Hypothesis testing for logistic regression

Modeling grad admissions

we need an interaction between GPA and rank!

Data: Grad application data

- `admit`: accepted to grad school? (0 = no, 1 = yes)
- `gre`: GRE score
- `gpa`: undergrad GPA
- `rank`: prestige of undergrad institution

New question: Does the relationship between GPA and the probability of acceptance depend on the prestige of a student's undergrad institution, after accounting for GRE score?

> How could we use hypothesis testing to investigate this research question? Discuss with your neighbor, then we will discuss as a group.

Model

$$Admit_i \sim Bernoulli(\pi_i)$$

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 GRE_i + \beta_2 GPA_i + \beta_3 Rank2_i$$
$$+ \beta_4 Rank3_i + \beta_5 Rank4_i + \beta_6 GPA_i \cdot Rank2_i$$
$$+\beta_7 GPA_i \cdot Rank3_i + \beta_8 GPA_i \cdot Rank4_i$$

> How would I interpret coefficients when I have an interaction?

β_1: change in log odds for a unit increase in GRE, holding GPA and rank fixed

β_2: change in log odds for a unit increase in GPA when Rank = 1, holding GRE fixed

$\beta_2 + \beta_6$: change in log odds for a unit increase in GPA when Rank = 2, holding GRE fixed

Hypotheses

$$Admit_i \sim Bernoulli(\pi_i)$$

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 GRE_i + \beta_2 GPA_i + \beta_3 Rank2_i + \beta_4 Rank3_i + \beta_5 Rank4_i + \beta_6 GPA_i \cdot Rank2_i + \beta_7 GPA_i \cdot Rank3_i + \beta_8 GPA_i \cdot Rank4_i$$

Question: Does the relationship between GPA and the probability of acceptance depend on the prestige of a student's undergrad institution, after accounting for GRE score?

> What are my null and alternative hypotheses?

H_0: $\beta_6 = \beta_7 = \beta_8 = 0$

H_A: at least one of $\beta_6, \beta_7, \beta_8 \neq 0$

Class activity, Part I

https://sta214-s23.github.io/class_activities/ca_lecture_9.html

Class activity

include both variables
and their interaction

Models:

```
m1 <- glm(admit ~ gre + gpa*as.factor(rank),
          data = grad_app, family = binomial)

m2 <- glm(admit ~ gre + gpa + as.factor(rank),
          data = grad_app, family = binomial)

m2$deviance - m1$deviance
```

```
## [1] 0.4054785
```

Class activity

$H_0 : \beta_6 = \beta_7 = \beta_8 = 0$

Test statistic: $G = 0.41$

Calculating a p-value:

```
pchisq(..., ..., lower.tail=F)
```

0.41 3

\# parameters

Class activity

$H_0 : \beta_6 = \beta_7 = \beta_8 = 0$

Test statistic: $G = 0.41$

Calculating a p-value:

```
pchisq(0.41, 3, lower.tail=F)
```

```
## [1] 0.9381691
```

Conclusion: we have very weak evidence against H_0

In this class: we will avoid p-value thresholds (e.g. 0.05)
we will avoid statistical significant

Likelihood ratio test for nested models

(LRT)

Goal: Compare full and reduced models

Steps:

- Calculate deviance for full and reduced models
- G = deviance for reduced - deviance for full
- p-value: $G \sim \chi^2_q$

parameters tested

$$G = -2\log L_{reduced} - (-2\log L_{full})$$

$$= 2\log\Big(\underbrace{\frac{L_{full}}{L_{reduced}}}_{\text{likelihood ratio!}}\Big)$$

(e.g. $\beta_1 > 0$)

. LRT can test nested models

. LRT cannot test one-side alternatives

Alternative: Wald tests for single parameters

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 GRE_i + \beta_2 GPA_i + \beta_3 Rank2_i$$
$$+ \beta_4 Rank3_i + \beta_5 Rank4_i + \beta_6 GPA_i \cdot Rank2_i$$
$$+\beta_7 GPA_i \cdot Rank3_i + \beta_8 GPA_i \cdot Rank4_i$$

Hypotheses: $H_0: \beta_1 = b$ (b is any value of interest) e.g. 0, 1, etc.

