STA221

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experiments with two factors

fire retardant example

There aren't many datasets that come with Chapter 26, so I found another to use as a motivating example.

In this dataset a new fire retardant treatment of cotton fabric is being tested. The name of this factor is `Bath`. There are two levels names `I` and `II`, which stand for "received the treatment" and "did not receive the new treatment" respectively.

(It's called "Bath" because the fabric is bathed in the treatment solution.)

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(It's called "Bath" because the fabric is bathed in the treatment solution.)

The amount of time it takes each cloth sample to start to burn is recorded.

Here is a numerical summary of the results:

Bath	n	Means	SD
I	24	16.875	5.921
II	24	9.154	5.670

but there is also another variable

The new treatment works. But there is also the matter of the efficiency with which the treatment can be applied.

There is another factor variable in this dataset: the number of “launderings”, which is the way a retardant treatment is applied.

This variable has two levels named 5 and 10, corresponding to the actual number of launderings.

Here is a summary of the results with respect to this variable:

Launderings	n	Means	SD
10	24	15.067	5.609
5	24	10.963	7.620

the effect of both variables

The new treatment is better than the old treatment (all else being equal). More laundering is better than less laundering (all else being equal).

But what about the combination of both variables? There are a few different ways they might combine to affect the results. We'll be concerned with these possibilities:

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A good graphical method to evaluate the relationship is called an *interaction plot*.

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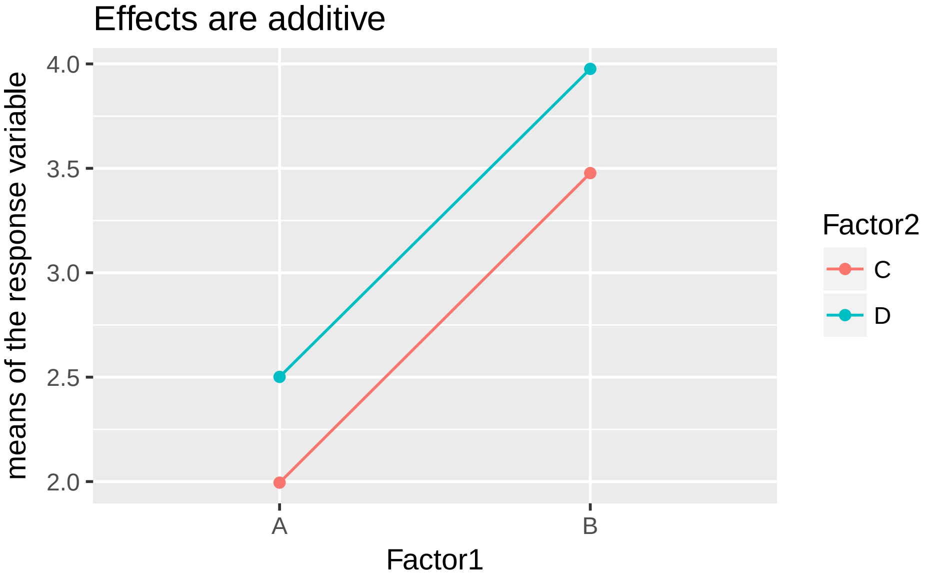
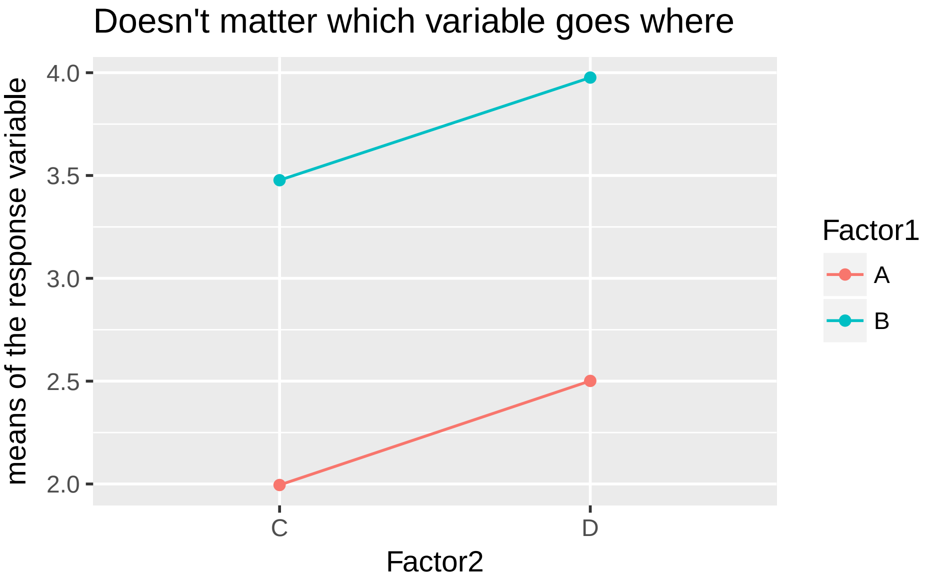
A good graphical method to evaluate the relationship is called an *interaction plot*.

