STA221

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statistical properties of b_1

Start with the basic simple linear regression model:

$$y = \beta_0 + \beta_1 x + \varepsilon$$

in which the error follows a $N(0, \sigma)$ distribution.

The slope estimator b_1 turns out to follow a normal distribtion with mean β_1 and standard deviation:

$$\frac{\sigma}{\sqrt{S_{xx}}}$$

(Recall $S_{xx} = \sum (x_i - \bar{x})^2$)

(Note: there is a typo on the first formula in section 24.2 - the s_x should not be under the $\sqrt{}$.)

relating various formulae to the simulation results

When I simulated thousands of b_1 from datasets with a variety of properties, we saw the following:

- histograms of simulated b_1 were symmetric and bell shaped. In fact, normal! Why? Let's look at the formula for b_1:

$$b_1 = \frac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sum(x_i - \overline{x})^2} = \sum d_i(y_i - \overline{y})$$

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  where the d_i are constants.
- Less variation when the underlying σ was smaller.
- Less variation when the x's were more spread out.
- Less variation when the sample size was larger.

statistical properties of b_1

We have:

$$\frac{b_1 - \beta_1}{\sigma/\sqrt{S_{xx}}} \sim N(0, 1)$$

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We can estimate σ using the "average" of the squared residuals:

$$s_e^2 = \frac{\sum(y_i - \hat{y}_i)^2}{n - 2}$$

statistical properties of b_1

Who wants to guess what distribution this will have:

$$\frac{b_1 - \beta_1}{s_e/\sqrt{S_{xx}}} \sim t_{n-2}$$

hypothesis testing for β_1

The principal hypothesis test concerns whether there is any linear relationship at all between x and y. The null hypothesis immediately presents itself:

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And it works the same way any other hypothesis test works. Use the data to compute:

$$\frac{b_1 - 0}{s_e / \sqrt{S_{xx}}}$$

and get the probability of being "further away" from H_0, according to the t_{n-2} distribution.

example - body fat versus weight
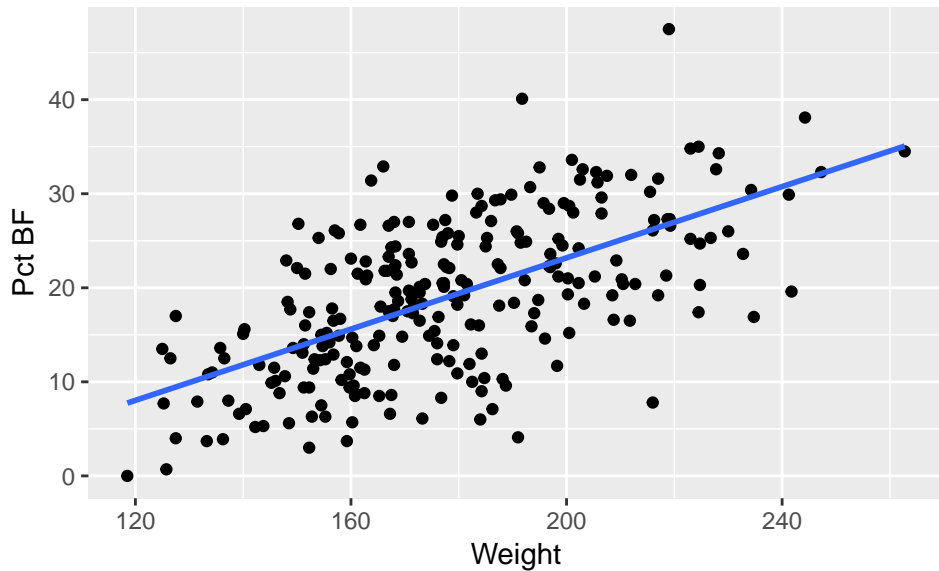
example - body fat versus weight

```
## 
## Coefficients:
##              Estimate Std. Error t value    Pr(>|t|)    
## (Intercept) -14.69314    2.76045  -5.323 0.000000229 ***
## Weight        0.18938    0.01533  12.357     < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## 
## Residual standard error: 6.538 on 248 degrees of freedom
## Multiple R-squared:  0.3811, Adjusted R-squared:  0.3786 
## F-statistic: 152.7 on 1 and 248 DF,  p-value: < 2.2e-16
```

