STA221

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Last edited: 2017-02-01 15:06

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- the error ε is random
- therefore, y is random (as the sum of a fixed part and a random part)

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- their means and variances

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  - "time series" methods are one way to deal with one type of non-independence.

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* with one exception TBA

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First, we'll look at the average value of b_0, using simulation. To do this I will start with a *fully known theoretical linear model*:

$$y = 2 + 0.75x + \varepsilon$$

with $\varepsilon \sim N(0, 1)$.

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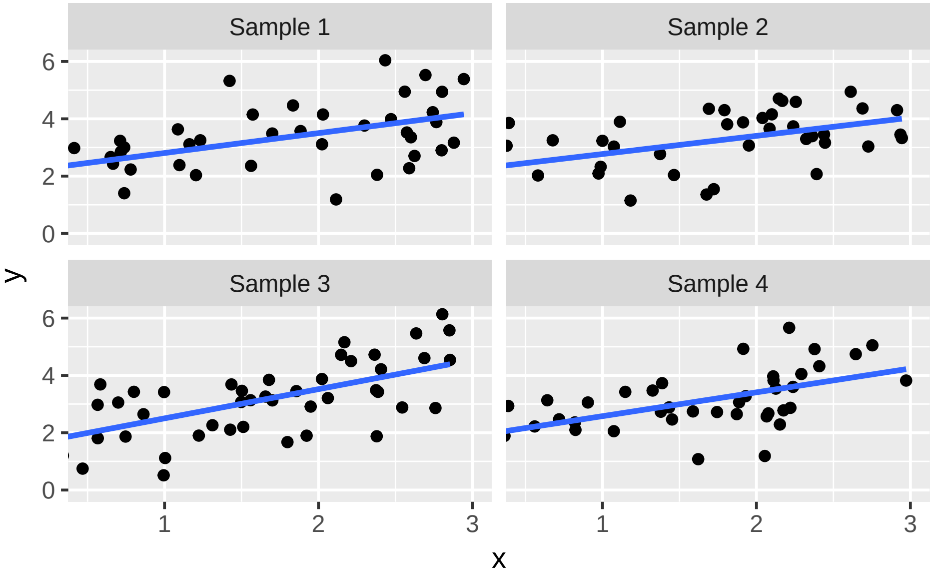
I will simulate fake datasets of size $n = 50$ from this model, compute the regression line for each dataset, and see what happens.

e.g. plots of four samples



properties of b_1 from 1000 samples

I would like to investigate the distribution of b_1 using simulation. So I will simulate 1000 replications, and see what happens.

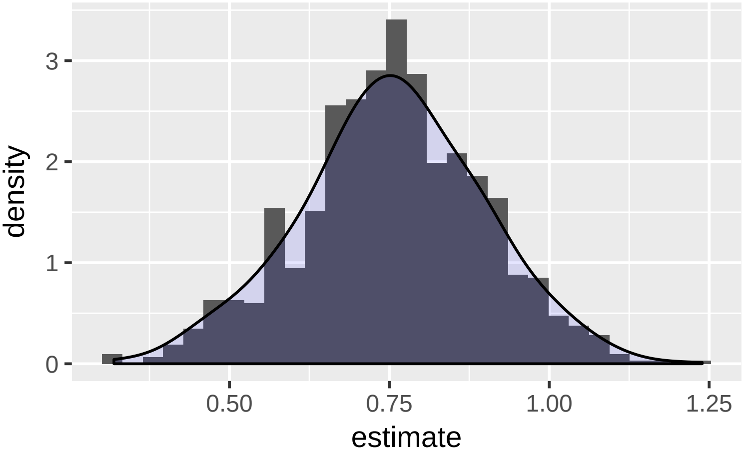
Here is a numerical summary of the 1000 simulated b_1 (and b_0 as well, since I have them):

term	Average	SD
(Intercept)	1.99499	0.22788
x	0.75465	0.14225

(Note: these numbers *change* every time I render the lecture notes - the simulation is embedded right in them.)

Conclusion: the average values of b_1 (and b_0) are the true values β_1 (and β_0).

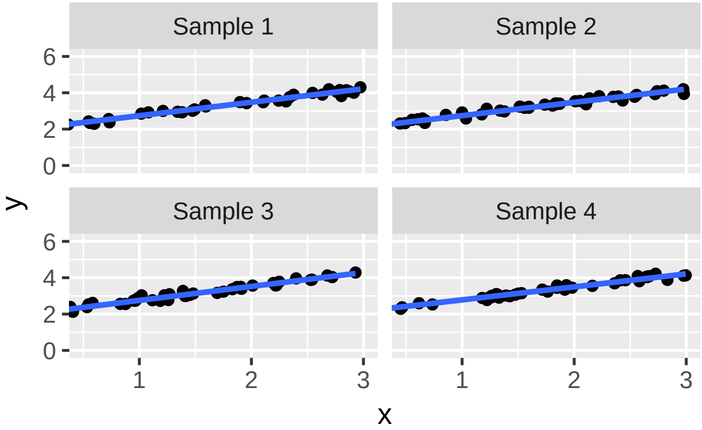
histogram of the simulated b_1



Looks symmetric and bell-shaped. Perhaps they have a normal distribution?

