STA221

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Last edited: 2017-01-16 14:00

goodness-of-fit testing ("Comparing Counts", Ch. 23 of textbook)

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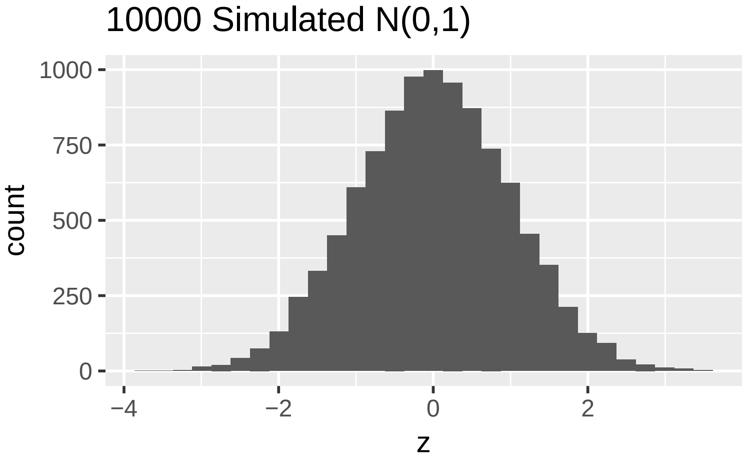
Why? Recall that a Binomial probability model is used to *count* the number of "*successes*" in n "trials".

Let's map "success" to the number 1 and "failure" to the number 0.

Counting 1s in a sequence of 0s and 1s is exactly equivalent to adding up all the 0s and 1s

detour 2.1 - what happens when you look at the square of a normal?

My compute can simulate random "draws" from a standard normal (N(0,1)) distribution, resulting in a histogram such as:



detour 2.2 - what what happens when you look at the square of a normal?

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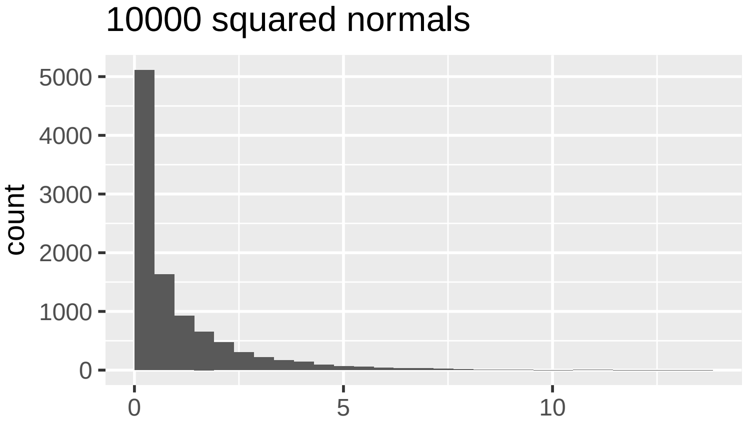

detour 2.3 - sum of squared normals?

I can simulate *two* columns of standard normals, square them *both*, add the results, and make a histogram of the result:

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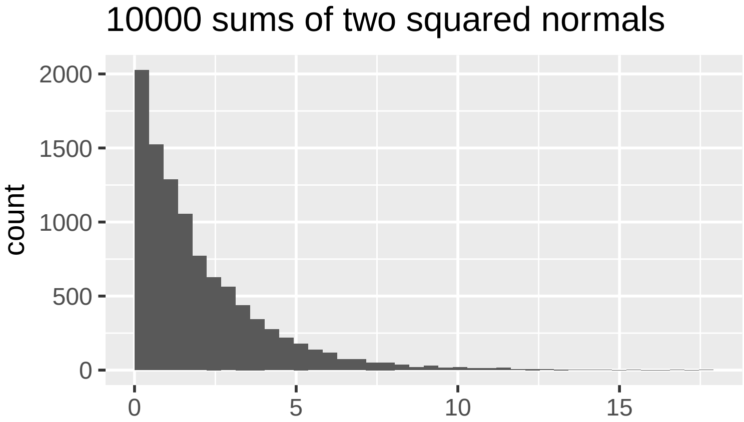
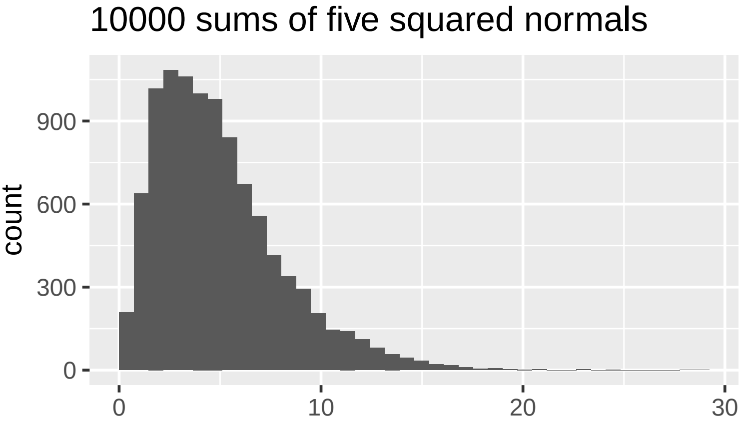


detour 2.4 - sum of many squared normals?

I can make several columns of normals, square them, add them up, and make a histogram. Here's the histogram with 5 columns of normals:



detour - the χ^2 family of distributions

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If you have n general $N(\mu, \sigma)$, say called $X_1, X_2, \ldots, X_n$, you could *standardize them*:

$$Z_i = \frac{X_i - \mu}{\sigma}$$

and then the sums of the squares of these Z_i will have a χ^2_n distribution.

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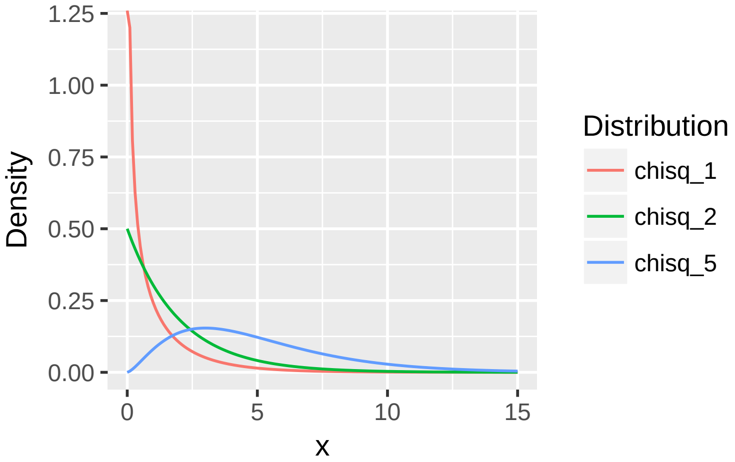
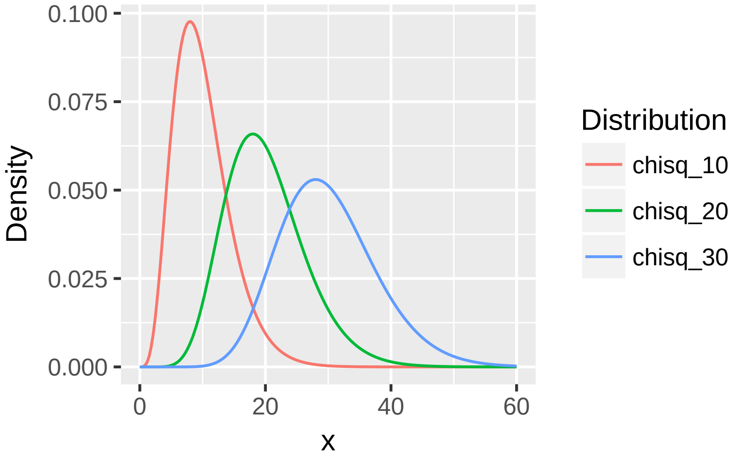
detour - pictures of some χ_n^2 distributions



detour - pictures of more χ^2_n distributions



Note: the average of a χ^2_n distribution is just n.

ever wonder why the sample variance is divided by $n-1$?

