STA221

Neil Montgomery

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The second condition is the main thing to check.

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  - the summary tables with counts are not datasets - they are summaries.
  - apparently some people try to apply the χ^2 procedures to other kinds of summary tables, which is why the readings emphasize this point as a warning.
- "randomization condition," which has more to do with the possibility of *inferring something about a larger population, or not* than anything to do with χ^2 tests per se.

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  - I did indeed analyse counts.
  - The analysis was not on any sample at all - I used *all* numbers ever drawn!
- All other examples satisfied the $E_{ij} \geq 5$ condition, which is the main thing that should always be verified and commented on.

post-hoc investigations of χ^2 tests using residuals

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These are approximately $N(0, 1)$, so one can glance at the cell-by-cell residuals to get information about which cells had the largest deviation from expected.

standardized residuals example - I (pipeline)

```
##         Pressure
## Leak  High  Low  Med  Sum
##   No    277  278  247  802
##   Yes    71   66   61  198
##   Sum   348  344  308 1000

##
##  Pearson's Chi-squared test
##
## data:  table(Leak, Pressure)
## X-squared = 0.16116, df = 2, p-value = 0.9226

##         Pressure
## Leak     High    Low    Med
##   No   -0.125  0.127 -0.001
##   Yes   0.253 -0.256  0.002
```

standardized residuals example - II (births)

```
## 
##  Pearson's Chi-squared test with Yates' continuity correction
## 
## data:  childbirth
## X-squared = 14.388, df = 1, p-value = 0.0001487


##          Breastfeeding
## Epidural    No    Yes
##      No  -1.728  1.418
##      Yes  2.429 -1.993
```

regression

linear models

Basic model: Output = Input + Noise

You've seen a few of these already; perhaps not expressed this way.

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The "one sample t-test" can be thought of a way to analyze data that can be modeled as:

$$Y_i = \mu + \varepsilon_i$$

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The "two sample t-test" could be modeled as:

$$Y_{ij} = \mu_i + \varepsilon_{ij}$$

where $i = 1, 2$ and the μ_i are the two population means. (There are a few ways to treat the ε_{ij}.)

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Let's consider the male body fat dataset that is discussed in the textbook (Chapter 24).

```
## Parsed with column specification:
## cols(
##   `Pct BF` = col_double(),
##   Age = col_integer(),
##   Weight = col_double(),
##   Height = col_double(),
##   Neck = col_double(),
##   Chest = col_double(),
##   Abdomen = col_double(),
##   waist = col_double(),
##   Hip = col_double(),
##   Thigh = col_double(),
##   Knee = col_double(),
```
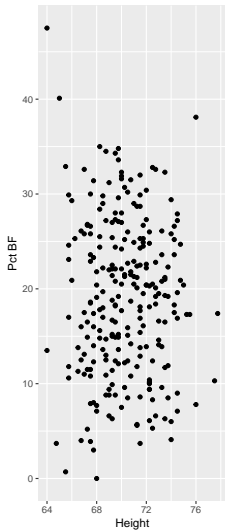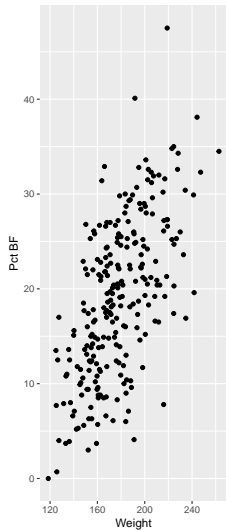
body fat EDA

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In general:

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where y and x are the variables and ε is the random noise.

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The *parameter* β_1 is the slope of the line and is of primary interest. (The parameter β_0 is the y-intercept and not normally of any interest.)

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- ▶ The "grid" could consist of as few as two values!

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- ▶ The data come in pairs $(y_1, x_1), (y_2, x_2), \ldots (y_n, x_n)$.
- ▶ For any "candidate" slope b_0^* and intercept b_1^* we could construct the set of "predictions" $\hat{y}_i = b_0^* + b_1^* x_i$ and their "residuals" $\varepsilon_i = y_i - \hat{y}_i$

more least squares

Here's the actual "least squares" part...

It is possible to find the unique slope and intercept that makes this sum of squared residuals:

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n}(y_i - (b_0^* + b_1^* y_i))^2$$

as small as possible.