To illustrate, I have simulated a dataset with two variables. `Factor1` has levels `A` and `B` while `Factor2` has levels `C` and `D`.

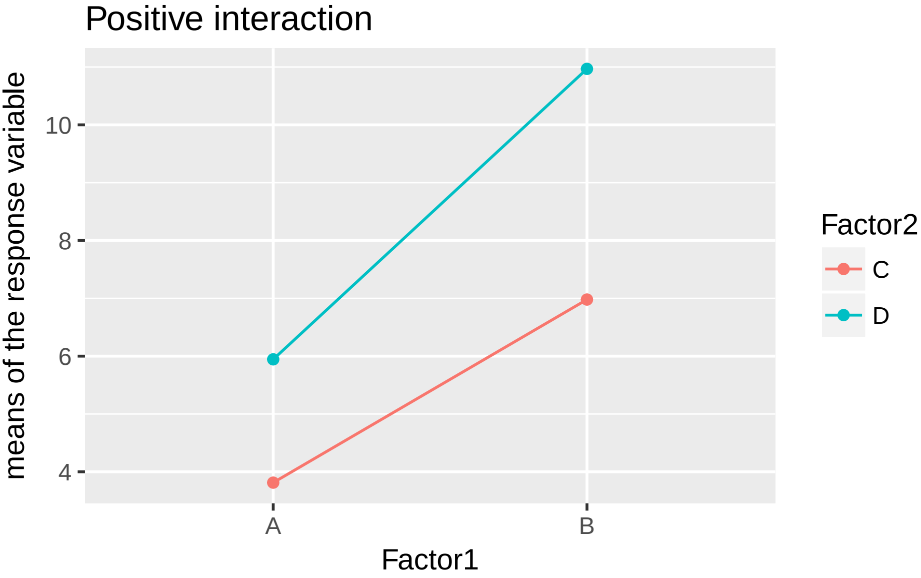
interaction plot example 1a - additive



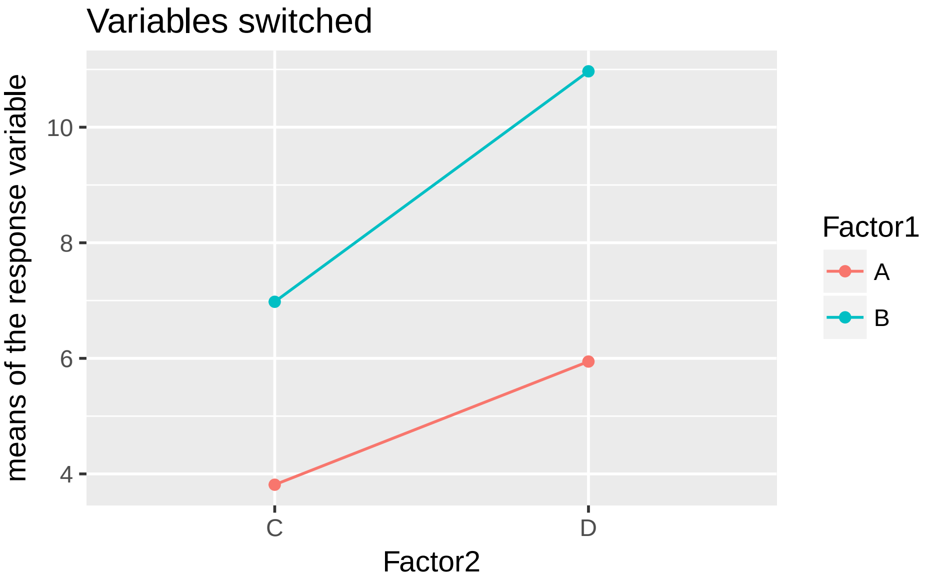