$H_A: \beta_1 \neq b$

or $\beta_1 > b$

or $\beta_1 < b$

Test statistic:

$$z = \frac{\hat{\beta}_1 - b}{SE(\hat{\beta}_1)} \sim N(0,1) \quad \text{under } H_0$$

Example

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 GRE_i + \beta_2 GPA_i + \beta_3 Rank2_i + \beta_4 Rank3_i + \beta_5 Rank4_i + \beta_6 GPA_i \cdot Rank2_i + \beta_7 GPA_i \cdot Rank3_i + \beta_8 GPA_i \cdot Rank4_i$$

$H_0 : \beta_1 = 0 \quad H_A : \beta_1 \neq 0$

corresponds to
$H_0: \beta = 0$

```
...
## Coefficients:
##                   Estimate Std. Error z value Pr(>|z|)
## (Intercept)      -4.985768   2.480668  -2.010   0.0444 *
## gre               0.002287   0.001102   2.075   0.0380 *
## gpa               1.089088   0.726130   1.500   0.1337
...
```

p-value for $H_A: \beta \neq 0$

$z = \frac{0.0023 - 0}{0.0011} \approx 2.075$ p-value $= 0.038$

Example

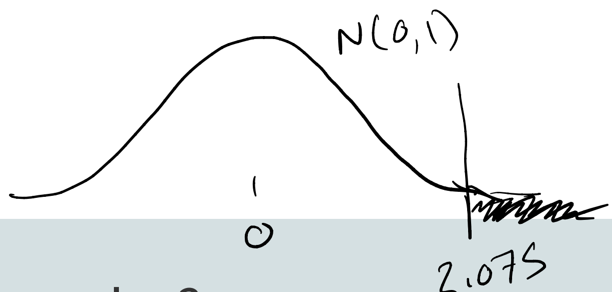
$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 GRE_i + \beta_2 GPA_i + \beta_3 Rank2_i$$
$$+ \beta_4 Rank3_i + \beta_5 Rank4_i + \beta_6 GPA_i \cdot Rank2_i$$
$$+ \beta_7 GPA_i \cdot Rank3_i + \beta_8 GPA_i \cdot Rank4_i$$

$H_0 : \beta_1 = 0 \quad H_A : \beta_1 > 0$

this is only the p-value when $H_0: \beta = 0$, $H_A: \beta \neq 0$

```
...
## Coefficients:
##                Estimate Std. Error z value Pr(>|z|)
## (Intercept)   -4.985768   2.480668  -2.010   0.0444 *
## gre            0.002287   0.001102   2.075   0.0380 *
## gpa            1.089088   0.726130   1.500   0.1337
...
```

$z = 2.075$



want $P(Z \geq 2.075)$

How do I calculate a p-value?

Example

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 GRE_i + \beta_2 GPA_i + \beta_3 Rank2_i + \beta_4 Rank3_i + \beta_5 Rank4_i + \beta_6 GPA_i \cdot Rank2_i + \beta_7 GPA_i \cdot Rank3_i + \beta_8 GPA_i \cdot Rank4_i$$

$$H_0 : \beta_1 = 0 \qquad H_A : \beta_1 > 0$$

```
...
## Coefficients:
##                          Estimate Std. Error z value Pr(>|z|)
## (Intercept)             -4.985768   2.480668  -2.010   0.0444 *
## gre                      0.002287   0.001102   2.075   0.0380 *
## gpa                      1.089088   0.726130   1.500   0.1337
...
```

```
pnorm(2.075, lower.tail=F)
```

$P(Z \geq 2.075)$

(right tail)

```
## [1] 0.01899327
```

$= \frac{0.038}{2}$

Wald tests vs. likelihood ratio tests

Wald test

- like t-tests
- test a single parameter
- some example hypotheses:
  - $H_0 : \beta_1 = 0$ vs. $H_A : \beta_1 \neq 0$
  - $H_0 : \beta_1 = 1$ vs. $H_A : \beta_1 > 1$