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The formula for the slope estimator b_1 turns out to be:

$$b_1 = \frac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sum(x_i - \overline{x})^2} = \frac{S_{xy}}{S_{xx}}$$

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The formula for the intercept is $b_0 = \overline{y} - b_1\overline{x}$

body fat examples

Here are the plots with the least squares regression lines added:
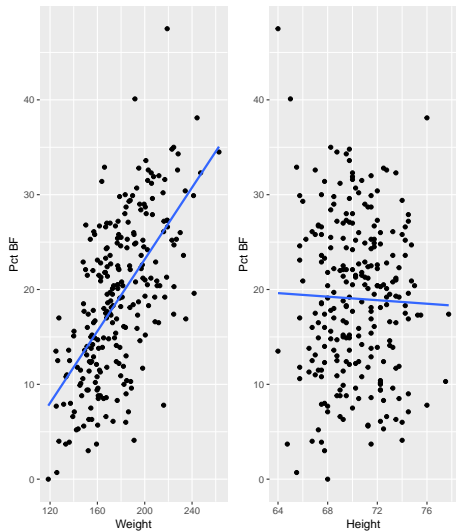
bodyfat calculation examples

Obviously don't do these by hand! Here is basic `R` regression output:

```
##
## Call:
## lm(formula = `Pct BF` ~ Weight, data = bodyfat)
##
## Coefficients:
## (Intercept)       Weight
##    -14.6931       0.1894

##
## Call:
## lm(formula = `Pct BF` ~ Height, data = bodyfat)
##
## Coefficients:
## (Intercept)       Height
##    25.58078      -0.09316
```

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- therefore, y is random (as the sum of a fixed part and a random part)

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  - ▶ violation of this requirement is **fatal** to any analysis.
  - ▶ transforming one or both variables is a possible remedy.

- ▶ independent observations in the dataset
  - ▶ hard to verify—usually assumed.
  - ▶ one type of non-independence can sometimes be detected by plotting values versus time or the order in which they were observed.
  - ▶ violation could be **fatal** but possibly not
  - ▶ "time series" methods are one way to deal with one type of non-independence.

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  - check using a special scatterplot involving the residuals, TBA

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These two assumptions can be rolled into one statement:

$$\varepsilon \sim N(0, \sigma)$$

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$$\varepsilon \sim N(0, \sigma)$$

* with one exception TBA

simulation for investigating b_0 and b_1

The properties of the slope parameter estimator b_1 is of most interest.

- ▶ What is its average value, variation, distribution?

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First, we'll look at the average value of b_0, using simulation. To do this I will start with a *fully known theoretical linear model*:

$$y = 2 + 0.75x + \varepsilon$$

with $\varepsilon \sim N(0, 1)$.

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with $\varepsilon \sim N(0, 1)$.

I will simulate fake datasets of size $n = 50$ from this model, compute the regression line for each dataset, and see what happens.

e.g. plots of four samples
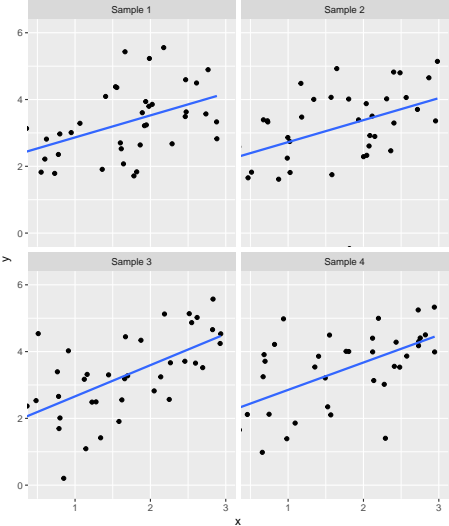
properties of b_1 from 1000 samples

I would like to investigate the distribution of b_1 using simulation. So I will simulate 1000 replications, and see what happens.
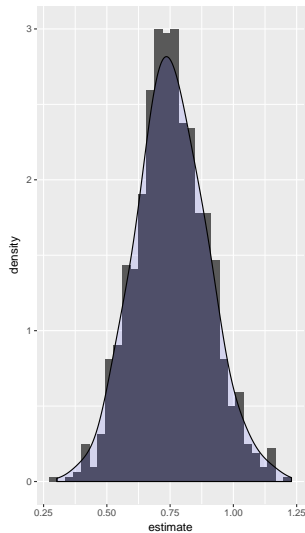
Here is a numerical summary of the 1000 simulated b_1 (and b_0 as well, since I have them):

	term	Average	SD
1	(Intercept)	2.00977	0.22943
2	x	0.74862	0.14093

(Note: these numbers *change* every time I render the lecture notes - the simulation is embedded right in them.)

