Quasi-Poisson models

Recap: Overdispersion

Overdispersion occurs when the response Y has higher variance than we would expect if Y followed a Poisson distribution.

Formally, let

$$\phi = \frac{\text{Variance}}{\text{Mean}}$$

Recap: Estimating overdispersion

The *Pearson residual* for observation i is defined as

$$e_{(P)i} = \frac{Y_i - \widehat{\lambda}_i}{\sqrt{\widehat{\lambda}_i}}$$

$$\widehat{\phi} = \frac{\sum\limits_{i=1}^{n} e_{(P)i}^2}{n - p}$$

- $p =$ number of parameters in model

Handling overdispersion

Overdispersion is a problem because our standard errors (for confidence intervals and hypothesis tests) are too low.

> If we think there is overdispersion, what should we do?

Adjusting the standard error

- In our data, $\widehat{\phi} = 1.83$
- This means our variance is 1.83 times bigger than it should be
- So our standard error is $\sqrt{1.83} = 1.35$ times bigger than it should be

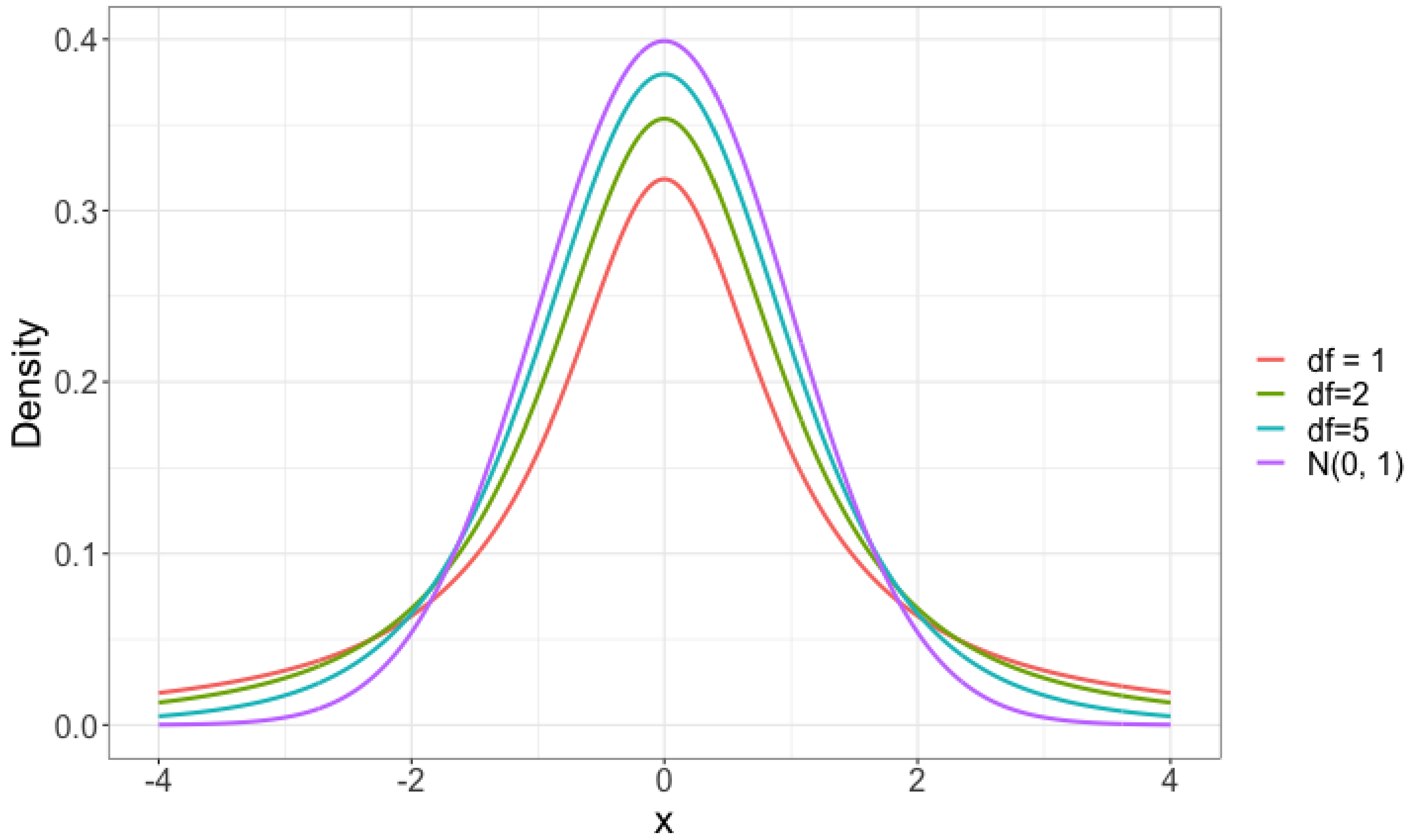
Adjusting the standard error in R

```
m2 <- glm(art ~ ., data = articles,
          family = quasipoisson)
```

```
...
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.304617   0.139273   2.187 0.028983 *
## femWomen    -0.224594   0.073860  -3.041 0.002427 **
## marMarried   0.155243   0.083003   1.870 0.061759 .
## kid5        -0.184883   0.054268  -3.407 0.000686 ***
## phd          0.012823   0.035700   0.359 0.719544
## ment         0.025543   0.002713   9.415  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for quasipoisson family taken to be 1.829006)
...
```

