STA221

Neil Montgomery

Last edited: 2017-03-01 11:56

the sample correlation coefficient

Recall this expression that is used in the formula for b_1:

$$S_{xy} = \sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})$$

the sample correlation coefficient

Recall this expression that is used in the formula for b_1:

$$S_{xy} = \sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})$$

This is a (crude) measure of the linear association between dataset variables with names x and y.

the sample correlation coefficient

Recall this expression that is used in the formula for b_1:

$$S_{xy} = \sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})$$

This is a (crude) measure of the linear association between dataset variables with names x and y.

It turns out to be a variation on something called a "sample covariance":

$$s_{xy} = \frac{\sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})}{n-1}$$

(I'm using the textbook's Chapter 6 notation which is inadvertently close to my own S_{xy} notation. Sorry!)

the sample correlation coefficient

The sample covariance s_{xy} depends on the unit of measurement for both variables, when all we care about is the strength of the relationship.

We can divide out the variation in both x and y to obtain what is called the *sample correlation coefficient:*

the sample correlation coefficient

The sample covariance s_{xy} depends on the unit of measurement for both variables, when all we care about is the strength of the relationship.

We can divide out the variation in both x and y to obtain what is called the *sample correlation coefficient:*

$$r = \frac{s_{xy}}{s_x s_y} = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$$

where s_x and s_y are the sample standard deviations of the x and y variables, and the final expression because the $n-1$ cancels top and bottom.

the sample correlation coefficient

The sample covariance s_{xy} depends on the unit of measurement for both variables, when all we care about is the strength of the relationship.

We can divide out the variation in both x and y to obtain what is called the *sample correlation coefficient:*

$$r = \frac{s_{xy}}{s_x s_y} = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$$

where s_x and s_y are the sample standard deviations of the x and y variables, and the final expression because the $n - 1$ cancels top and bottom.

The sample mean estimates the mean... The sample variance estimates the variance... The sample correlation coefficient does in fact estimate a true, unknown "correlation coefficient", which is called ρ, but whose details we will not investigate, other than to point out that it is a number that assesses the strenght of the linear relationship between two distributions.

properties of the sample correlation coefficient

It is symmetric in x and y. There is not (necessarily) an "input" and an "output" variable.

properties of the sample correlation coefficient

It is symmetric in x and y. There is not (necessarily) an "input" and an "output" variable.

$$(r)^2 = R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

properties of the sample correlation coefficient

It is symmetric in x and y. There is not (necessarily) an “input” and an “output” variable.

$$(r)^2 = R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

$$-1 \leq r \leq 1$$

properties of the sample correlation coefficient

It is symmetric in x and y. There is not (necessarily) an "input" and an "output" variable.

$$(r)^2 = R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

$$-1 \leq r \leq 1$$

The sample correlation coefficient is only suitable when the relationship is linear, and is susceptible to all the same shortcomings as any regression model.

properties of the sample correlation coefficient

It is symmetric in x and y. There is not (necessarily) an "input" and an "output" variable.

$$(r)^2 = R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

$$-1 \leq r \leq 1$$

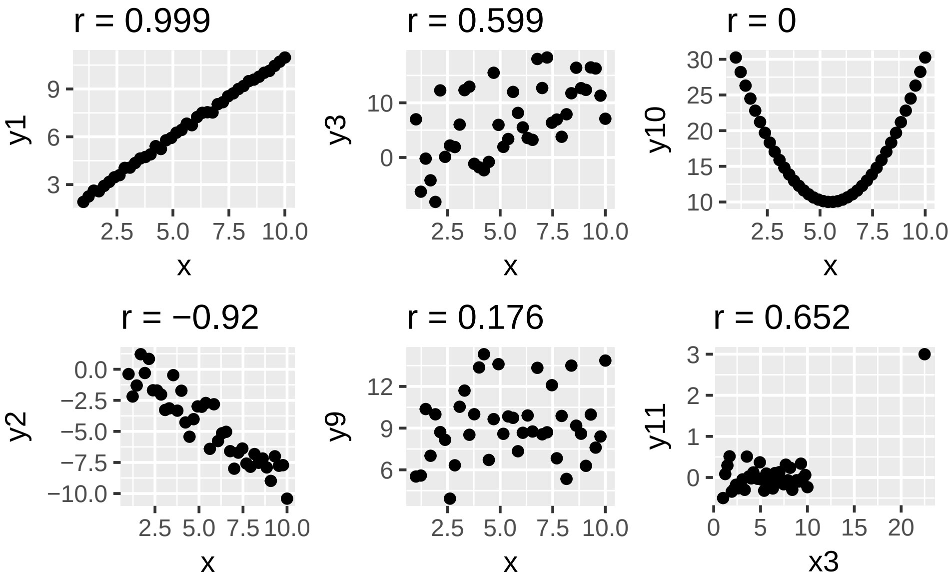
The sample correlation coefficient is only suitable when the relationship is linear, and is susceptible to all the same shortcomings as any regression model.

