STA221

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statistical properties of b_1

Start with the basic simple linear regression model:

$$y = \beta_0 + \beta_1 x + \varepsilon$$

in which the error follows a $N(0, \sigma)$ distribution.

The slope estimator b_1 turns out to follow a normal distribtion with mean β_1 and standard deviation:

$$\frac{\sigma}{\sqrt{S_{xx}}}$$

(Recall $S_{xx} = \sum (x_i - \overline{x})^2$)

(Note: there is a typo on the first formula in section 24.2 - the s_x should not be under the $\sqrt{\ }$.)

relating various formulae to the simulation results

When I simulated thousands of b_1 from datasets with a variety of properties, we saw the following:

- histograms of simulated b_1 were symmetric and bell shaped. In fact, normal! Why? Let's look at the formula for b_1:

$$b_1 = \frac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sum(x_i - \overline{x})^2} = \sum d_i(y_i - \overline{y})$$

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- Less variation when the underlying σ was smaller.
- Less variation when the x's were less spread out.
- Less variation when the sample size was larger.

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We have:

$$\frac{b_1 - \beta_1}{\sigma / \sqrt{S_{xx}}} \sim N(0, 1)$$

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We can estimate σ using the "average" of the squared residuals:

$$s_e = \frac{\sum(y_i - \hat{y}_i)^2}{n-2}$$

statistical properties of b_1

Who wants to guess what distribution this will have:

$$\frac{b_1 - \beta_1}{s_e/\sqrt{S_{xx}}} \sim ???$$

hypothesis testing for β_1

The principal hypothesis test concerns whether there is any linear relationship at all between x and y. The null hypothesis immediately presents itself:

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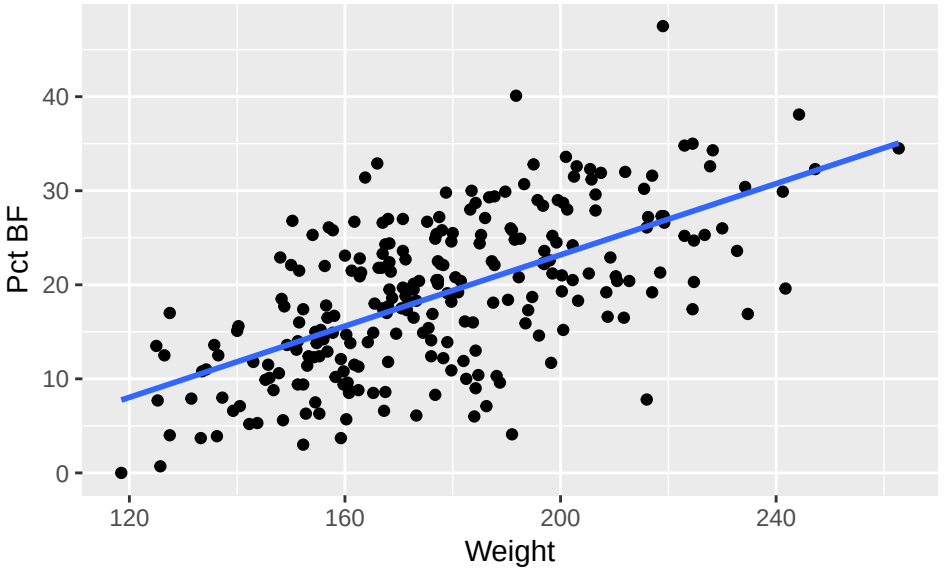
$$H_0 : \beta_1 = 0$$

And it works the same way any other hypothesis test works. Use the data to compute:

$$\frac{b_1 - 0}{s_e/\sqrt{S_{xx}}}$$

and get the probability of being "further away" from H_0, according to the ??? distribution.