Look at the formula for sample variance:

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}$$

The numerator is a sum of n squares, but the denominator is $n-1$. Why?

pictures of $\sum_{i=1}^{5}(x_i - \overline{x})^2$

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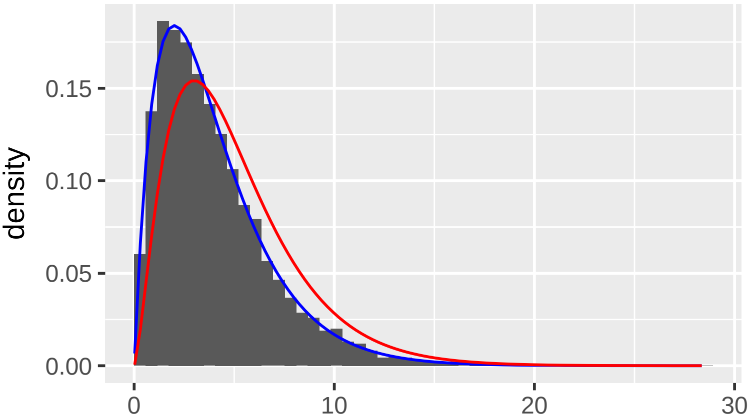
Here it is, with the χ^2_4 distribution in blue and the χ^2_5 in red:



a heuristic explanation

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So given $\overline{x}$ and *any* $n-1$ of the n raw values, I can calculate that other raw value.

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So given $\overline{x}$ and *any* $n-1$ of the n raw values, I can calculate that other raw value.

We say s^2 (given $\overline{x}$) only has $n-1$ degrees of freedom.

is there evidence that something doesn't follow a given distribution?

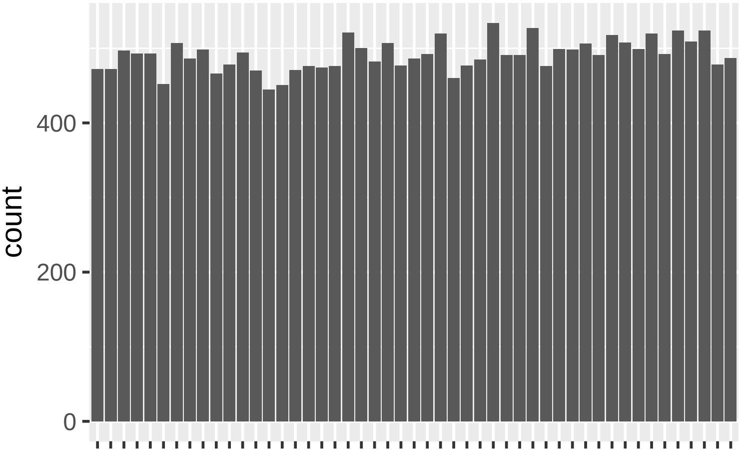
is a lottery “fair”

Lotto 6/49 is a Canadian lottery in which 49 identical balls are mixed together and 7 are selected, now twice per week. People can win money based on how many of the numbers they have out of the 6 on their ticket.

I found a list of every number ever picked here:
`http://portalseven.com/lottery/canada_lotto_649.jsp`

```
## # A tibble: 3,437 × 8
##                 date  num1  num2  num3  num4  num5  num6 bonus
##                <chr> <dbl> <dbl> <dbl> <dbl> <dbl> <dbl> <dbl>
## 1 Sat, Jan 14, 2017     1     6    19    30    32    44    33
## 2 Wed, Jan 11, 2017    24    34    36    38    42    43    30
## 3  Sat, Jan 7, 2017     1    10    18    19    23    27    48
## 4  Wed, Jan 4, 2017     2    11    13    23    35    48    30
## 5 Sat, Dec 31, 2016     3     5    14    18    26    28    40
## # ... with 3,432 more rows
```

all 49 numbers should appear with roughly the same frequency



categorical data, cells, observed cell counts

The dataset (now) consists of one variable called `numbers`. This is a *categorical*, or *factor* variable with 49 possible *levels*. There are 24050 observations.

