Beginning linear mixed effects models

Data: flipped classrooms?

- A *flipped classroom* involves students watching lectures at home, and doing activities during class time
- There is debate about the pros and cons of this teaching method
- Here we will look at simulated data from an experiment with flipped classrooms

Data: flipped classrooms?

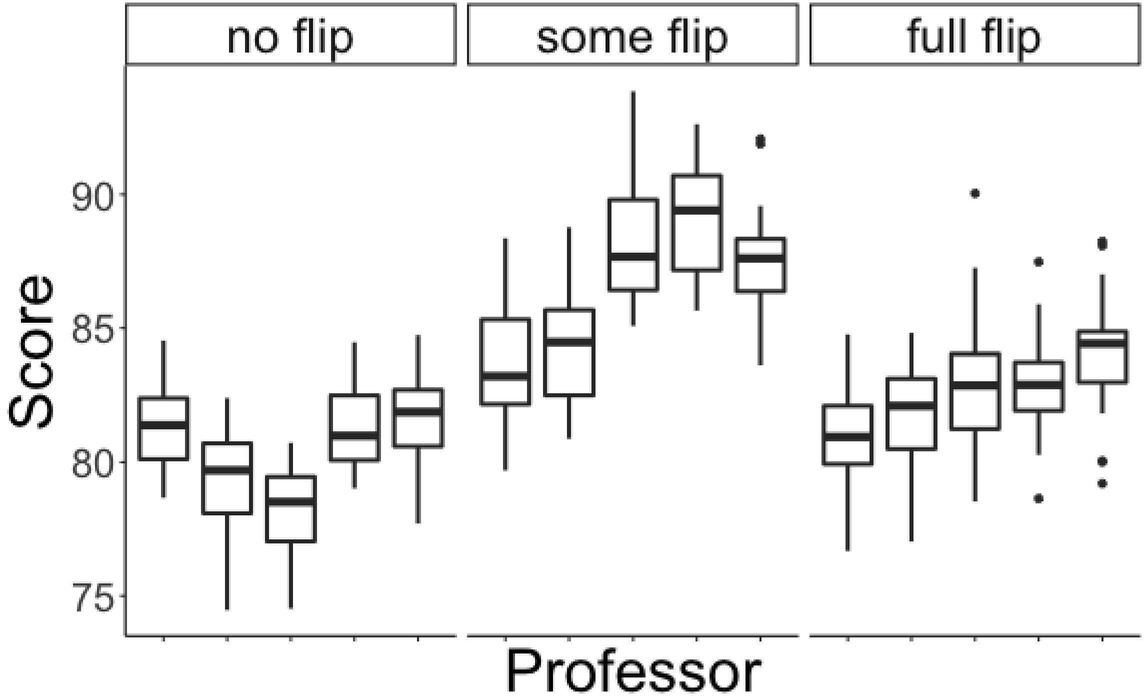
- 15 classes of introductory statistics
- 25 students in each class (so 375 students total)
- Each class taught by a different professor
- Each professor randomly assigned a teaching style: No flip, Some flip, and Fully flipped
- At the end of the semester, we give all the students in all the classes the same exam, and compare their results

Data: flipped classrooms?

Data set has 375 rows (one per student), and the following variables:

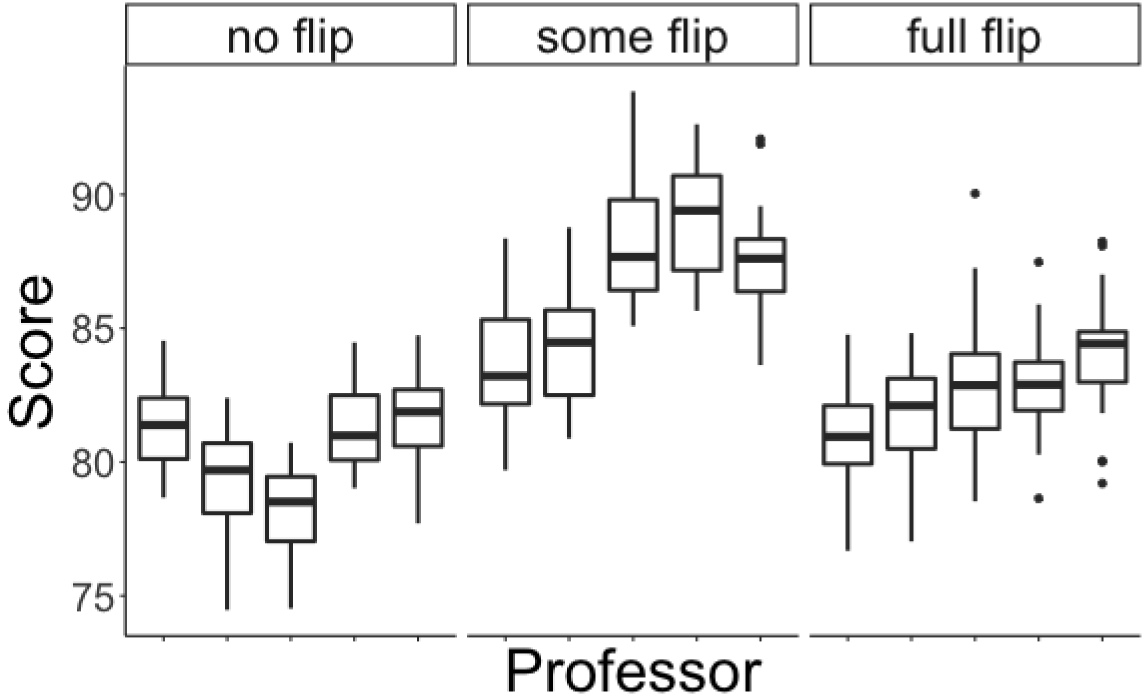
- `professor`: which professor taught the class (1 -- 15)
- `style`: which teaching style the professor used (no flip, some flip, fully flipped)
- `score`: the student's score on the final exam

Considering results



> What do you notice about the scores?

Considering results



- There may be some differences between styles
- There may be some differences between professors

Considering results

> Suppose we notice that, on average, students in the “Some Flipped” classes have higher scores than students in the “Fully Flipped” classes. What might explain this difference?

Considering results

> Suppose we notice that, on average, students in the “Some Flipped” classes have higher scores than students in the “Fully Flipped” classes. What might explain this difference?

- The "Some Flipped" method may lead to higher test results.
- The professors assigned to teach "Some Flipped" may teach in such a way that their scores are higher than those in the "Fully Flipped" group (more experience, etc.).
- The students in the "Some Flipped" classes may have been stronger than those in the "Fully Flipped" group.

Different effects

- *Effect of interest* (*treatment effect*): The "Some Flipped" method may lead to higher test results; *the treatment imposed by the researchers has an effect on the outcome.*

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Different effects

- *Effect of interest* (*treatment effect*): The "Some Flipped" method may lead to higher test results; *the treatment imposed by the researchers has an effect on the outcome.*
- *Group effect*: The professors assigned to teach "Some Flipped" may have had an impact on the test scores; *the group the students are in has an effect on the outcome.*
- *Individual effect*: The students in the "Some Flipped" classes may have been stronger than those in the "Fully Flipped" group; *the individuals' characteristics or abilities have an effect on the outcome.*

Writing down a model

Score is a continuous response, so we can go back to linear models:

$$Score_i = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + \varepsilon_i$$

$$\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2_\varepsilon)$$

Which effects does this model capture?

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> Which effects does this model capture?

- Treatment effect (β_0 is the average score in the no flip group, and β_1 and β_2 tell us how the score changes in the other groups)
- Individual effect (ε_i is the difference from the mean for student i)

Assumptions

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$\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2_\varepsilon)$

> What does this model assume about group effects (differences between professors)?

Assumptions

$$Score_i = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + \varepsilon_i$$

$$\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2_\varepsilon)$$

> What does this model assume about group effects (differences between professors)?

That there are no systematic differences between professors (i.e., no group effects)

Assumptions

$$Score_i = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + \varepsilon_i$$

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> What does this model assume about correlation within a class?

Assumptions

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> What does this model assume about correlation within a class?

That there is no correlation between student scores within the same class

> Is this a good assumption?

Writing down a model

$$Score_i = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + \varepsilon_i$$

$$\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2_\varepsilon)$$

> How can I incorporate systematic differences between classes?

Writing down a model

$$Score_i = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + \varepsilon_i$$

$$\varepsilon_i \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$$

> How can I incorporate systematic differences between classes?