Conclusion: the average values of b_1 (and b_0) are the true values β_1 (and β_0).

histogram of the simulated b_1

Looks symmetric and bell-shaped. Perhaps they have a normal distribution?

change σ from 1 to 0.1

I will simulate again, but this time with $\varepsilon \sim N(0, 0.1)$. Four example plots:
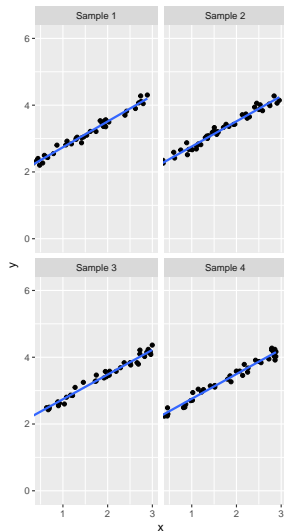
properties of b_1 from 1000 samples ($\sigma = 0.1$ version)

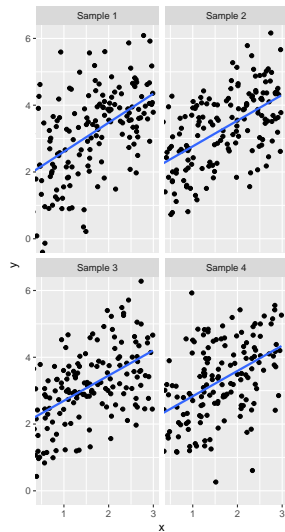The averages and SDs of the 1000 estimators:

term	Average	SD
(Intercept)	1.99989	0.02264
x	0.75006	0.01424

The histogram looks the same.

Conclusion: when the *inherent underlying noise is smaller* the parameter estimators are *more accurate*.

put σ back to 1; increase the sample size to $n = 200$

Four sample plots:

properties of b_1 from 1000 samples ($n = 200$ version)
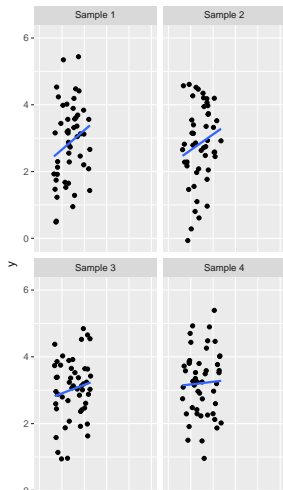
The averages and SDs of the 1000 estimators:

term	Average	SD
(Intercept)	2.00508	0.11322
x	0.74845	0.06919

The histogram looks the same.

Conclusion: when the *sample size is larger* the parameter estimators are *more accurate*.

back to $n = 50$; properties of b_1 when the x values are less spread out

This one is a little more subtle. It turns out the x values affect the accuracy of the parameter estimates. I re-simulate with less spread in the x values. Four sample plots with x values 4 times "less spread out":

properties of b_1 (x less spread version)

The averages and SDs of the 1000 estimators:

term	Average	SD
(Intercept)	2.03038	0.70632
x	0.72287	0.55236

The histogram looks the same.

Conclusion: when the *x values are less spread out* the parameter estimators are *less accurate*.

statistical properties of b_1

Start with the basic simple linear regression model:

$$y = \beta_0 + \beta_1 x + \varepsilon$$

in which the error follows a $N(0, \sigma)$ distribution.

The slope estimator b_1 turns out to follow a normal distribtion with mean β_1 and standard deviation:

$$\frac{\sigma}{\sqrt{S_{xx}}}$$

(Recall $S_{xx} = \sum(x_i - \bar{x})^2$)

(Note: there is a typo on the first formula in section 24.2 - the s_x should not be under the $\sqrt{}$.)

statistical properties of b_1

Therefore we have:

$$\frac{b_1 - \beta_1}{\sigma/\sqrt{S_{xx}}} \sim N(0, 1)$$

and p-values and confidence intervals come from this—BAM we're done.

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and p-values and confidence intervals come from this—BAM we're done.

Except we would never know the true value of σ. This is the third simple regression parameter—a nuisance we'll have to deal with.