interaction plot example 1b - additive



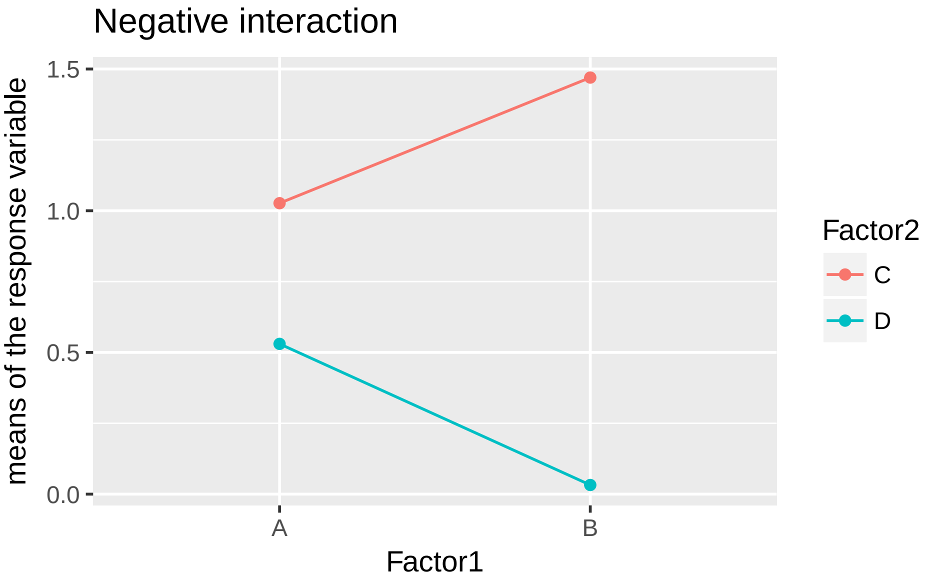
interaction plot example 2a - positive interaction



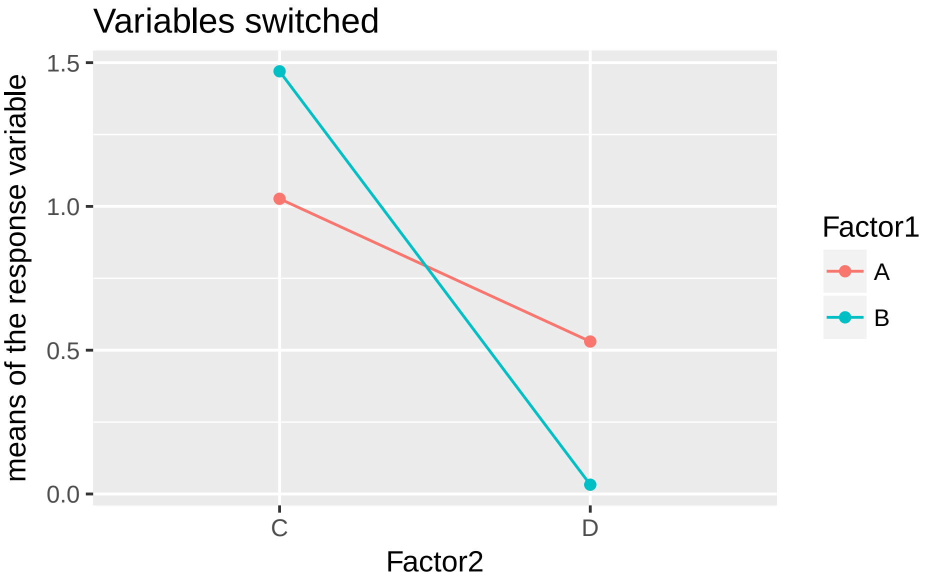
interaction plot example 2b - positive interaction