Add a variable for the different professors:

$$Score_i = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + \\ \beta_3 \text{Class2}_i + \cdots + \beta_{16} \text{Class15}_i + \varepsilon_i$$

$$\varepsilon_i \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$$

Writing down a model

$$Score_i = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i +$$
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$\varepsilon_i \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$

> How many parameters did we add to the model to capture class differences?

Writing down a model

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> How many parameters did we add to the model to capture class differences?

14 (β_3,..., β_{16})

Writing down a model

$$Score_i = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + \\ \beta_3 \text{Class2}_i + \cdots + \beta_{16} \text{Class15}_i + \varepsilon_i$$

$$\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2_\varepsilon)$$

> Do we want to do inference on β_3,..., β_{16} ?

Writing down a model

$$Score_i = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + \\ \beta_3 \text{Class2}_i + \cdots + \beta_{16} \text{Class15}_i + \varepsilon_i$$

$\varepsilon_i \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$

> Do we want to do inference on β_3,..., β_{16} ?

No -- we only care about inference for the treatment effect parameters (β_1 and β_2)

> Can we do something *different* to capture group effects?

Our first mixed effects model

Linear model:

$$
\begin{gathered}
Score_i = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + \\
\beta_3 \text{Class2}_i + \cdots + \beta_{16} \text{Class15}_i + \varepsilon_i
\end{gathered}
$$

$\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2_\varepsilon)$

Linear mixed effects model: Let $Score_{ij}$ be the score of student j in class i

$$
Score_{ij} = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + u_i + \varepsilon_{ij}
$$

$\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2_\varepsilon) \qquad u_i \overset{iid}{\sim} N(0, \sigma^2_u)$

Anatomy of the mixed effects model

Linear mixed effects model: Let $Score_{ij}$ be the score of student j in class i

$$Score_{ij} = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + u_i + \varepsilon_{ij}$$

$$\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2) \qquad u_i \overset{iid}{\sim} N(0, \sigma_u^2)$$

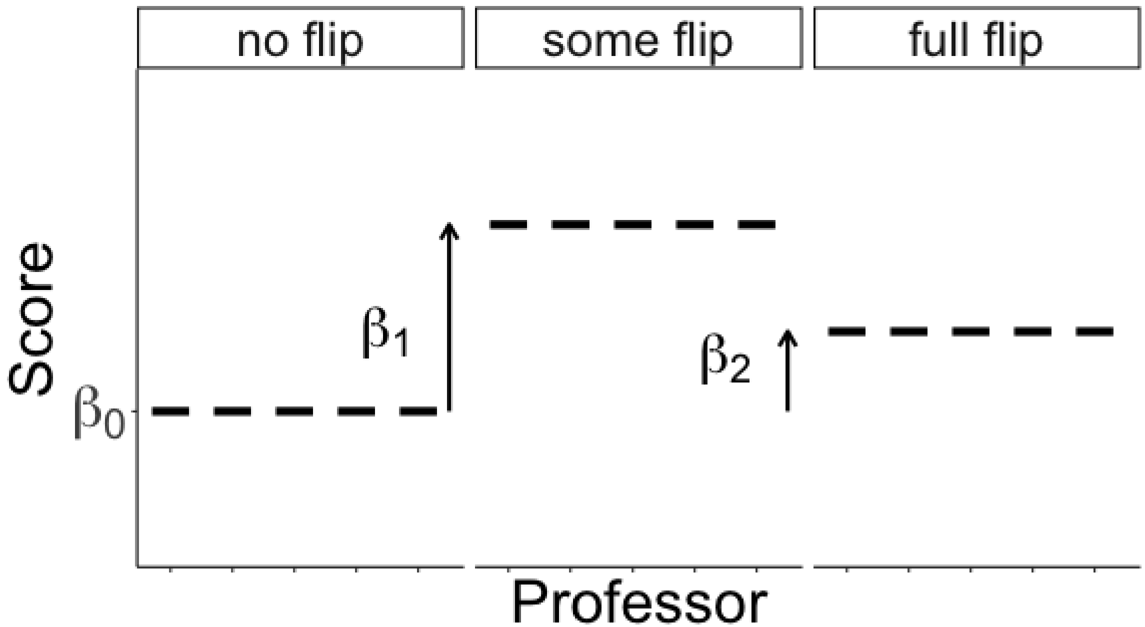
- $\beta_0, \beta_1, \beta_2$: **fixed effect** terms (representing treatment effect)
- u_i : **random effect** terms (representing group effects)
- ε_{ij} : **noise** terms (representing individual effects)