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A categorical variable is summarized by producing a table of *observed cell counts* (notation: O_i). In this case:

```
## # A tibble: 49 × 2
##   numbers   O_i
##    <fctr> <int>
## 1       1   472
## 2       2   472
## 3       3   497
## 4       4   493
## 5       5   493
## # ... with 44 more rows
```

expected cell counts

If Lotto $6/49$ is actually fair, each number would appear with probability $1/49 = 0.0204$ each.

After 24050 numbers have been selected, we would expect to see:

$$24050 \cdot \frac{1}{49} = 490.82$$

of each number.

These are called *expected cell counts* — calulated under the assumption of fairness as defined in this example. (Notation: E_i)

measuring the deviation from the assumption of fairness

Each O_i is a count (i.e. a sum of 0s and 1s), which will have an approximate normal distribution. It turns out:

$$\frac{O_i - E_i}{\sqrt{E_i}}$$

has a standard normal distribution, as long as there are enough 1s in the sample.

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The overall deviation is measured as:

$$\sum_{i=1}^{n} \left(\frac{O_i - E_i}{\sqrt{E_i}} \right)^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

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We say

$$\sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

has $n-1$ degrees of freedom, and it follows (approximately) a χ^2_{n-1} distribution.

let's measure the deviation

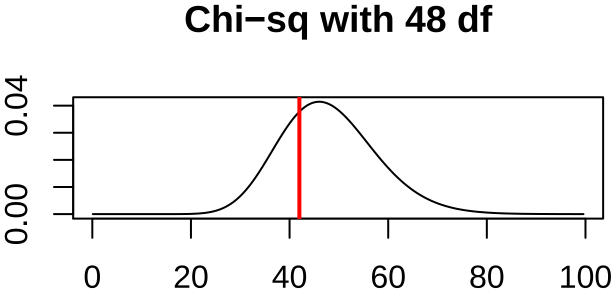
Here are the first few deviations (with $(O_i - E_i)^2/E_i$ called `D_i` for short):

```
## # A tibble: 49 × 4
##   numbers   O_i    E_i         D_i
##    <fctr> <int>  <dbl>       <dbl>
## 1       1   472 490.82 0.721634000
## 2       2   472 490.82 0.721634000
## 3       3   497 490.82 0.077813455
## 4       4   493 490.82 0.009682572
## 5       5   493 490.82 0.009682572
## # ... with 44 more rows
```

The sum of the `D_i` column is 41.99. Is this number surprising?

surprising, compared to what?

We know we should compare this number with the χ^2_{48} distribution. Here we can see we are not surprised. There is no evidence that Lotto 6/49 is unfair.



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In the Lotto example, technically this statement is:

$$H_0 : p_1 = p_2 = \cdots = p_{49} = \frac{1}{49}$$

But usually we just make H_0 a simple written statement:

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The alternative hypothesis is the negation of the null. We don't normally bother to write it down.