example - body fat versus weight



example - body fat versus weight

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##
## Coefficients:
##              Estimate Std. Error t value    Pr(>|t|)
## (Intercept) -14.69314    2.76045  -5.323 0.000000229 ***
## Weight        0.18938    0.01533  12.357     < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 6.538 on 248 degrees of freedom
## Multiple R-squared:  0.3811, Adjusted R-squared:  0.3786
## F-statistic: 152.7 on 1 and 248 DF,  p-value: < 2.2e-16
```

this table translated into formulae

Coefficients:

	Estimate	Std. Error	t value	Pr(>|t|)
b_0	(Not	often	very	relevant)
b_1	$\frac{S_{xy}}{S_{xx}}$	$\frac{s_e}{\sqrt{S_{xx}}}$	$\frac{b_0 - 0}{s_e/\sqrt{S_{xx}}}$	the p-value

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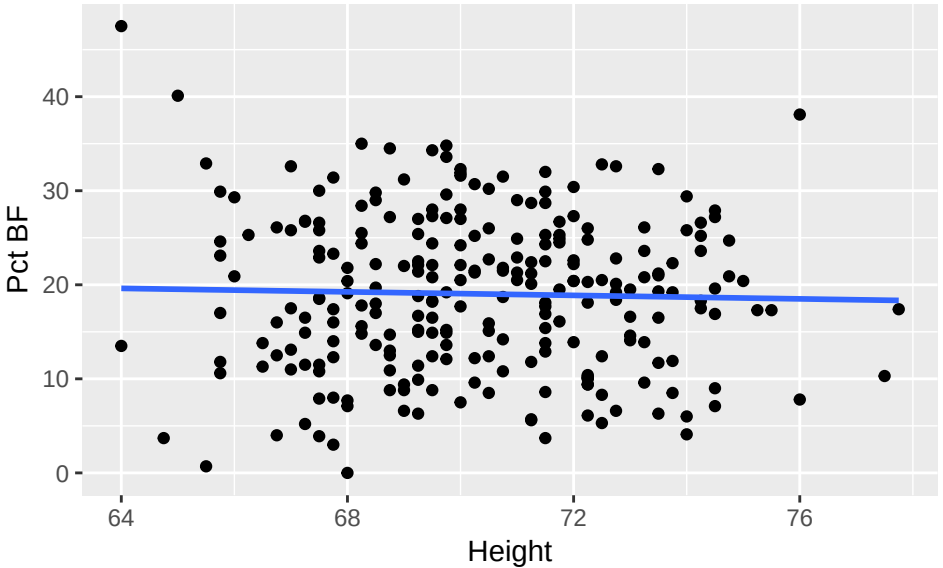
Residual standard error: s_e on $n-2$ degrees of freedom

Other stuff at the bottom not yet explained...

example - body fat versus height