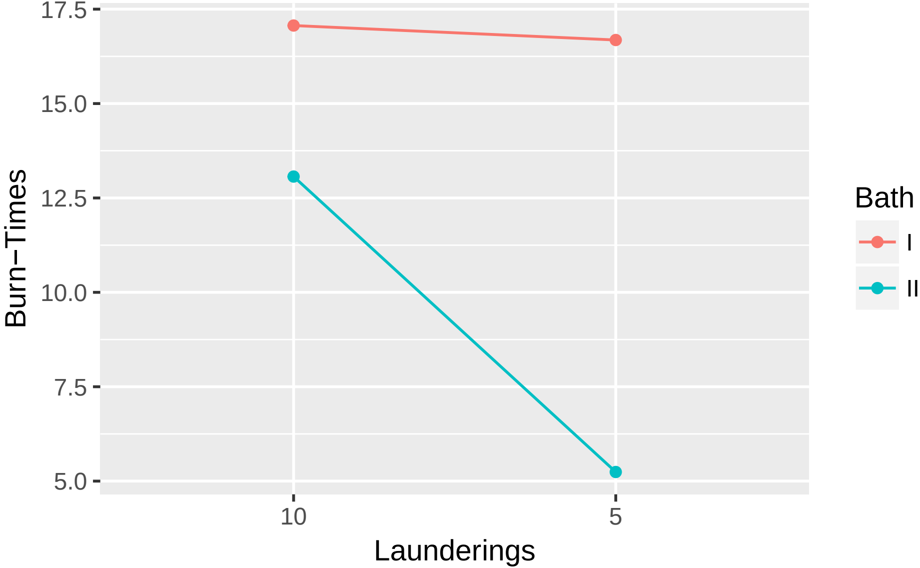
interaction plot example 3a - negative interaction



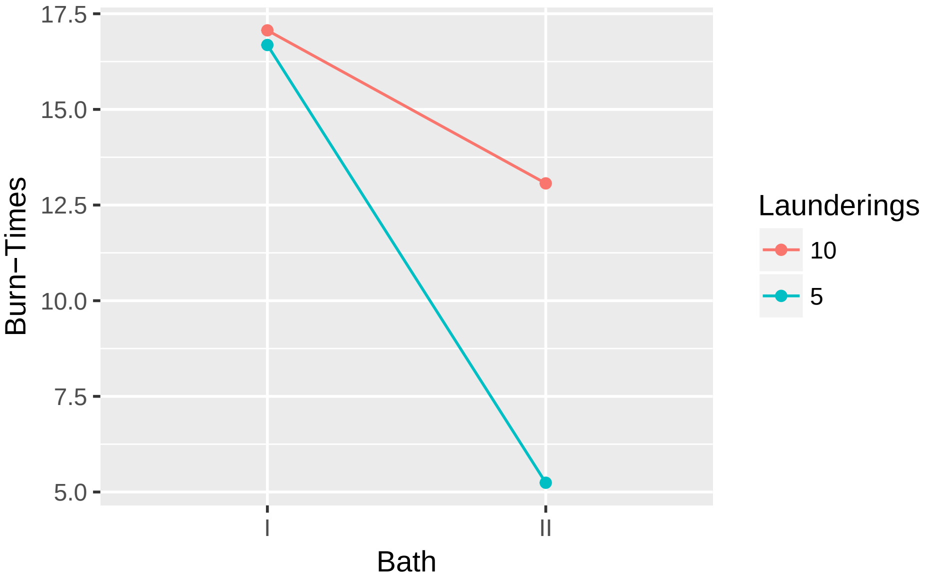
interaction plot example 3a - negative interaction



interaction plot of the fire retardant data



fire data - variables switched



tentative conclusion from fire data | new models to consider

It seems that the new treatment is just as good, even after only 5 launderings, so it is also more efficient.

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Here are the models we will consider:

$$y_{ijk} = \mu + \tau_i + \gamma_j + \varepsilon_{ijk} \qquad \text{no interaction assumed}$$

$$y_{ijk} = \mu + \tau_i + \gamma_j + (\tau\gamma)_{ij} + \varepsilon_{ijk} \qquad \text{general model}$$

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ε_{ijk} is random noise, assumed to be $N(0, \sigma)$.

the analysis

We have new sums of squares decompositions. The details can get brutal - so we'll stay symbolic.

We'll call the factor variables `A` and `B`, with I and J levels respectively.

