STA221

Neil Montgomery

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multiple regression

regression with more than one input variable

The Universal Statistical Model:

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The most important stastical model (in my opinion) is the linear regression model with more than one "x" variable. For example, with 3 input variables:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon$$

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We treat y as random. The inputs are not random. They can be whatever you like, even functions of one another, with one technical limitation*.

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So, for example, the following is a valid multiple regression model:

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This kind of "polynomial" model is good for fitting some types of non-linear relationships between y and a single x.

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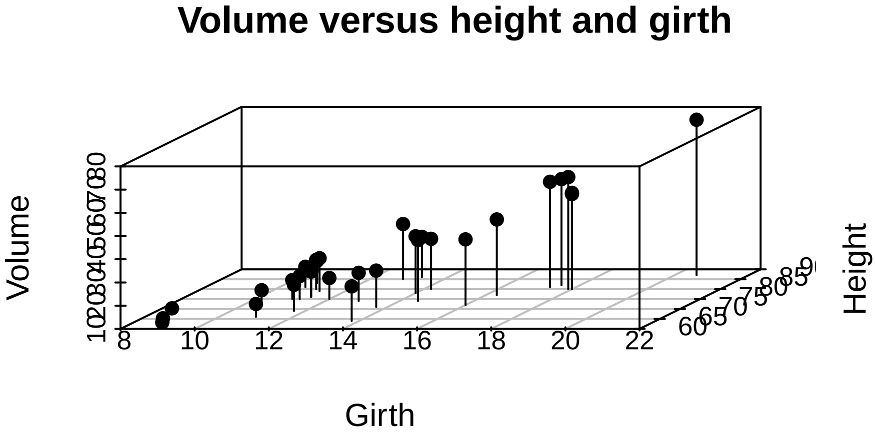
*A variable cannot be a linear function of other variables in the model.

what is being accomplished in multiple regression?

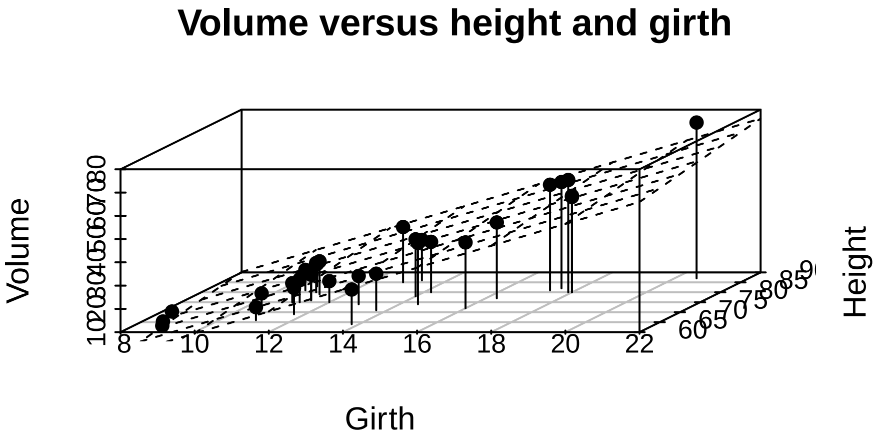
R comes with some sample datasets. One is called `trees` and has variables Girth, Height, and Volume. Here's a peek at the data:

```
## # A tibble: 31 × 3
##   Girth Height Volume
##   <dbl>  <dbl>  <dbl>
## 1   8.3     70   10.3
## 2   8.6     65   10.3
## 3   8.8     63   10.2
## 4  10.5     72   16.4
## 5  10.7     81   18.8
## # ... with 26 more rows
```

what is being accomplished in multiple regression?



multiple regression fits a surface to the points



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  - Parameter testing and estimation

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  - Model selection: which variables?
  - "Multicollinearity" (highly correlated inputs)

parameter interpretation

The multiple regression model:

$$y = \beta_0 + \beta_1 x_1 + \ldots \beta_k x_k + \varepsilon, \quad \varepsilon \sim N(0, \sigma)$$

has many parameters.

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The β_i from $i \in \{1, \ldots, k\}$ are the slope parameters, and have a different interpretation than before.

slope parameter interpretation

β_i is:

- the change in y

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That bold, italic statement should echo in your mind any time you think of anything to do with β_i.

trees example

We might want to model $y =$`Volume` (the amount of wood) as a linear model of the input variables $x_1 =$`Girth` and $x_2 =$`Height`, as follows:

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The computer uses the method of "least squares", like before. A full treatment of the analysis requires matrix algebra.

fitted values | residuals

Here's the first row of the `trees` data:

Girth	Height	Volume
8.3	70	10.3

We could call these values y_1, x_{11}, and x_{21}

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For a dataset with n rows (the sample size), there is a fitted value and residual for each row.

trees data fitted model

Here's what R produces:

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -57.9877     8.6382  -6.713 2.75e-07
## Girth         4.7082     0.2643  17.816  < 2e-16
## Height        0.3393     0.1302   2.607   0.0145
##
## Residual standard error: 3.882 on 28 degrees of freedom
## Multiple R-squared:  0.948,  Adjusted R-squared:  0.9442
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individual slope parameter hypothesis testing

The usual hypothesis test for a single parameter:

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given all the other x's in the model

the overall hypothesis test

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Null hypothesis can be expressed as:

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It is also possible to test any subset of these parameters, such as:

$$H_0 : \beta_1 = \beta_2 = 0$$

although at the moment it's not clear why this might be a good idea.

estimating σ

This works the same as with simple regression, in which we used $\sqrt{MSE}$ where:

$$MSE = \frac{\sum_{j=1}^{n} (y_j - \hat{y}_j)^2}{n-2}$$

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There was only one input variable, so another way to think of this was "sample size minus the number of input variables, then minus 1."

estimating σ

In multiple regression, nothing changes. Use $\sqrt{MSE}$, where:

$$MSE = \frac{\sum_{j=1}^{n} (y_j - \hat{y}_j)^2}{n - (k+1)}$$

hypothesis testing for β_i

The computer produces the estimate b_i, which has these properties:

$$E(b_i) = \beta_i$$
$$\mathrm{Var}(b_i) = \sigma \cdot c_i$$

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Just like before, we get:

$$\frac{b_i - \beta_i}{\sqrt{MSE}\sqrt{c_i}} \sim t_{n-k+1}$$

hypothesis testing for β_i in the trees example

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the overall F test

"Is there any linear relationship between y and the input variables?"

Based on the same, original SS decomposition.

$$\text{variation in the } y \;=\; \text{variation due to the model} \;+\; \text{variation due to error}$$

$$\sum (y_i - \overline{y})^2 = \qquad\qquad + $$

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The p-value then comes from **CORRECTED 2017-04-08**:

$$\frac{SS_{Regression}/k}{SS_{Error}/(n-k-1)}=\frac{MSR}{MSE}\sim F_{k,n-k-1}$$

the overall F test - trees example

The information is in the usual R output:

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##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
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One can obtain an "ANOVA" table from this information:

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
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	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Regression	2	7684.16	3842.08	254.97	1.07×10^{-18}
Error	28	421.92	15.07		

model assumptions and calculation requirements

Model:

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Pretty much the same as with simple regression.

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First, there's the independence assumption, which can't really be verified without knowledge of the data collection itself (common violation - repeated measures.)

The main ones to worry about are:

1. The linear model is appropriate (fatal if violated).

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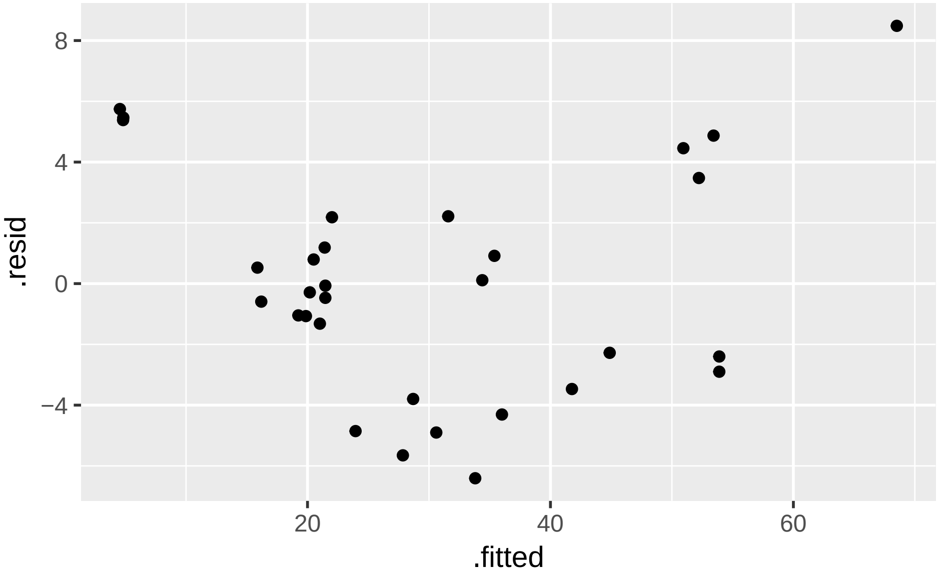
1\. and 2. are verified with a plot of residuals versus fitted values, and 3. is verified with a normal quantile plot of the residuals.

residuals versus fitted values - trees example (fatal)



not surprising, since the model was obviously wrong

If you really wanted to model the $y =$`Volume` of wood using $x_1 =$`Girth` and $x_2 =$`Height`, you need to include the square of `Girth`, because of the volume-of-a-cylinder formula $V = \pi r^2 h$.