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## Residual standard error: 8.307 on 248 degrees of freedom
## Multiple R-squared:  0.0008637,  Adjusted R-squared:  -0.003165
## F-statistic: 0.2144 on 1 and 248 DF,  p-value: 0.6438
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confidence interval for the true slope β_1

95% confidence intervals are all pretty much the same, based on:

$$\frac{\text{estimator} - \text{parameter}}{SE(\text{estimator})} \sim \text{ something symmetric and bell shaped}$$

resulting in a formula:

$$\text{estimator} \pm \text{“2”} SE(\text{estimator})$$

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In the case of β_1 we have:

$$\frac{b_1 - \beta_1}{s_e/\sqrt{S_{xx}}} \sim ???$$

result in a 95% C.I. of:

$$b_1 \pm ???_{???} \frac{s_e}{\sqrt{S_{xx}}}$$

example C.I.'s for β_1 - body fat versus weight and height

Since $n = 250$, our value of "2" is in this case: 1.9695757

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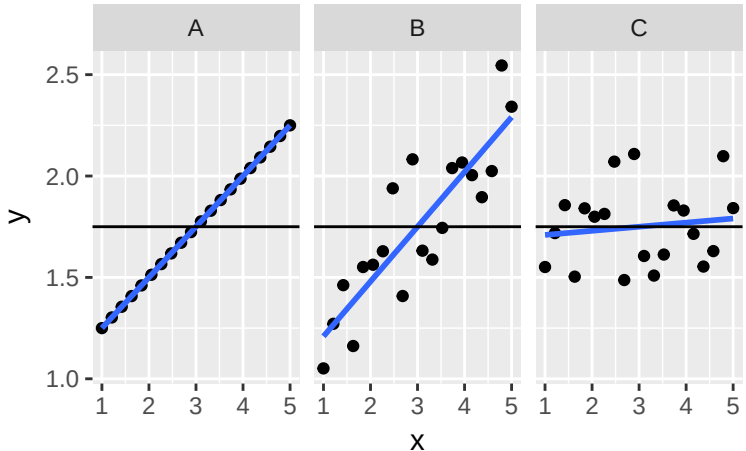
A = all "model" | B = "typical" | C = all "error":



R^2 conceptual building blocks; a "sum-of-squares" decomposition

variation in the y $=$ variation due to the model $+$ variation due to error

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$$\sum (y_i - \overline{y})^2 = \sum (y_i - \overline{y})^2 + \sum (\hat{y}_i - \hat{y}_i)^2$$

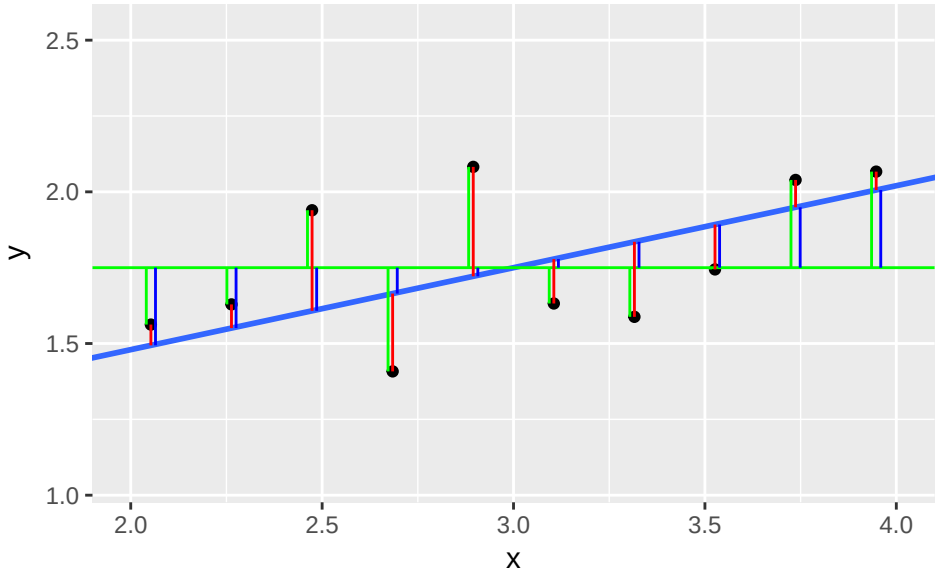
R^2 conceptual building blocks; a "sum-of-squares" decomposition

variation in the y = variation due to the model + variation due to error

$$\sum(y_i - \overline{y})^2 = \sum(y_i - \overline{y})^2 + \sum(\hat{y}_i - \hat{y}_i)^2$$

$$SS_{Total} = SS_{Regression} + SS_{Error}$$

sum of squares decomposition, graphically



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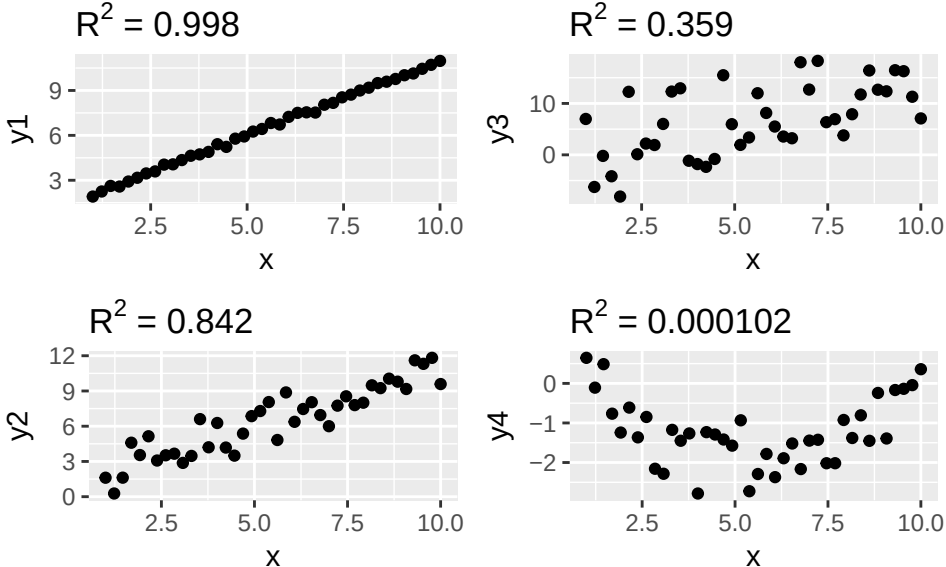