We can estimate σ using the "average" of the squared residuals:

$$s_e = \frac{\sum (y_i - \hat{y}_i)^2}{n - 2}$$

statistical properties of b_1

Who wants to guess what distribution this will have:

$$\frac{b_1 - \beta_1}{s_e / \sqrt{S_{xx}}}$$

statistical properties of b_1

Who wants to guess what distribution this will have:

$$\frac{b_1 - \beta_1}{s_e/\sqrt{S_{xx}}} \sim t_{n-2}$$

hypothesis testing for β_1

The principal hypothesis test concerns whether there is any linear relationship at all between x and y. The null hypothesis immediately presents itself:

$$H_0 : \beta_1 = 0$$

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$$H_0 : \beta_1 = 0$$

And it works the same way any other hypothesis test works. Use the data to compute:

$$\frac{b_1 - 0}{s_e/\sqrt{S_{xx}}}$$

and get the probability of being "further away" from H_0, according to the t_{n-2} distribution.

example - body fat versus weight
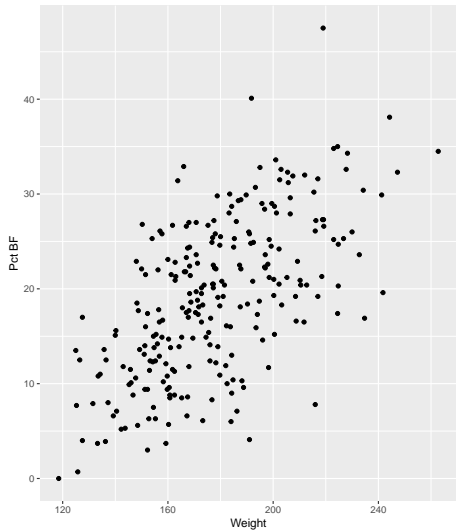
example - body fat versus weight

```
## 
## Coefficients:
##              Estimate Std. Error t value   Pr(>|t|)
## (Intercept) -14.69314    2.76045  -5.323 0.000000229 ***
## Weight        0.18938    0.01533  12.357     < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## 
## Residual standard error: 6.538 on 248 degrees of freedom
## Multiple R-squared:  0.3811, Adjusted R-squared:  0.3786
## F-statistic: 152.7 on 1 and 248 DF,  p-value: < 2.2e-16
```

this table translated into formulae

```
Coefficients:

              Estimate   Std. Error   t value     Pr(>|t|)
```

	Estimate	Std. Error	t value	Pr(>|t|)
b_0	(Not	often	very	relevant)
b_1	$\frac{S_{xy}}{S_{xx}}$	$\frac{s_e}{\sqrt{S_{xx}}}$	$\frac{b_1-0}{s_e/\sqrt{S_{xx}}}$	the p-value

```
---
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                                          (s_e/√S_xx)
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b_0 (Not   often   very   relevant)

b_1 $\dfrac{S_{xy}}{S_{xx}}$ $\dfrac{s_e}{\sqrt{S_{xx}}}$ $\dfrac{b_1-0}{s_e/\sqrt{S_{xx}}}$ the p-value

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A line of questionable utility.

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```

A line of questionable utility.

Residual standard error: s_e on $n - 2$ degrees of freedom

```
Coefficients:

                Estimate   Std. Error   t value      Pr(>|t|)
        b_0       (Not        often        very       relevant)
        b_1   $\frac{S_{xy}}{S_{xx}}$   $\frac{s_e}{\sqrt{S_{xx}}}$   $\frac{b_1 - 0}{s_e/\sqrt{S_{xx}}}$   the p-value
---

A line of questionable utility.

Residual standard error:  $s_e$ on $n-2$ degrees of freedom

Other stuff at the bottom not yet explained...
```

example - body fat versus height
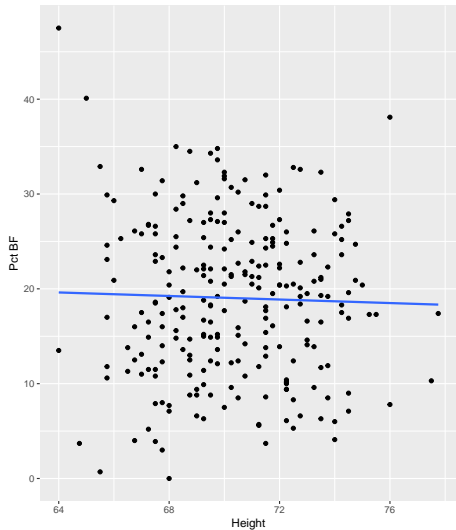
example - body fat versus height

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## 
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 25.58078   14.15400    1.807   0.0719 .
## Height      -0.09316    0.20119   -0.463   0.6438
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## 
## Residual standard error: 8.307 on 248 degrees of freedom
## Multiple R-squared:  0.0008637,  Adjusted R-squared:  -0.003165
## F-statistic: 0.2144 on 1 and 248 DF,  p-value: 0.6438
```

