on the denominator degrees of freedom?

Class activity

> What is the approximate denominator df, using Satterthwaite's method?

Class activity

> Do we have evidence for a relationship between overall satisfaction and price?

Class activity

$$Price_{ij} = \beta_0 + \beta_1 Satisfaction_{ij} + u_i + \varepsilon_{ij}$$

$u_i \overset{iid}{\sim} N(0, \sigma_u^2) \quad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$

where $Price_{ij}$ is the price of rental j in neighborhood i

> What assumptions are we making in this mixed effects model?

Assumptions

$$Price_{ij} = \beta_0 + \beta_1 Satisfaction_{ij} + u_i + \varepsilon_{ij}$$

$u_i \overset{iid}{\sim} N(0, \sigma_u^2) \quad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$

- **Shape:**
  - the overall relationship between satisfaction and price is linear
  - The slope is the *same* for each neighborhood

Assumptions

$$Price_{ij} = \beta_0 + \beta_1 Satisfaction_{ij} + u_i + \varepsilon_{ij}$$

$$u_i \overset{iid}{\sim} N(0, \sigma_u^2) \qquad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$$

- **Independence:**
  - random effects are independent
  - observations within neighborhoods are independent after accounting for the random effect (i.e., the random effect captures the correlation within neighborhoods)
  - observations from different neighborhoods are independent

Assumptions

$$Price_{ij} = \beta_0 + \beta_1 Satisfaction_{ij} + u_i + \varepsilon_{ij}$$

$$u_i \overset{iid}{\sim} N(0, \sigma_u^2) \qquad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$$

- **Normality:** Both $u_i \sim N(0, \sigma_u^2)$ and $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$
- **Constant variance:**
  - ε_{ij} has the same variance σ_ε^2 regardless of satisfaction or neighborhood