Keep in mind it is *one number* that is being used to summarize an entire empirical bivariate relationship. And it isn't even the *best* number.

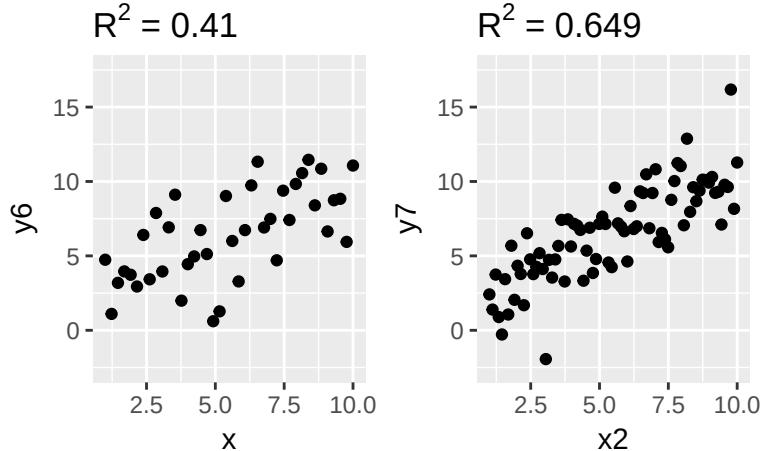
Some R^2 values



Another limitation: sample size effect

Both simulated datasets are from the ***same underlying model***

(happens to be $y = 1 + 1 \cdot x + \varepsilon$ with $\varepsilon \sim N(0, 2)$)



regression model assumption (etc.) verification

recap model and calculation requirements

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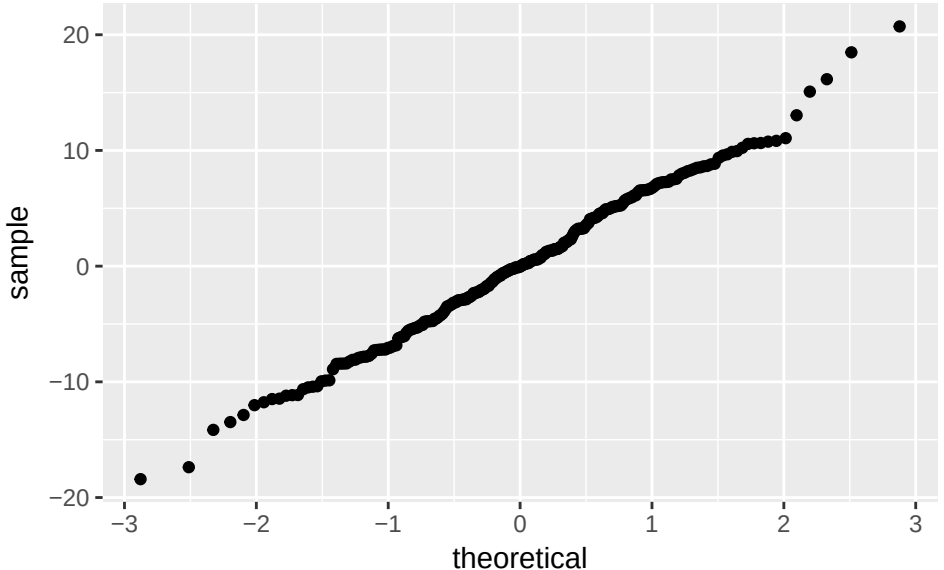
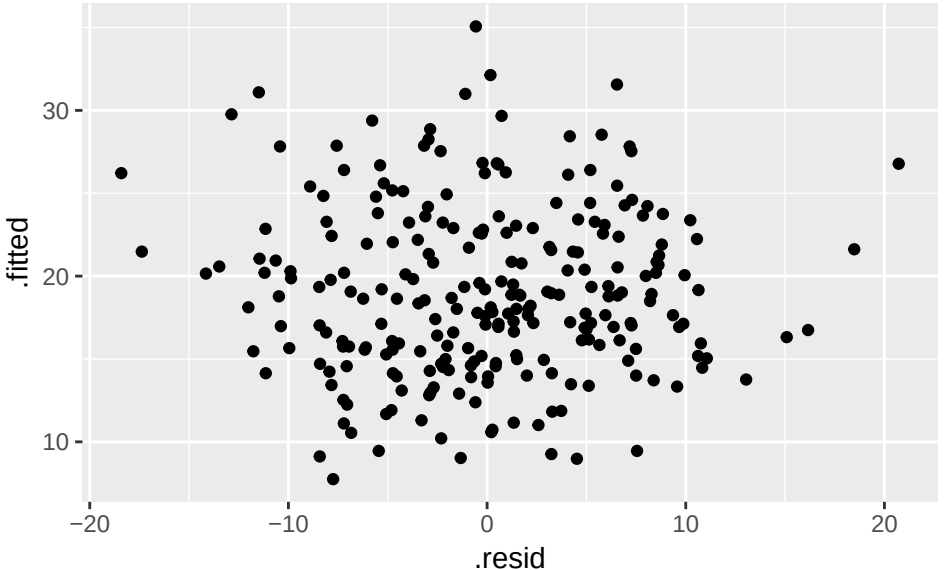
We will verify graphically, using various plots of the *residuals* $\hat{\varepsilon}_i = y_i - \hat{y}_i$

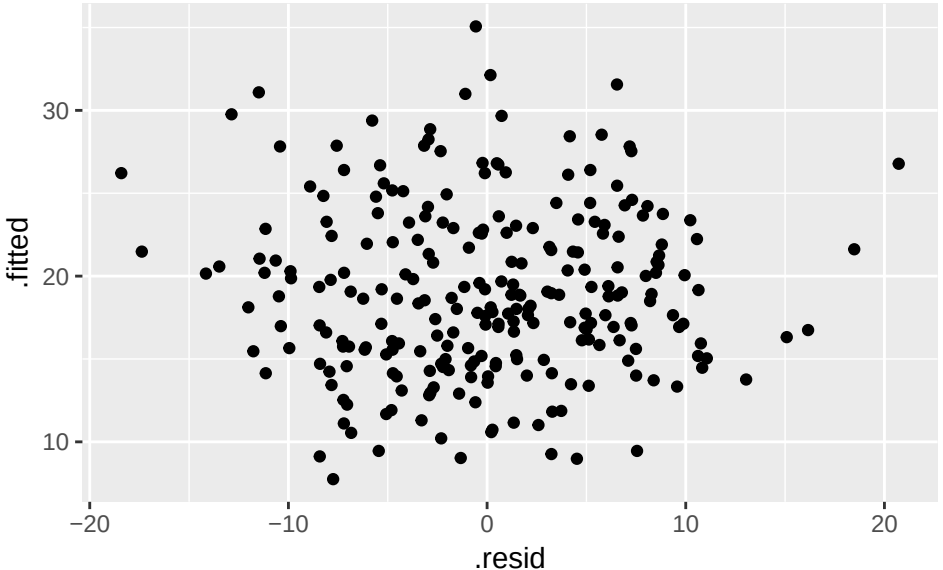
verify normality with normal quantile (or normal probability) plot of $\hat{\varepsilon}_i$



verify linearity with plot of $\hat{\varepsilon}_i$ versus $\hat{y}_i$



verify equal variance with (same!) plot of $\hat{\varepsilon}_i$ versus $\hat{y}_i$