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Compute the *observed value of the test statistic*:

$$\chi^2_{\text{obs}} = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = 41.99$$

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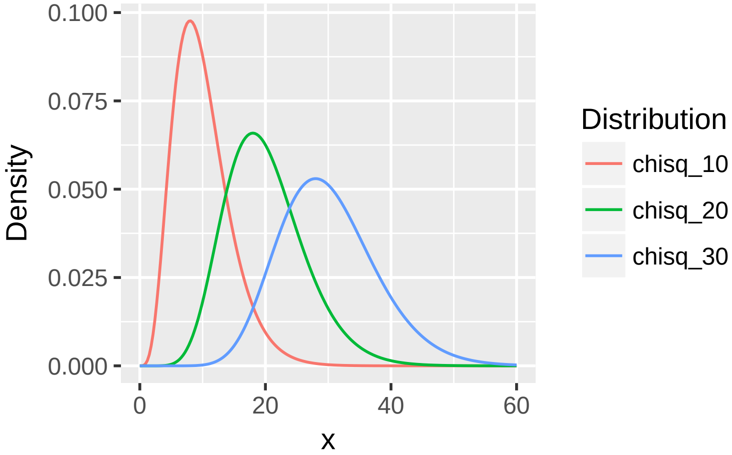
Calculate the p-value based on χ^2_{obs} being approximately χ^2_{n-1}.

goodness-if-fit testing p-value

A p-value is the *probability of observing a more extreme value*, in the sense of being further from where the null hypothesis “lives”, which is where in this case?



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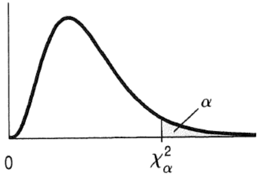
On tests you'll need to use a table. Here's a close-up of the book's table:

Table X

Values of χ^2_α

Right-tail probability / df	0.10	0.05	0.025	0.01	0.005
1	2.706	3.841	5.024	6.635	7.879
2	4.605	5.991	7.378	9.210	10.597
3	6.251	7.815	9.348	11.345	12.838
4	7.779	9.488	11.143	13.277	14.860
5	9.236	11.070	12.833	15.086	16.750
6	10.645	12.592	14.449	16.812	18.548
7	12.017	14.067	16.013	18.475	20.278
8	13.362	15.507	17.535	20.090	21.955
9	14.684	16.919	19.023	21.666	23.589
10	15.987	18.307	20.483	23.209	25.188
11	17.275	19.675	21.920	24.725	26.757
12	18.549	21.026	23.337	26.217	28.300
13	19.812	22.362	24.736	27.688	29.819



goodness-of-fit testing p-value (from table)

28	37.916	41.337	44.461	48.278	50.994
29	39.087	42.557	45.722	59.588	52.336
30	40.256	43.773	46.979	50.892	53.672
40	51.805	55.759	59.342	63.691	66.767
50	63.167	67.505	71.420	76.154	79.490
60	74.397	79.082	83.298	88.381	91.955
70	85.527	90.531	95.023	100.424	104.213

From a table the best you can do is to estimate the p-value.

All this together is called the "χ^2 goodness-of-fit test."

applications of χ^2 goodness-of-fit testing to two-way tables

contingency tables

Recall the gas pipelines data:

```
## # A tibble: 1,000 × 4
##      Leak  Size Material Pressure
##    <fctr> <ord>   <fctr>   <fctr>
## 1      No  1.75  Aldyl A     High
## 2      No  1.75  Aldyl A      Med
## 3      No     1  Aldyl A      Low
## 4     Yes   1.5    Steel      Med
## 5      No     1    Steel     High
## # ... with 995 more rows
```

The (only?) suitable numerical summary for two categorical/factor variables at a time is a so-called contingency table, or two-way table.

two-way table for “Leak” and “Pressure”

	High	Low	Med	Sum
No	277	278	247	802
Yes	71	66	61	198
Sum	348	344	308	1000

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This question is answered using a *test of homogeneity*.

Question 2: are the rows and columns *independent*?

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The mechanics of both tests are identical. Only the interpretation is (slightly) different.

two-way table again

Count version:

	High	Low	Med	Sum
No	277	278	247	802
Yes	71	66	61	198
Sum	348	344	308	1000

Proportion version:

	High	Low	Med	Sum
No	0.277	0.278	0.247	0.802
Yes	0.071	0.066	0.061	0.198
Sum	0.348	0.344	0.308	1.000