change σ from 1 to 0.1

I will simulate again, but this time with $\varepsilon \sim N(0, 0.1)$. Four example plots:



properties of b_1 from 1000 samples ($\sigma = 0.1$ version)

The averages and SDs of the 1000 estimators:

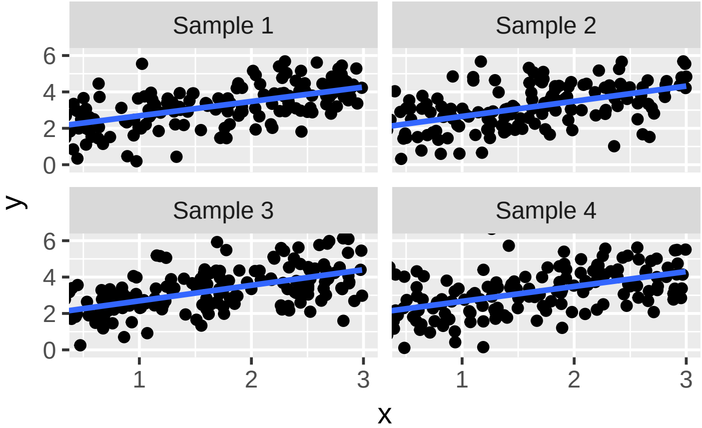
term	Average	SD
(Intercept)	1.99950	0.02333
x	0.75039	0.01432

The histogram looks the same.

Conclusion: when the *inherent underlying noise is smaller* the parameter estimators are *more accurate*.

put σ back to 1; increase the sample size to $n = 200$

Four sample plots:



properties of b_1 from 1000 samples ($n = 200$ version)

The averages and SDs of the 1000 estimators:

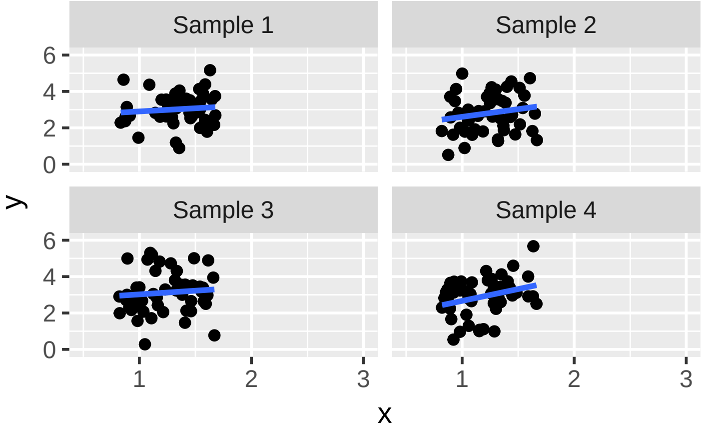
term	Average	SD
(Intercept)	2.00363	0.11522
x	0.74960	0.06986

The histogram looks the same.

Conclusion: when the *sample size is larger* the parameter estimators are *more accurate*.

back to $n = 50$; properties of b_1 when the x values are less spread out

This one is a little more subtle. It turns out the x values affect the accuracy of the parameter estimates. I re-simulate with less spread in the x values. Four sample plots with x values 4 times “less spread out”:



properties of b_1 (x less spread version)

The averages and SDs of the 1000 estimators:

term	Average	SD
(Intercept)	2.01478	0.73753
x	0.73827	0.57449

The histogram looks the same.

Conclusion: when the x *values are less spread out* the parameter estimators are *less accurate*.

statistical properties of b_1

Start with the basic simple linear regression model:

$$y = \beta_0 + \beta_1 x + \varepsilon$$

in which the error follows a $N(0, \sigma)$ distribution.

The slope estimator b_1 turns out to follow a normal distribtion with mean β_1 and standard deviation:

$$\frac{\sigma}{\sqrt{S_{xx}}}$$

(Recall $S_{xx} = \sum(x_i - \overline{x})^2$)

(Note: there is a typo on the first formula in section 24.2 - the s_x should not be under the $\sqrt{}$.)

statistical properties of b_1

Therefore we have:

$$\frac{b_1 - \beta_1}{\sigma/\sqrt{S_{xx}}} \sim N(0,1)$$

and p-values and confidence intervals come from this—BAM we're done.

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Except we would never know the true value of σ. This is the third simple regression parameter—a nuisance we'll have to deal with.

We can estimate σ using the "average" of the squared residuals:

$$s_e = \frac{\sum (y_i - \hat{y}_i)^2}{n-2}$$

statistical properties of b_1

Who wants to guess what distribution this will have:

$$\frac{b_1 - \beta_1}{s_e/\sqrt{S_{xx}}} \sim ???$$

hypothesis testing for β_1

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$$H_0 : \beta_1 = 0$$

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$$H_0 : \beta_1 = 0$$

And it works the same way any other hypothesis test works. Use the data to compute:

$$\frac{b_1 - 0}{s_e/\sqrt{S_{xx}}}$$

and get the probability of being "further away" from H_0, according to the ??? distribution.

example - body fat versus weight