Anatomy of the mixed effects model

$$Score_{ij} = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + u_i + \varepsilon_{ij}$$

$$\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2) \qquad u_i \overset{iid}{\sim} N(0, \sigma_u^2)$$

Part 1: Fixed effects (treatment effects)

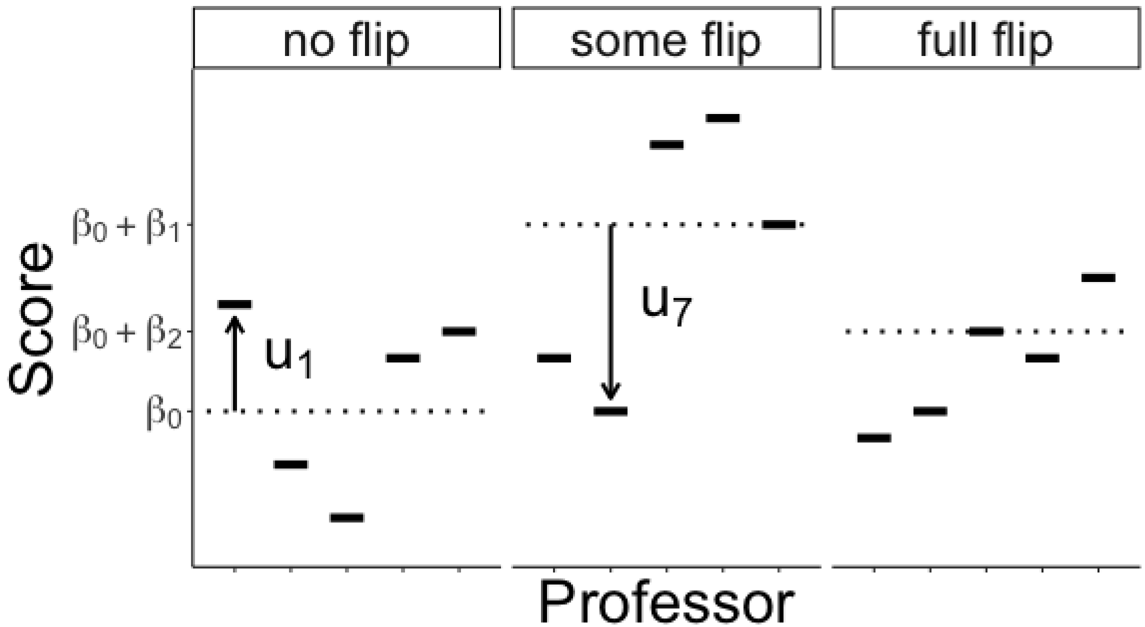


Anatomy of the mixed effects model

$$Score_{ij} = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + u_i + \varepsilon_{ij}$$

$$\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2_\varepsilon) \qquad u_i \overset{iid}{\sim} N(0, \sigma^2_u)$$

Part 2: Random effects (group effects)