- Allowing ϕ to be different from 1 means we are using a *quasi-likelihood* (in this case, a *quasi-Poisson*)

t-distribution



Calculating a confidence interval

```
...
## (Intercept)  0.304617   0.139273   2.187 0.028983 *
## femWomen    -0.224594   0.073860  -3.041 0.002427 **
## marMarried   0.155243   0.083003   1.870 0.061759 .
## kid5        -0.184883   0.054268  -3.407 0.000686 ***
## phd          0.012823   0.035700   0.359 0.719544
## ment         0.025543   0.002713   9.415  < 2e-16 ***
...
```

New confidence interval for β_4:

$$0.0128 \pm t^*_{n-p} \cdot 0.0357$$

```r
qt(0.025, df=909, lower.tail=F)
```

```
## [1] 1.962577
```

$$0.0128 \pm 1.96 \cdot 0.0357 = (-0.0572,\ 0.0828)$$

Adjusting the standard error in R

Poisson:

```
...
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept)  0.304617   0.102981   2.958   0.0031 **
## femWomen    -0.224594   0.054613  -4.112 3.92e-05 ***
## marMarried   0.155243   0.061374   2.529   0.0114 *
## kid5        -0.184883   0.040127  -4.607 4.08e-06 ***
...
```

Quasi-Poisson:

```
...
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.304617   0.139273   2.187 0.028983 *
## femWomen    -0.224594   0.073860  -3.041 0.002427 **
## marMarried   0.155243   0.083003   1.870 0.061759 .
## kid5        -0.184883   0.054268  -3.407 0.000686 ***
...
```

Back to simulations

```
n <- 1000
nsim <- 500
contains_beta <- rep(0, nsim)
for(i in 1:nsim){
  x <- rnorm(n, sd = 0.5)
  y2 <- rnbinom(n, size=0.5, mu=exp(x))

  m2 <- glm(y2 ~ x, family = poisson)

  upper <- summary(m2)$coefficients[2,1] +
      1.96*summary(m2)$coefficients[2,2]
  lower <- summary(m2)$coefficients[2,1] -
      1.96*summary(m2)$coefficients[2,2]

  contains_beta[i] <- upper > 1 && lower < 1
}

mean(contains_beta)
```

```
## [1] 0.642
```

Adjusting for overdispersion

```
n <- 1000
nsim <- 500
contains_beta <- rep(0, nsim)
for(i in 1:nsim){
  x <- rnorm(n, sd = 0.5)
  y2 <- rnbinom(n, size=0.5, mu=exp(x))

  m2 <- glm(y2 ~ x, family = quasipoisson)

  upper <- summary(m2)$coefficients[2,1] +
      qt(0.025, n-2, lower.tail = F)*summary(m2)$coefficients[2,2]
  lower <- summary(m2)$coefficients[2,1] -
      qt(0.025, n-2, lower.tail = F)*summary(m2)$coefficients[2,2]

  contains_beta[i] <- upper > 1 && lower < 1
}

mean(contains_beta)
```

```
## [1] 0.926
```

Class activity

https://sta214-s23.github.io/class_activities/ca_lecture_22.html

Class activity

```
...
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -1.30445    0.34161  -3.818 0.000274 ***
## regionMW     0.09754    0.48893   0.199 0.842417
## regionNE     0.76268    0.42117   1.811 0.074167 .
## regionSE     0.87237    0.42175   2.068 0.042044 *
## regionSW     0.50708    0.50973   0.995 0.323027
## regionW      0.20934    0.51242   0.409 0.684055
...
```

> What confidence interval should I calculate to compare western and central schools?