confidence interval for the true slope β_1

95% confidence intervals are all pretty much the same, based on:

$$\frac{\text{estimator} - \text{parameter}}{SE(\text{estimator})} \sim \text{ something symmetric and bell shaped}$$

resulting in a formula:

$$\text{estimator} \pm \text{"2"} SE(\text{estimator})$$

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resulting in a formula:

$$\text{estimator} \pm \text{``2''} SE(\text{estimator})$$

In the case of β_1 we have:

$$\frac{b_1 - \beta_1}{s_e/\sqrt{S_{xx}}} \sim t_{n-2}$$

result in a 95% C.I. of:

$$b_1 \pm t^*_{n-2} \frac{s_e}{\sqrt{S_{xx}}}$$

example C.I.'s for β_1 - body fat versus weight and height

Since $n = 250$, our value of "2" is in this case: 1.9695757

```
##
## Coefficients:
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## Weight        0.18938    0.01533  12.357      < 2e-16


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A = all "model" | B = "typical" | C = all "error":
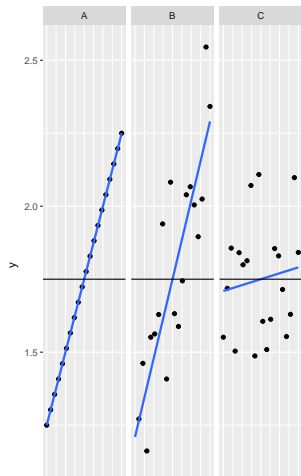
R^2 conceptual building blocks; a "sum-of-squares" decomposition

variation in the y = variation due to the model + variation due to error

$$\sum (y_i - \bar{y})^2 = \qquad\qquad +$$

R^2 conceptual building blocks; a "sum-of-squares" decomposition

variation in the y = variation due to the model + variation due to error

$$\sum (y_i - \overline{y})^2 = \sum (\hat{y}_i - \overline{y})^2 +$$

R^2 conceptual building blocks; a "sum-of-squares" decomposition

variation in the y $=$ variation due to the model $+$ variation due to error

$$\sum (y_i - \overline{y})^2 = \sum (\hat{y}_i - \overline{y})^2 + \sum (y_i - \hat{y}_i)^2$$

R^2 conceptual building blocks; a "sum-of-squares" decomposition

variation in the y = variation due to the model + variation due to error

$$\sum(y_i - \overline{y})^2 = \sum(\hat{y}_i - \overline{y})^2 + \sum(y_i - \hat{y}_i)^2$$
$$SS_{Total} = SS_{Regression} + SS_{Error}$$

sum of squares decomposition, graphically
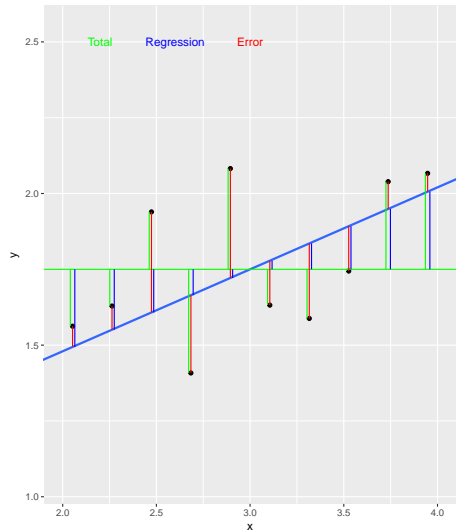
R^2 definition

