STA221

Neil Montgomery

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the sample correlation coefficient

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It turns out to be a variation on something called a "sample covariance":

$$s_{xy} = \frac{\sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})}{n - 1}$$

(I'm using the textbook's Chapter 6 notation which is inadvertently close to my own S_{xy} notation. Sorry!)

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The sample mean estimates the mean... The sample variance estimates the variance... The sample correlation coefficient does in fact estimate a true, unknown "correlation coefficient", which is called ρ, but whose details we will not investigate, other than to point out that it is a number that assesses the strenght of the linear relationship between two distributions.

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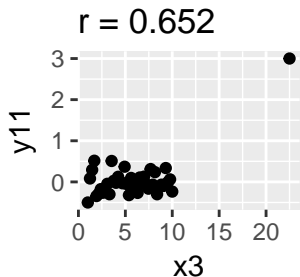
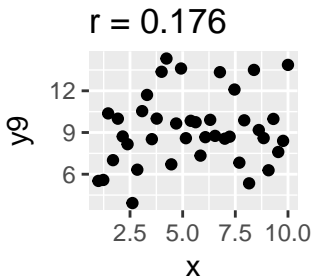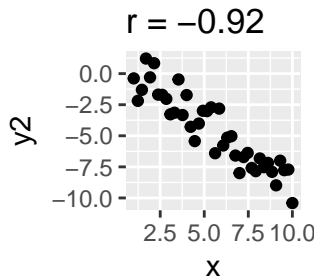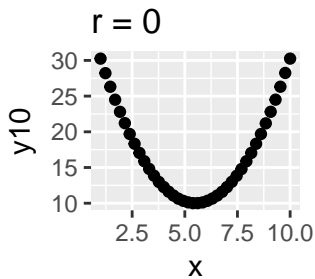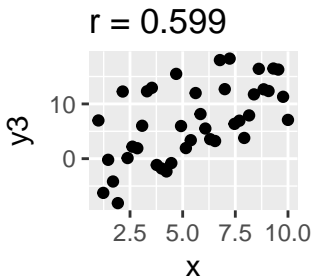$$(r)^2 = R^2 = \frac{SST}{SSR} = 1 - \frac{SSE}{SSR}$$

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$$r = b_1 \sqrt{\frac{S_{xy}}{S_{xx}}}$$

where b_1 is the slope estimator with x is "input"...

examples
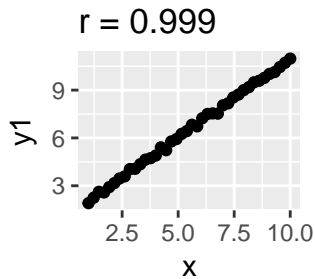
inference for correlation coefficient

Since b_1 has a normal distribtion, it might not come as a surprise the r also has a normal distribution. In fact:

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(Note: confidence interval is also possible, but this is best left to the computer.)

bodyfat example

Recall the dataset:

```
## # A tibble: 250 × 15
##   `Pct BF`   Age Weight Height  Neck Chest Abdomen    waist    Hip
##      <dbl> <int>  <dbl>  <dbl> <dbl> <dbl>   <dbl>    <dbl>  <dbl>
## 1     12.3    23 154.25  67.75  36.2  93.1    85.2 33.54331   94.5
## 2      6.1    22 173.25  72.25  38.5  93.6    83.0 32.67717   98.7
## 3     25.3    22 154.00  66.25  34.0  95.8    87.9 34.60630   99.2
## 4     10.4    26 184.75  72.25  37.4 101.8    86.4 34.01575  101.2
## 5     28.7    24 184.25  71.25  34.4  97.3   100.0 39.37008  101.9
## # ... with 245 more rows, and 6 more variables: Thigh <dbl>,
## #   Knee <dbl>, Ankle <dbl>, Bicep <dbl>, Forearm <dbl>, Wrist <dbl>
```

I wonder if the correlation between Neck and Chest circumferences is non-zero.

example - correlation matrix

A very useful information display is a "correlation matrix". Focus on the nine displayed
variables, excluding Age:

```
##            Pct BF Weight  Height  Neck Chest Abdomen waist   Hip
## Pct BF     1.0000  0.617 -0.0294 0.489 0.701   0.824 0.824 0.633
## Weight     0.6173  1.000  0.5129 0.810 0.891   0.874 0.874 0.933
## Height    -0.0294  0.513  1.0000 0.325 0.224   0.187 0.187 0.397
## Neck       0.4885  0.810  0.3247 1.000 0.769   0.728 0.728 0.708
## Chest      0.7007  0.891  0.2236 0.769 1.000   0.910 0.910 0.825
## Abdomen    0.8237  0.874  0.1867 0.728 0.910   1.000 1.000 0.861
## waist      0.8237  0.874  0.1867 0.728 0.910   1.000 1.000 0.861
## Hip        0.6327  0.933  0.3967 0.708 0.825   0.861 0.861 1.000
```