In the additive case we get, assuming a common sample size of n for each combination of factor levels (strongly encouraged in practice!):

$$SS_{Total} = SS_A + SS_B + SS_{Error}$$

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Everything has χ^2 distributions. The degrees of freedom add up (N is the grand sample size):

$$N - 1 = (I - 1) + (J - 1) + (N - I - J + 1)$$

fire example - no interaction (?!)

```
##             Df Sum Sq Mean Sq F value    Pr(>F)
## Bath         1  715.3   715.3  23.954 0.0000131
## Launderings  1  202.1   202.1   6.769    0.0125
## Residuals   45 1343.8    29.9
```

sums of squares - a few details

The total sum of squares SS_{Total} is (as always) $(N-1)$ times the sample varianace of the response variable:

$$\sum_{i,j,k}\left(y_{ijk}-\overline{\overline{y}}\right)^2$$

The treatment sums of squares will be:

$$SS_A = nI\sum_i\left(\overline{y}_{i\cdot\cdot}-\overline{\overline{y}}\right)^2$$

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The key thing to note is that each treatment sum of squares is computed using the average over all levels of the other treatment. . .

. . . which only makes sense when there is no interaction.

error sum of squares

For the sake of completeness:

$$SS_{Error} = \sum_{i,j,k} \left(y_{ijk} - \overline{y}_{ij\cdot} \right)^2$$

Note that $y_{ijk} - \overline{y}_{ij\cdot}$ is also called a "residual".

model assumptions

Mostly the same as with one treatment factor, with the same verification techniques.

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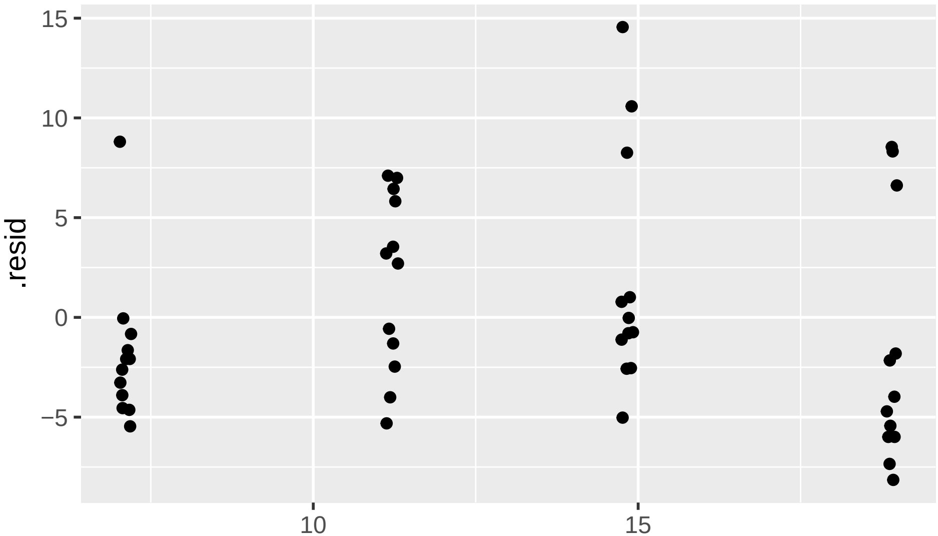
When $n = 1$, the lack of interaction is also an *assumption*.

fire retardant model assumptions - equal variance

Plot of residuals versus "fitted values" (in this case, just the group averages):