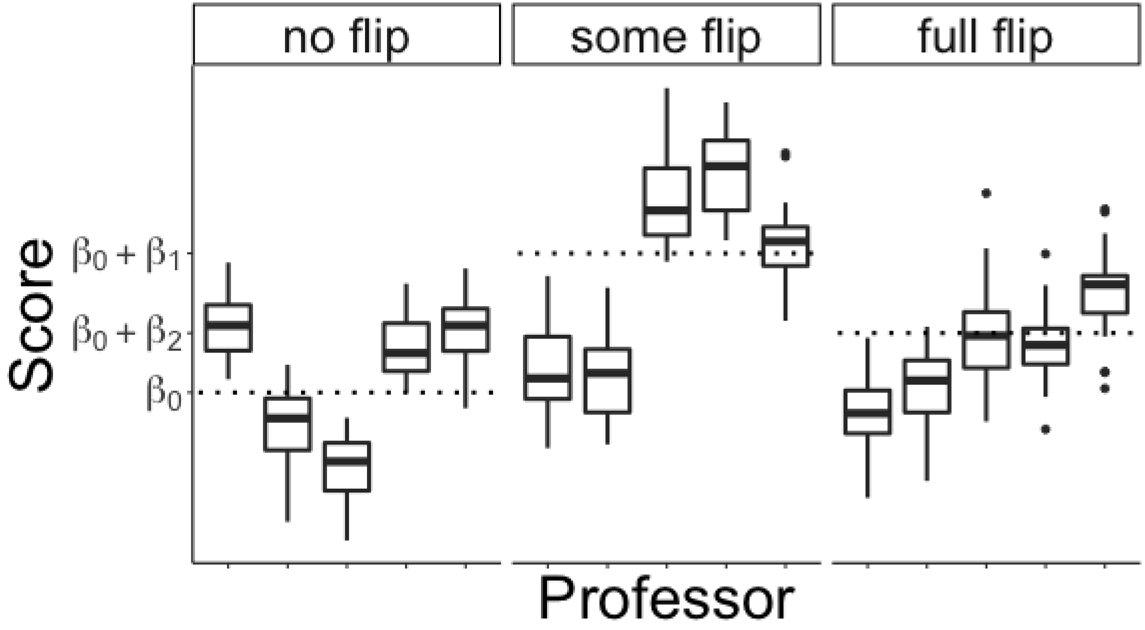


Anatomy of the mixed effects model

$$Score_{ij} = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + u_i + \varepsilon_{ij}$$

$$\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2) \qquad u_i \overset{iid}{\sim} N(0, \sigma_u^2)$$

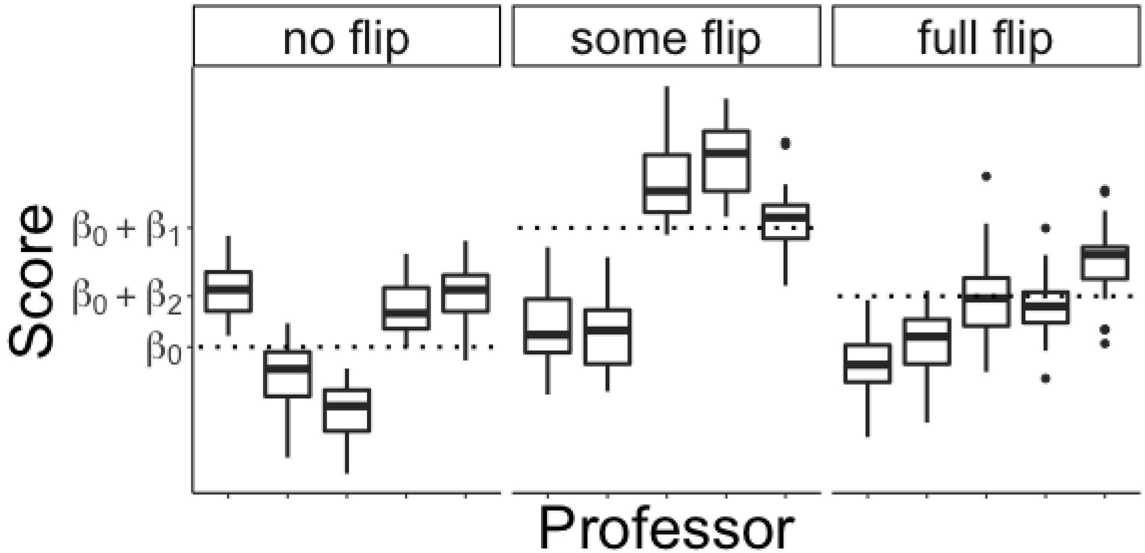
Part 3: Noise (individual effects)



Understanding variance parameters

$$Score_{ij} = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + u_i + \varepsilon_{ij}$$

$$\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2) \qquad u_i \overset{iid}{\sim} N(0, \sigma_u^2)$$



How does the picture change if I increase σ_ε^2 ?

Increasing σ_ε^2

$$Score_{ij} = \beta_0 + \beta_1 \text{SomeFlipped}_i + \beta_2 \text{FullyFlipped}_i + u_i + \varepsilon_{ij}$$