Likelihood ratio test

- like nested F-tests
- test one or more parameters
- some example hypotheses:
  - $H_0 : \beta_1 = 0$ vs. $H_A : \beta_1 \neq 0$

$H_0 : \beta_1 = \beta_2 = 0$

H_A: at least one of $\beta_1, \beta_2 \neq 0$

p-values are different, because test statistics and distributions are different

Confidence intervals

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 GRE_i + \beta_2 GPA_i + \beta_3 Rank2_i$$
$$+ \beta_4 Rank3_i + \beta_5 Rank4_i + \beta_6 GPA_i \cdot Rank2_i$$
$$+ \beta_7 GPA_i \cdot Rank3_i + \beta_8 GPA_i \cdot Rank4_i$$

Now we want a "reasonable range" of values for β_1.

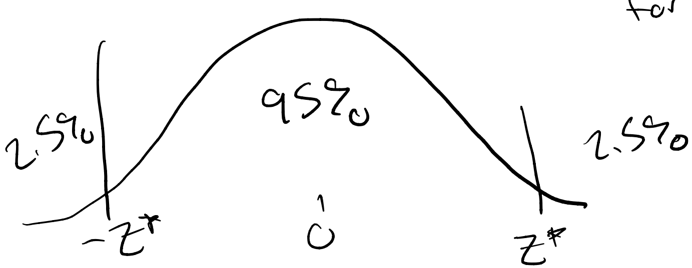
Confidence interval:

$$\hat{\beta}_1 \pm z^* \underbrace{SE(\hat{\beta}_1)}_{\text{standard error}}$$

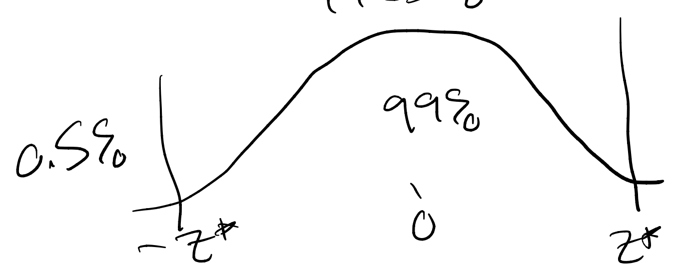
estimate

critical value for $N(0,1)$

95% CI



99 CI %



Computing z^*

Example: for a 95% confidence interval, $z^* = 1.96$

quantiles of a normal distribution

```
qnorm(0.025, lower.tail=F)
```

amount in each tail

```
## [1] 1.959964
```

Example: for a 99% confidence interval, $z^* = 2.58$:

```
qnorm(0.005, lower.tail=F)
```

```
## [1] 2.575829
```

Confidence interval

```
...
## Coefficients:
##                    Estimate Std. Error z value Pr(>|z|)
## (Intercept)       -4.985768   2.480668  -2.010   0.0444 *
## gre                0.002287   0.001102   2.075   0.0380 *
## gpa                1.089088   0.726130   1.500   0.1337
...
```

95% confidence interval for β_1:

$$0.0023 \pm 1.96\,(0.0011) = (0.0001,\ 0.0045)$$

we are 95% confident that $\beta_1 \in (0.0001, 0.0045)$

i.e.,

we are 95% confident that a one-unit increase in GRE is associated with an increase in the log odds of admission by between 0.0001 and 0.0045, holding GPA & rank fixed

Class activity, Part II

https://sta214-s23.github.io/class_activities/ca_lecture_9.html

Class activity

```
...
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  1.73743    0.08499   20.44   <2e-16 ***
## WBC         -0.36085    0.01243  -29.03   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
...
```

$-0.361 \pm 2.576(0.012)$

$= (-0.392, -0.330)$

99% CI for change in odds (e^{β_1}):

$(e^{-0.392}, e^{-0.330}) = (0.676, 0.719)$