estimation and prediction with regression models

estimate the mean response at a new x value

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What's the "obvious" best guess using the data?

$$\hat{\mu}_\nu = b_0 + b_1 x_\nu$$

estimate the mean response—with confidence

A confidence interval will be as usual based on:

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$$s_e \sqrt{\frac{1}{n} + \frac{(x_\nu - \overline{x})^2}{S_{xx}}}$$

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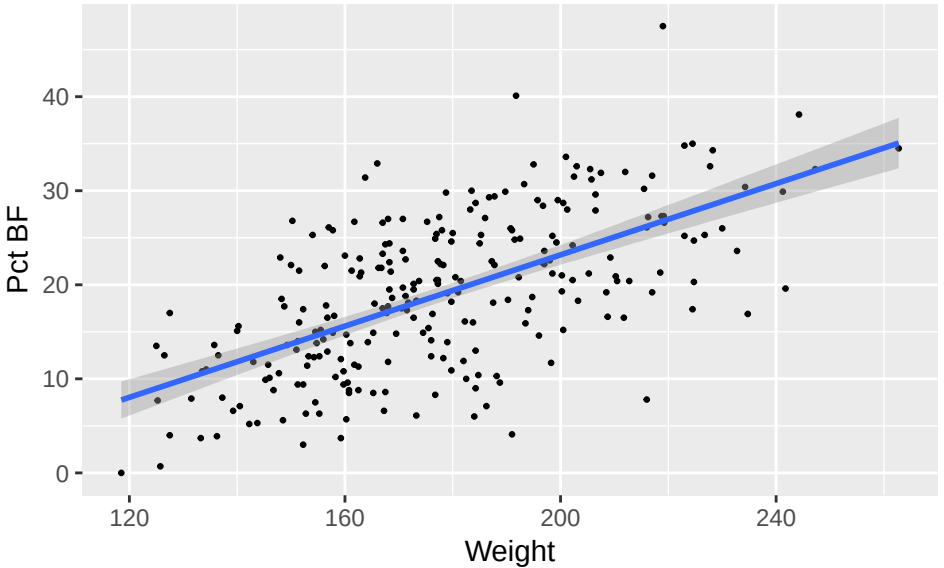
The standard error of $\hat{\mu}_\nu - \mu_\nu$ is:

$$s_e\sqrt{\frac{1}{n} + \frac{(x_\nu - \overline{x})^2}{S_{xx}}}$$

So the 95% C.I. for the mean response at x_ν will be:

$$\hat{\mu}_\nu \pm \text{“2”} s_e\sqrt{\frac{1}{n} + \frac{(x_\nu - \overline{x})^2}{S_{xx}}}$$

picture of 95% CI for mean response - weight model



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The *same* guess as the estimate for μ_ν.