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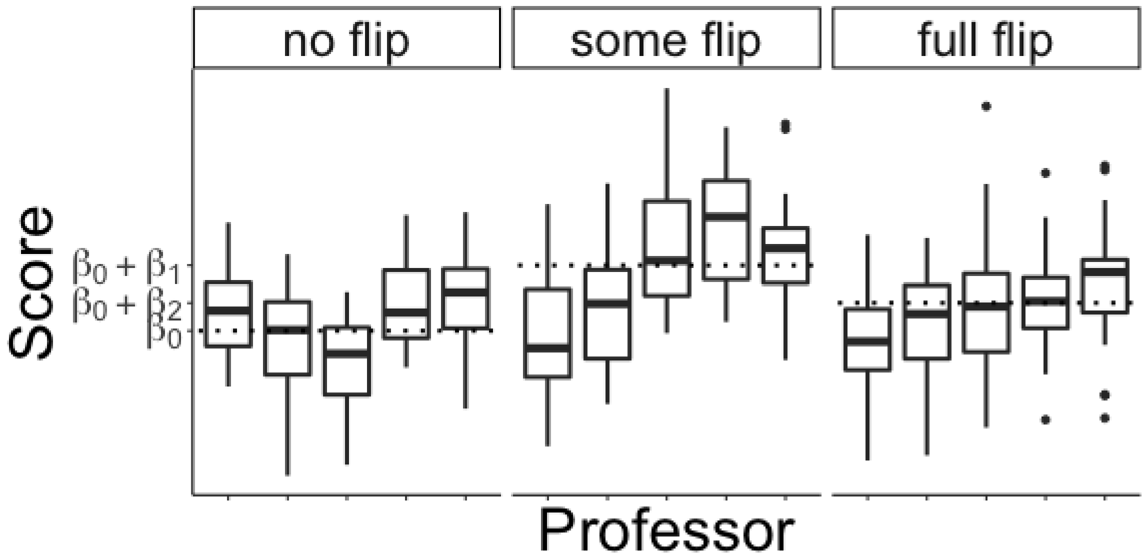

How does the picture change if I increase σ_u^2?

Increasing σ_u^2

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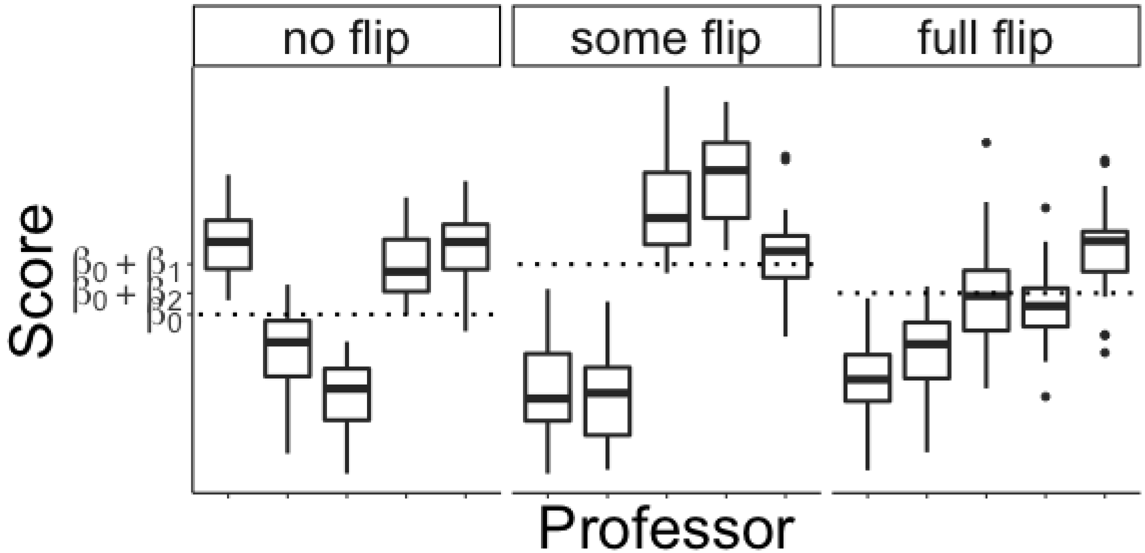
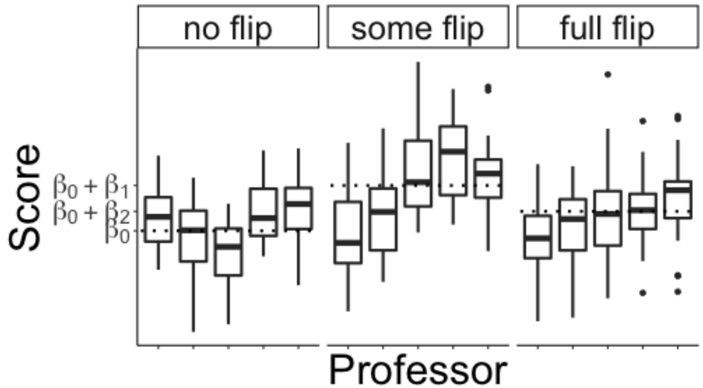
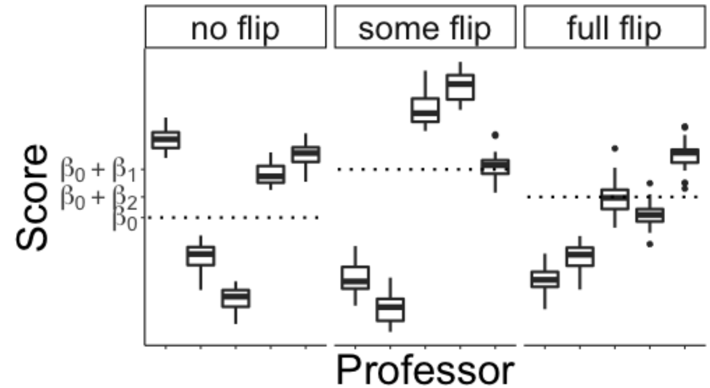
$$\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2) \qquad u_i \overset{iid}{\sim} N(0, \sigma_u^2)$$