$$R^2 = \frac{SS_{Regression}}{SS_{Total}} = 1 - \frac{SS_{Error}}{SS_{Total}}$$

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Although it is not a coefficient, and it does not really determine anything. It's just a mildly useful number.

Keep in mind it is *one number* that is being used to summarize an entire empirical bivariate relationship. And it isn't even the *best* number.
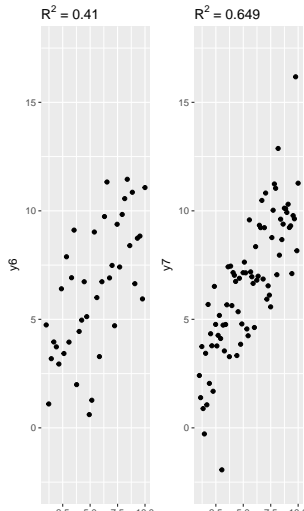
Some R^2 values
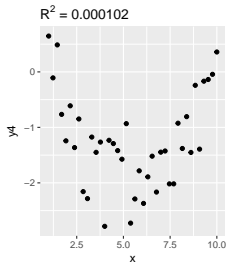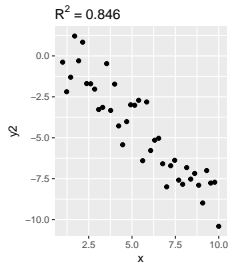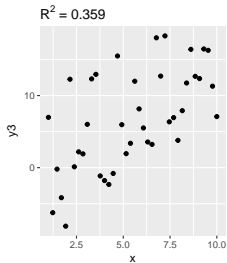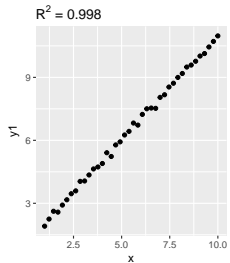
Another limitation: sample size effect

Both simulated datasets are from the **same underlying model**

(happens to be $y = 1 + 1 \cdot x + \varepsilon$ with $\varepsilon \sim N(0, 2)$)

regression model assumption (etc.) verification

recap model and calculation requirements

The model is:

$$y = \beta_0 + \beta_1 x + \varepsilon \text{ with } \varepsilon \sim N(0, \sigma)$$

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- Linear relationship between y and x.

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- Variation plus/minus the line is of constant magnitude.

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Also, the observations should be independent, but this is hard to verify (a plot of values versus time/order could be appropriate.)

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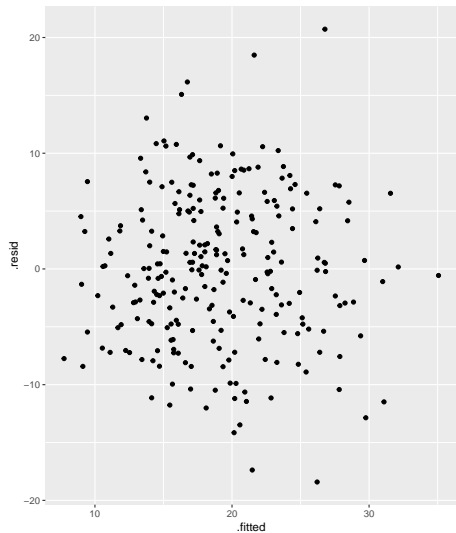- ▶ Error has a normal distribution.

Also, the observations should be independent, but this is hard to verify (a plot of values versus time/order could be appropriate.)

We will verify graphically, using various plots of the *residuals* $\hat{\varepsilon}_i = y_i - \hat{y}_i$

verify normality with normal quantile (or normal probability) plot of $\hat{\varepsilon}_i$
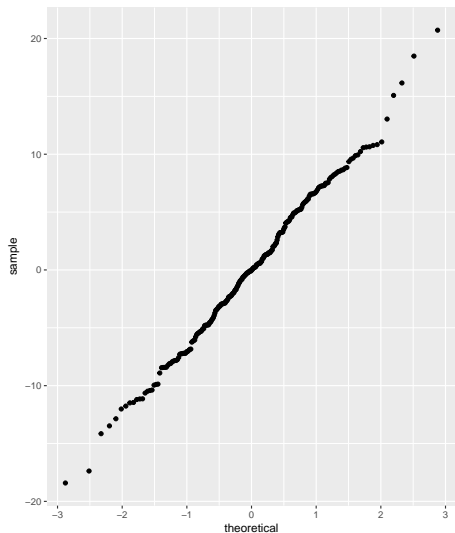
verify linearity with plot of $\hat{\varepsilon}_i$ versus $\hat{y}_i$

verify equal variance with (same!) plot of $\hat{\varepsilon}_i$ versus $\hat{y}_i$