So let's fit the model:

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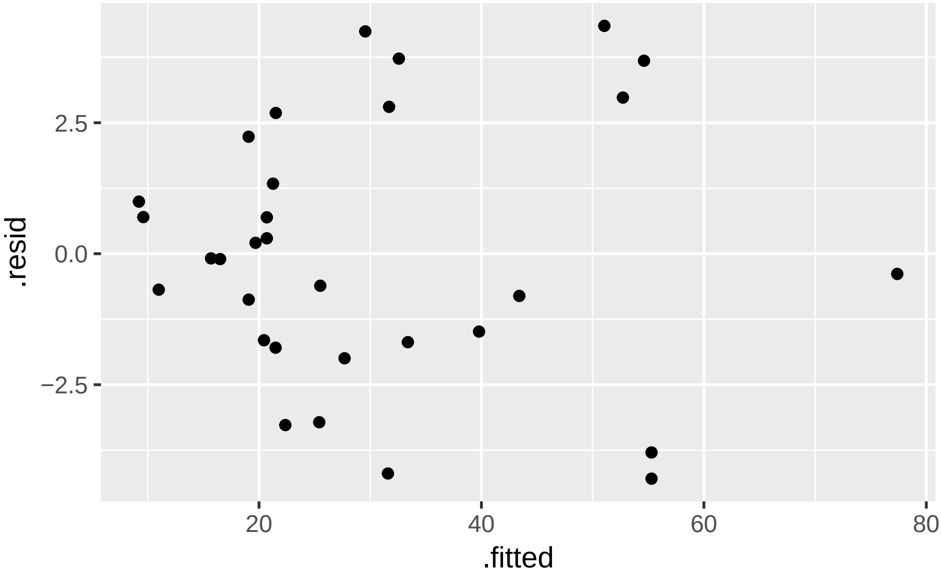
   the original β's are at least conceptually similar.
2. When adding squares of variables (etc.), usually best to keep the original in the model as well.

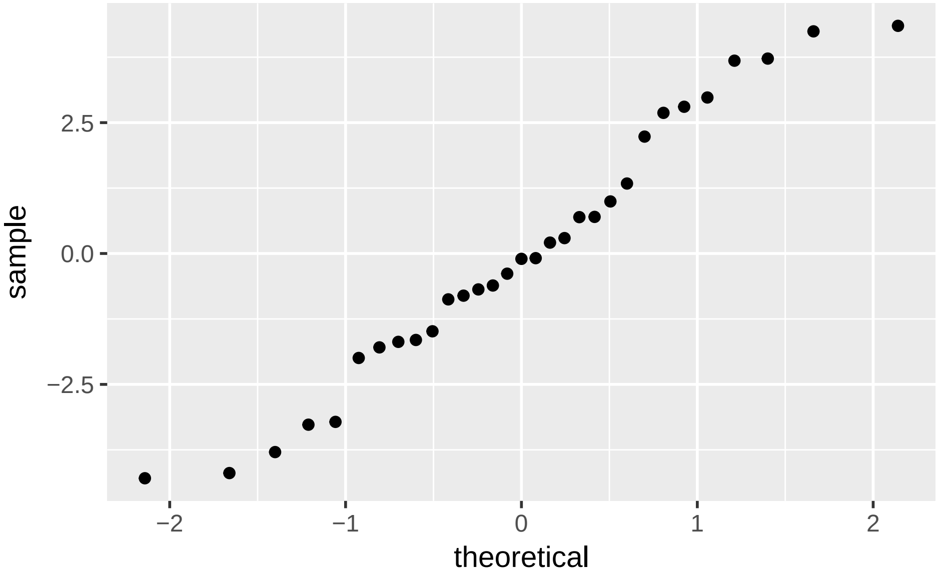
new trees model fit

```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)   -9.9204    10.0791   -0.98  0.33373
## Girth         -2.8851     1.3099   -2.20  0.03634
## I(Girth^2)     0.2686     0.0459    5.85  3.1e-06
## Height         0.3764     0.0882    4.27  0.00022
##
## Residual standard error: 2.6 on 27 degrees of freedom
## Multiple R-squared:  0.977,  Adjusted R-squared:  0.975
## F-statistic:  383 on 3 and 27 DF,  p-value: <2e-16
```

new trees model resids v. fits



normal quantile plot of residuals



towards an "adjusted" R^2

R^2 comes from dividing SS_{Total} through the SS decomposition:

$$SS_{Total} = SS_{Regression} + SS_{Error}$$

The definition $R^2 = SSR/SST = 1 - SSE/SST$ is the same no matter how many input variables there are.

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For example, I can add a pure nonsense x_4 variable to the trees data and fit the "bigger" model.

trees vs. trees plus nonsense

The last best model we had:

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```

With a `Nonsense` (randomly generated) variable added:

```
##
## Coefficients:
```

adjusting R^2 for the number of input variables

A more fair (but still not perfect) single-number-summary of a multiple regression fit is:

$$R^2_{adj} = 1 - \frac{MS_{Error}}{MS_{Total}}$$

where MS_{Total} is just another name for the sample variance of the output y values:

$$MS_{Total} = \frac{SS_{Total}}{n-1} = \frac{\sum\limits_{i=1}^{n} (y_i - \overline{y})^2}{n-1}$$