this table translated into formulae

```
Coefficients:

                 Estimate   Std. Error   t value        Pr(>|t|)

        b_0       (Not         often        very        relevant)

        b_1     \frac{S_{xy}}{S_{xx}}   \frac{s_e}{\sqrt{S_{xx}}}   \frac{b_1-0}{s_e/\sqrt{S_{xx}}}   the p-value

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Residual standard error: s_e on $n - 2$ degrees of freedom

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Residual standard error: s_e on $n - 2$ degrees of freedom

Other stuff at the bottom not yet explained...

example - body fat versus height
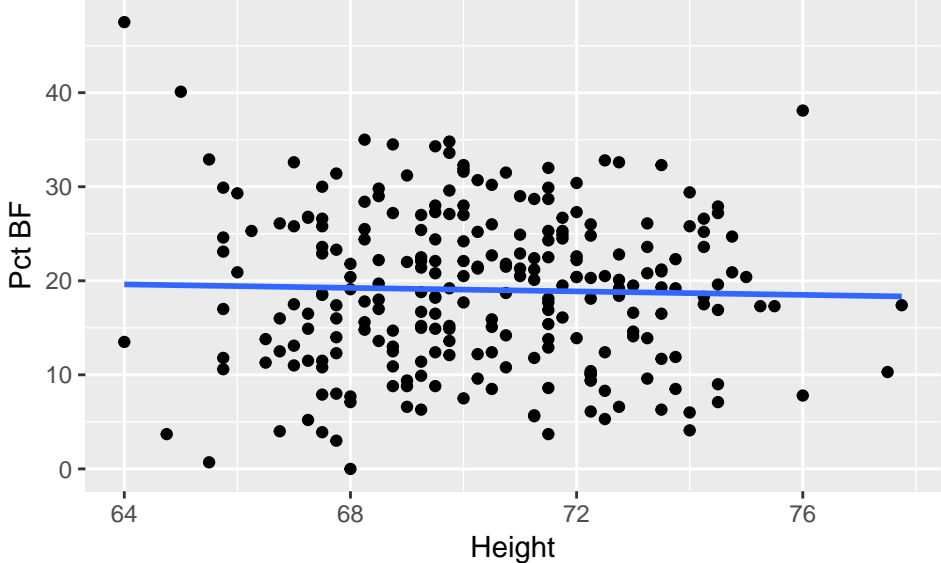
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##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 25.58078   14.15400    1.807   0.0719 .
## Height      -0.09316    0.20119   -0.463   0.6438
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## 
## Residual standard error: 8.307 on 248 degrees of freedom
## Multiple R-squared:  0.0008637,  Adjusted R-squared:  -0.003165
## F-statistic: 0.2144 on 1 and 248 DF,  p-value: 0.6438
```

confidence interval for the true slope β_1

95% confidence intervals are all pretty much the same, based on:

$$\frac{\text{estimator} - \text{parameter}}{SE(\text{estimator})} \sim \text{ something symmetric and bell shaped}$$

resulting in a formula:

$$\text{estimator} \pm \text{``2''} SE(\text{estimator})$$

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In the case of β_1 we have:

$$\frac{b_1 - \beta_1}{s_e/\sqrt{S_{xx}}} \sim t_{n-2}$$

result in a 95% C.I. of:

$$b_1 \pm t_{n-2}^* \frac{s_e}{\sqrt{S_{xx}}}$$

example C.I.'s for β_1 - body fat versus weight and height

Since $n = 250$, our value of "2" is in this case: 1.9695757

```
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A = all "model" | B = "typical" | C = all "error":
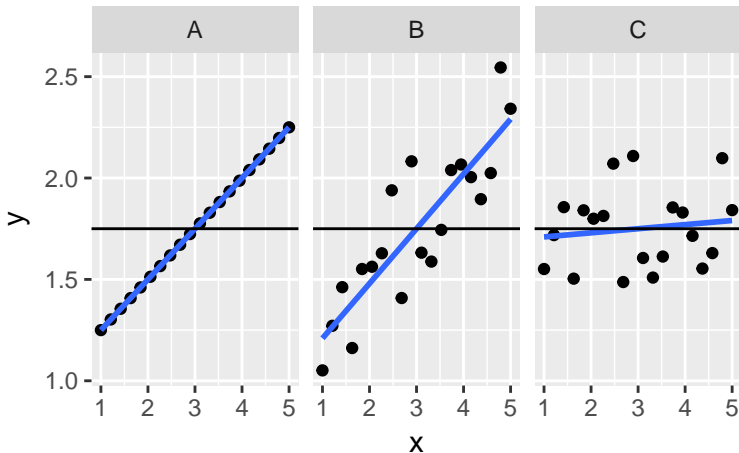
R^2 conceptual building blocks; a "sum-of-squares" decomposition

variation in the y = variation due to the model + variation due to error

$$\sum(y_i - \bar{y})^2 = \qquad +$$

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variation in the y = variation due to the model + variation due to error

$$\sum (y_i - \overline{y})^2 = \sum (\hat{y}_i - \overline{y})^2 +$$

R^2 conceptual building blocks; a "sum-of-squares" decomposition

variation in the y $=$ variation due to the model $+$ variation due to error

$$\sum(y_i - \overline{y})^2 = \sum(\hat{y}_i - \overline{y})^2 + \sum(y_i - \hat{y}_i)^2$$

R^2 conceptual building blocks; a "sum-of-squares" decomposition

variation in the y = variation due to the model + variation due to error

$$\sum(y_i - \overline{y})^2 = \sum(\hat{y}_i - \overline{y})^2 + \sum(y_i - \hat{y}_i)^2$$
$$SS_{Total} = SS_{Regression} + SS_{Error}$$

sum of squares decomposition, graphically

R^2 definition

$$R^2 = \frac{SS_{Regression}}{SS_{Total}} = 1 - \frac{SS_{Error}}{SS_{Total}}$$

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Although it is not a coefficient, and it does not really determine anything. It's just a mildly useful number.