Neck versus Chest
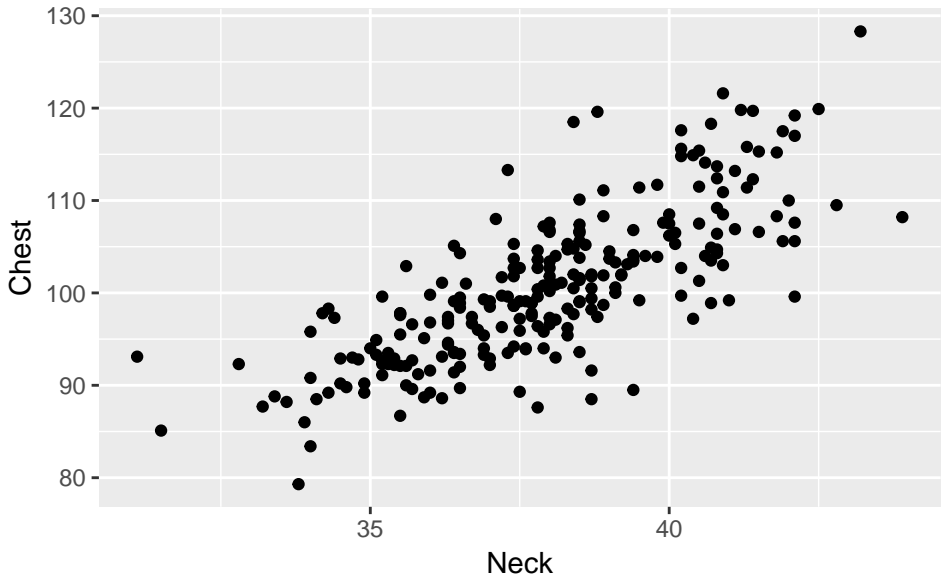
correlation analysis

```
## 
##  Pearson's product-moment correlation
## 
## data:  Neck and Chest
## t = 20, df = 200, p-value <2e-16
## alternative hypothesis: true correlation is not equal to 0
## 95 percent confidence interval:
##  0.713 0.815
## sample estimates:
##    cor
## 0.769
```

another example: Pct BF versus Height

Recall:

```
## 
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  25.5808    14.1540    1.81    0.072
## Height       -0.0932     0.2012   -0.46    0.644
## 
## Residual standard error: 8.31 on 248 degrees of freedom
## Multiple R-squared:  0.000864,   Adjusted R-squared:  -0.00317
## F-statistic: 0.214 on 1 and 248 DF,  p-value: 0.644
```

compare p-value of 0.644 for $H_0 : \beta_1 = 0$

Now the correlation analysis:

```
##
##  Pearson's product-moment correlation
##
## data:  Pct BF and Height
## t = -0.463, df = 248, p-value = 0.644
## alternative hypothesis: true correlation is not equal to 0
## 95 percent confidence interval:
##  -0.1528991  0.0950239
## sample estimates:
##         cor
## -0.0293896
```

Not a coincidence! The conclusion must be identical.

the analysis of designed experiments

Formal definitions: factor, level

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In reality any dataset with one categorical "input" variable and one numerical "output" variable will be analysed the same as a formally designed experiment.

Typical dataset...

Truck.ID	Oil	Viscosity
HT 265	Volvo	25.5
HT 372	Castrol	25.7
HT 572	Komatsu	25.6
HT 908	Volvo	24.7
HT 201	Castrol	26.5
HT 898	Komatsu	25.4
HT 944	Volvo	24.4
HT 660	Castrol	22.8
HT 629	Komatsu	26.1
HT 61	Volvo	25.0
HT 205	Castrol	25.0
HT 176	Komatsu	25.9

One factor notation, models

"Balanced" case with equal sample size n for each of k levels for $N = nk$ total.

Levels:	1	2	...	i	...	k
	y_{11}	y_{21}	...	y_{i1}	...	y_{k1}
	y_{12}	y_{22}	...	y_{i2}	...	y_{k2}
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
	y_{1n}	y_{2n}	...	y_{in}	...	y_{kn}
Sample average:	$\overline{y}_1$	$\overline{y}_2$	...	$\overline{y}_i$	...	$\overline{y}_k$

Grand overall average: $\overline{\overline{y}}$

Models:

$$y_{ij} = \mu_i + \varepsilon_{ij}, \qquad \varepsilon_{ij} \text{ i.i.d. } N(0, \sigma^2)$$

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \qquad \sum \alpha_i = 0 \qquad \varepsilon_{ij} \text{ i.i.d. } N(0, \sigma^2)$$

The main question

The main question is $H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$ versus the negation (equivalently: all the $\alpha_i = 0$.)

In other words "is the variation among all the y_{ij} due to the factor variable, or just due to random chance?". The analysis even follows this logic.

The variation among the y_{ij} is quantified as (as usual?):

$$(N - 1) \cdot s_y^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} (y_{ij} - \overline{\overline{y}})^2$$

We will split this up into the "factor" part and the "random chance" part (like done in regression).