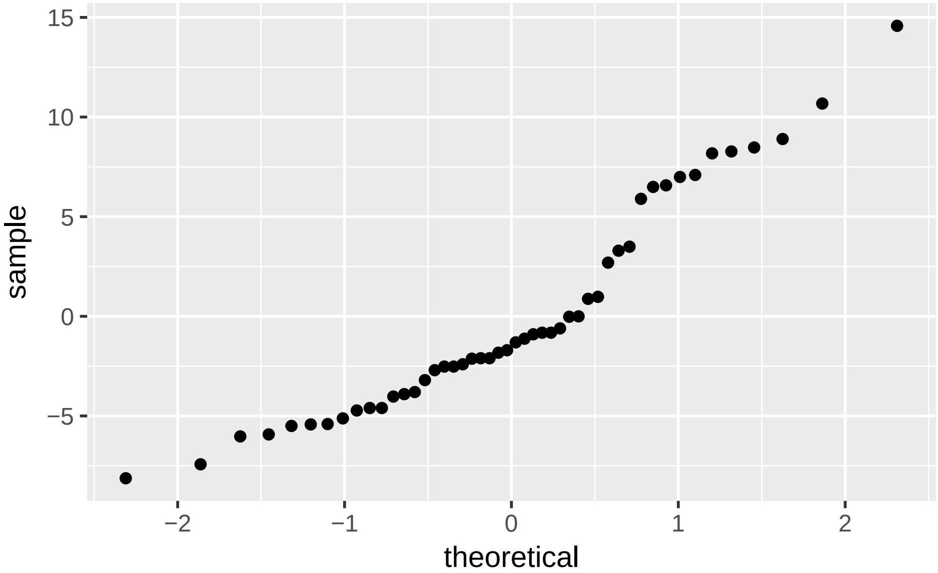


fire retardant model assumptions - equal variance

Since $n = 12$ Levene's test also works:

```
## Levene's Test for Homogeneity of Variance (center = median)
##       Df F value Pr(>F)
## group  3  0.7138  0.549
##       44
```

fire retardant model assumptions - normality



the general model and analysis (with interaction)

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The sum of squares decomposition is now:

$$SS_{Total} = SS_A + SS_B + SS_{AB} + SS_{Error}$$

The interaction sum of squares is:

$$n \sum_{i,j} \left(y_{ijk} - \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot j \cdot} + \overline{\overline{y}} \right)^2$$

Small when additive; large when not.

degrees of freedom - balanced case

$$
\begin{aligned}
SS_{Total} &= SS_A + SS_B + SS_{AB} + SS_{Error} \\
(N-1) &= (I-1) + (J-1) + (I-1)(J-1) + IJ(n-1)
\end{aligned}
$$

Note: $IJ(n-1) = N - IJ$

We get (in addition):

$$\frac{SS_{AB}/(I-1)(J-1)}{SS_{Error}/IJ(n-1)} \sim F_{(I-1)(J-1),IJ(n-1)}$$

If there is evidence for interaction, do not try to interpret the “main effects”.

flame retardant with interaction

Flame retardant example, with interaction (and without):

```
##                  Df Sum Sq Mean Sq F value     Pr(>F)
## Bath              1  715.3   715.3  26.726 0.00000549
## Launderings       1  202.1   202.1   7.552    0.00866
## Bath:Launderings  1  166.1   166.1   6.207    0.01657
## Residuals        44 1177.7    26.8
```

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##             Df Sum Sq Mean Sq F value    Pr(>F)
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Also, the ANOVA table wouldn't work!

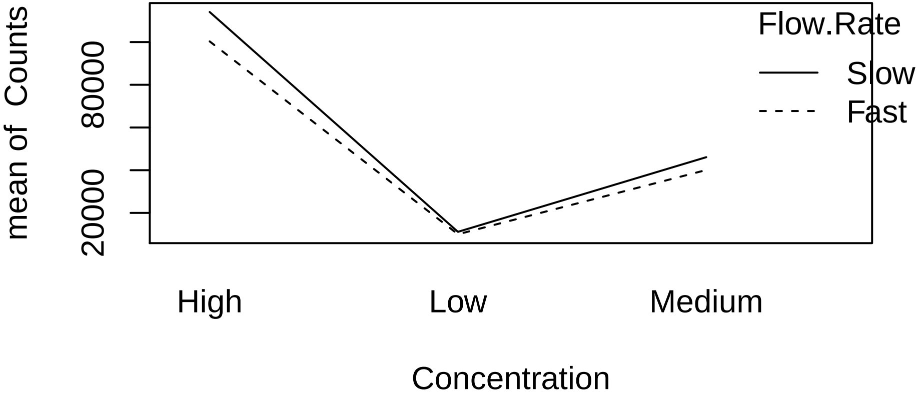
The technique is to use SS_{AB} as the error sum of squares.

another overall example

Chromatography example.

Two factors: flow rate (fast and slow); Concentration (low, med, high)

interaction plot



analysis

```
##                              Df      Sum Sq     Mean Sq  F value
## Flow.Rate                     1   364008333   364008333   29.645
## Concentration                 2 48365460080 24182730040 1969.424
## Flow.Rate:Concentration       2   203032027   101516013    8.267
## Residuals                    24   294698040    12279085
##                                            Pr(>F)
## Flow.Rate                               0.0000135
## Concentration           < 0.0000000000000002
## Flow.Rate:Concentration                   0.00186
## Residuals
```

assumptions

```
## Levene's Test for Homogeneity of Variance (center = median)
##       Df F value Pr(>F)
## group  5  1.5612 0.2089
##       24
```

assumptions