Keep in mind it is *one number* that is being used to summarize an entire empirical bivariate relationship. And it isn't even the *best* number.

Some R^2 values
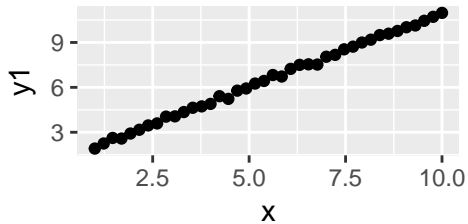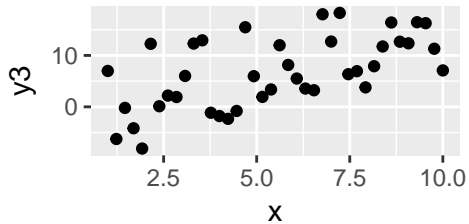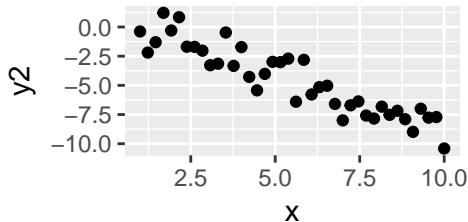
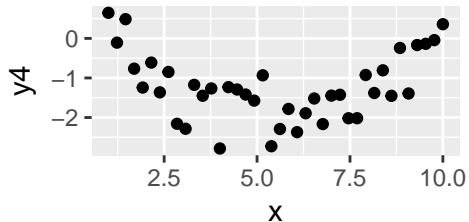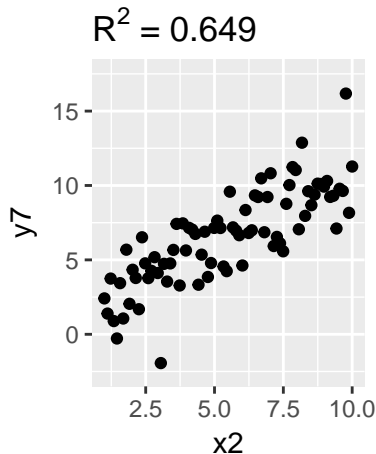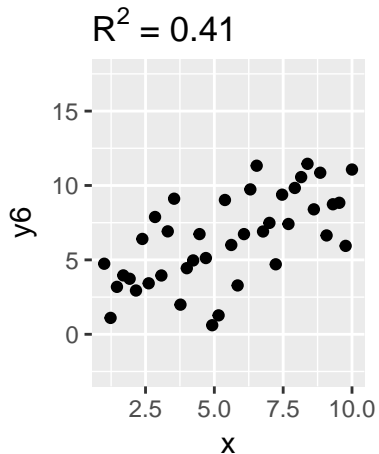## Another limitation: sample size effect

Both simulated datasets are from the *same underlying model*

(happens to be $y = 1 + 1 \cdot x + \varepsilon$ with $\varepsilon \sim N(0, 2)$)

regression model assumption (etc.) verification

recap model and calculation requirements

The model is:

$$y = \beta_0 + \beta_1 x + \varepsilon \text{ with } \varepsilon \sim N(0, \sigma)$$

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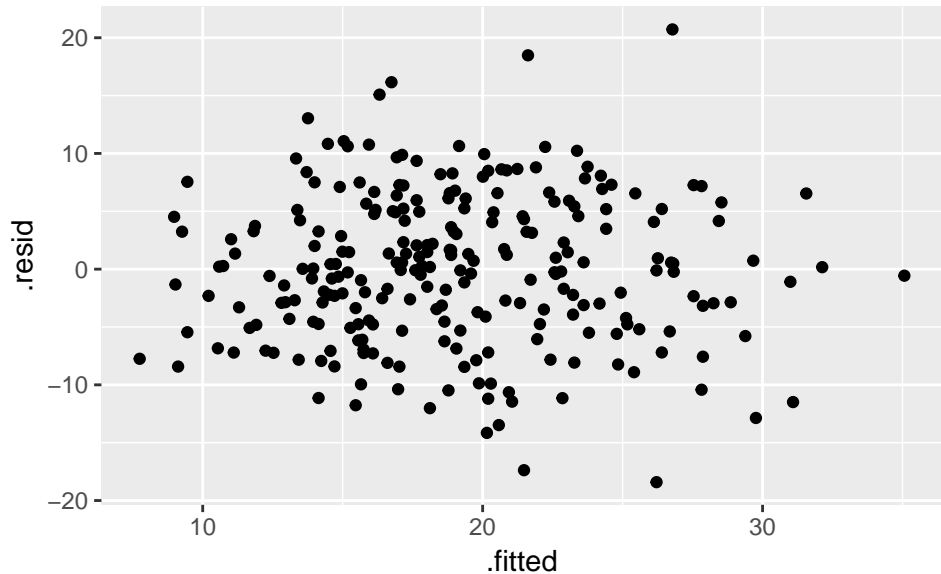- Error has a normal distribution.

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We will verify graphically, using various plots of the *residuals* $\hat{\varepsilon}_i = y_i - \hat{y}_i$

verify normality with normal quantile (or normal probability) plot of $\hat{\varepsilon}_i$
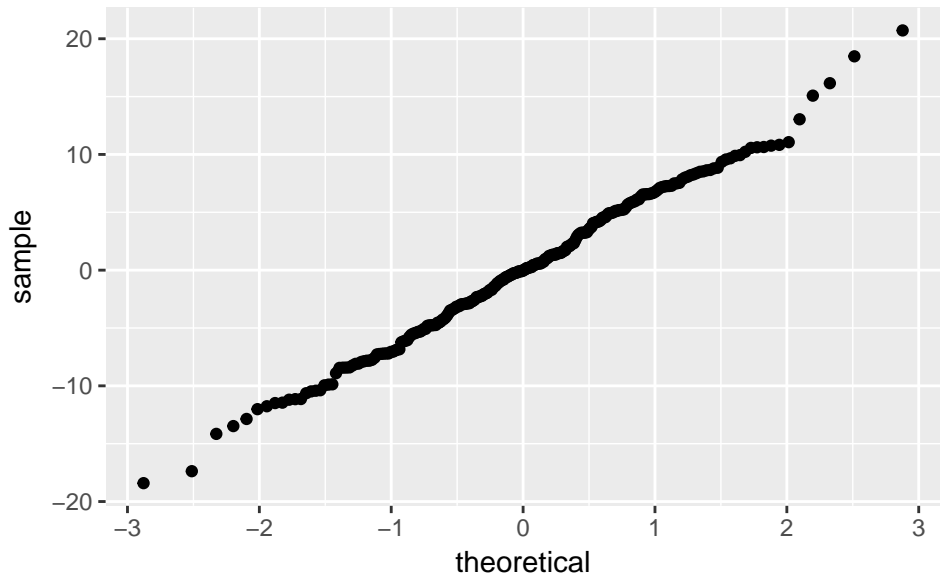
verify linearity with plot of $\hat{\varepsilon}_i$ versus $\hat{y}_i$

verify equal variance with (same!) plot of $\hat{\varepsilon}_i$ versus $\hat{y}_i$