σ_u^2 vs. σ_ε^2





σ_ε^2 is large relative to σ_u^2

σ_ε^2 is small relative to σ_u^2

- Observations within a group are *more correlated* when σ_ε^2 is small relative to σ_u^2
- **Intra-class correlation:** $\dfrac{\sigma_u^2}{\sigma_u^2 + \sigma_\varepsilon^2}$

Class activity

https://sta214-f22.github.io/class_activities/ca_lecture_25.html

Class activity

> Why is a mixed effect model useful for this data?

Class activity

> Why is a mixed effect model useful for this data?

- There is probably variation between neighborhoods, which we need to account for
- But we don't care about comparing neighborhoods. We just want to look at price and overall satisfaction

Class activity

> What is the population model?

Class activity

> What is the population model?

$$Price_{ij} = \beta_0 + \beta_1 Satisfaction_{ij} + u_i + \varepsilon_{ij}$$

$u_i \overset{iid}{\sim} N(0, \sigma_u^2) \quad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$

where $Price_{ij}$ is the price of rental j in neighborhood i

Class activity

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> What are the effect of interest, group effect, and individual effect?

Class activity

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where $Price_{ij}$ is the price of rental j in neighborhood i

> What are the effect of interest, group effect, and individual effect?

- effect of interest: β_1 (slope for relationship between satisfaction and price)
- group effect: u_i (random effect for neighborhood)
- individual effect: ε_{ij} (variation between rentals in a neighborhood)

Class activity

$$Price_{ij} = \beta_0 + \beta_1 Satisfaction_{ij} + u_i + \varepsilon_{ij}$$

$u_i \overset{iid}{\sim} N(0, \sigma_u^2) \quad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$

where $Price_{ij}$ is the price of rental j in neighborhood i

> What assumptions are we making in this mixed effects model?

Assumptions

$$Price_{ij} = \beta_0 + \beta_1 Satisfaction_{ij} + u_i + \varepsilon_{ij}$$

$$u_i \overset{iid}{\sim} N(0, \sigma_u^2) \qquad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$$

- **Shape:**
  - the overall relationship between satisfaction and price is linear
  - The slope is the *same* for each neighborhood

Assumptions

$$Price_{ij} = \beta_0 + \beta_1 Satisfaction_{ij} + u_i + \varepsilon_{ij}$$

$u_i \overset{iid}{\sim} N(0, \sigma^2_u) \quad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2_\varepsilon)$

- **Independence:**
  - random effects are independent
  - observations within neighborhoods are independent after accounting for the random effect (i.e., the random effect captures the correlation within neighborhoods)
  - observations from different neighborhoods are independent

Assumptions

$$Price_{ij} = \beta_0 + \beta_1 Satisfaction_{ij} + u_i + \varepsilon_{ij}$$

$$u_i \overset{iid}{\sim} N(0, \sigma_u^2) \qquad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$$

- **Normality:** Both $u_i \sim N(0, \sigma_u^2)$ and $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$
- **Constant variance:**
  - ε_{ij} has the same variance σ_ε^2 regardless of satisfaction or neighborhood