Class activity

```
...
## regionSE      0.87237    0.42175   2.068 0.042044 *
## regionSW      0.50708    0.50973   0.995 0.323027
## regionW       0.20934    0.51242   0.409 0.684055
...
```

```
qt(0.025, 75, lower.tail=F)
```

```
## [1] 1.992102
```

95% confidence interval for β_5:

$$0.209 \pm 1.99 \cdot 0.512 = (-0.81,\ 1.23)$$

95% confidence interval for e^{β_5}:

$$(e^{-0.81}, e^{1.23}) = (0.44,\ 3.42)$$

Comparing Poisson and quasi-Poisson

Poisson:

- Mean = λ_i
- Variance = λ_i

quasi-Poisson:

- Mean = λ_i
- Variance = $\phi\lambda_i$
- Variance is a linear function of the mean

> What if we want variance to depend on the mean in a different way?

Introducing the negative binomial

If $Y_i \sim NB(\theta, p)$, then Y_i takes values $y = 0, 1, 2, 3, \ldots$ with probabilities

$$P(Y_i = y) = \frac{(y + \theta - 1)!}{y!(\theta - 1)!}(1 - p)^\theta p^y$$

- $\theta > 0, \quad p \in [0, 1]$
- Mean = $\dfrac{p\theta}{1 - p} = \mu$
- Variance = $\dfrac{p\theta}{(1 - p)^2} = \mu + \dfrac{\mu^2}{\theta}$
- Variance is a *quadratic* function of the mean

Mean and variance for a negative binomial variable

If $Y_i \sim NB(\theta, p)$, then

- Mean = $\dfrac{p\theta}{1-p} = \mu$
- Variance = $\dfrac{p\theta}{(1-p)^2} = \mu + \dfrac{\mu^2}{\theta}$

> How is θ related to overdispersion?

Negative binomial regression

$$Y_i \sim NB(\theta,\ p_i)$$

$$\log(\mu_i) = \beta_0 + \beta_1 X_i$$

- $\mu_i = \dfrac{p_i \theta}{1 - p_i}$
- Note that θ is the same for all i
- Note that just like in Poisson regression, we model the average count
  - Interpretation of βs is the same as in Poisson regression

Comparing Poisson, quasi-Poisson, negative binomial

Poisson:

- Mean = λ_i
- Variance = λ_i

quasi-Poisson:

- Mean = λ_i
- Variance = $\phi\lambda_i$

negative binomial:

- Mean = μ_i
- Variance = $\mu_i + \frac{\mu_i^2}{\theta}$

In R

```
m3 <- glm.nb(art ~ ., data = articles)
```

```
...
##               Estimate Std. Error z value Pr(>|z|)
## (Intercept)   0.256144   0.137348   1.865 0.062191 .
## femWomen     -0.216418   0.072636  -2.979 0.002887 **
## marMarried    0.150489   0.082097   1.833 0.066791 .
## kid5         -0.176415   0.052813  -3.340 0.000837 ***
## phd           0.015271   0.035873   0.426 0.670326
## ment          0.029082   0.003214   9.048  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
##
## (Dispersion parameter for Negative Binomial(2.2644) fami
...
```

$\hat{\theta} = 2.264$

In R

```
...
##               Estimate Std. Error z value Pr(>|z|)
## (Intercept)  0.256144   0.137348   1.865 0.062191 .
## femWomen    -0.216418   0.072636  -2.979 0.002887 **
## marMarried   0.150489   0.082097   1.833 0.066791 .
## kid5        -0.176415   0.052813  -3.340 0.000837 ***
## phd          0.015271   0.035873   0.426 0.670326
## ment         0.029082   0.003214   9.048  < 2e-16 ***
...
```

> How do I interpret the estimated coefficient -0.176?

quasi-Poisson vs. negative binomial

quasi-Poisson:

- linear relationship between mean and variance
- easy to interpret $\widehat{\phi}$
- same as Poisson regression when $\phi = 1$
- simple adjustment to estimated standard errors
- estimated coefficients same as in Poisson regression

negative binomial:

- quadratic relationship between mean and variance
- we get to use a likelihood, rather than a quasi-likelihood
- Same as Poisson regression when θ is very large and p is very small