CORRECTED!

$$r = b_1\sqrt{\frac{S_{yy}}{S_{xx}}}$$

where b_1 is the slope estimator with x is "input"...

examples



r = 0.999
y1
9
6
3
2.5 5.0 7.5 10.0
x
r = 0.599
y3
10
0
2.5 5.0 7.5 10.0
x
r = 0
y10
30
25
20
15
10
2.5 5.0 7.5 10.0
x
r = −0.92
y2
0.0
−2.5
−5.0
−7.5
−10.0
2.5 5.0 7.5 10.0
x
r = 0.176
y9
12
9
6
2.5 5.0 7.5 10.0
x
r = 0.652
y11
3
2
1
0
0 5 10 15 20
x3

inference for correlation coefficient

Since b_1 has a normal distribtion, it might not come as a surprise the r also has a normal distribution. In fact:

$$\frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \sim t_{n-2}$$

inference for correlation coefficient

Since b_1 has a normal distribtion, it might not come as a surprise the r also has a normal distribution. In fact:

$$\frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \sim t_{n-2}$$

So it is possible to do hypothesis testing for $H_0 : \rho = 0$ versus $H_a : \rho \neq 0$.

inference for correlation coefficient

Since b_1 has a normal distribtion, it might not come as a surprise the r also has a normal distribution. In fact:

$$\frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \sim t_{n-2}$$

So it is possible to do hypothesis testing for $H_0 : \rho = 0$ versus $H_a : \rho \neq 0$.

(Note: confidence interval is also possible, but this is best left to the computer.)

bodyfat example

Recall the dataset:

```
## # A tibble: 250 × 15
##   `Pct BF`   Age Weight Height  Neck Chest Abdomen    waist   Hip
##      <dbl> <int>  <dbl>  <dbl> <dbl> <dbl>   <dbl>    <dbl> <dbl>
## 1     12.3    23 154.25  67.75  36.2  93.1    85.2 33.54331  94.5
## 2      6.1    22 173.25  72.25  38.5  93.6    83.0 32.67717  98.7
## 3     25.3    22 154.00  66.25  34.0  95.8    87.9 34.60630  99.2
## 4     10.4    26 184.75  72.25  37.4 101.8    86.4 34.01575 101.2
## 5     28.7    24 184.25  71.25  34.4  97.3   100.0 39.37008 101.9
## # ... with 245 more rows, and 6 more variables: Thigh <dbl>,
## #   Knee <dbl>, Ankle <dbl>, Bicep <dbl>, Forearm <dbl>, Wrist <dbl>
```

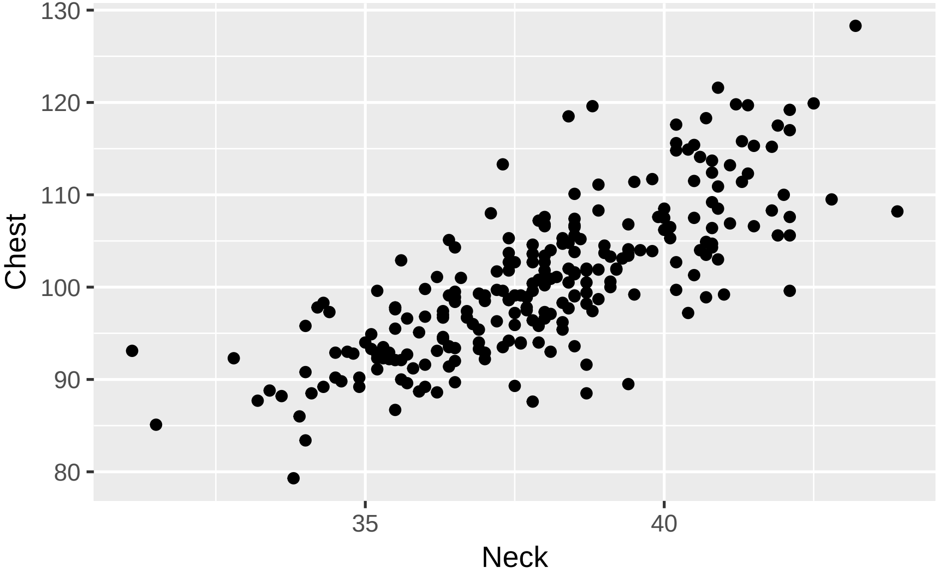
I wonder if the correlation between `Neck` and `Chest` circumferences is non-zero.

example - correlation matrix

A very useful information display is a "correlation matrix". Focus on the nine displayed variables, excluding `Age`:

```
##            Pct BF Weight  Height  Neck Chest Abdomen waist   Hip
## Pct BF     1.0000  0.617 -0.0294 0.489 0.701   0.824 0.824 0.633
## Weight     0.6173  1.000  0.5129 0.810 0.891   0.874 0.874 0.933
## Height    -0.0294  0.513  1.0000 0.325 0.224   0.187 0.187 0.397
## Neck       0.4885  0.810  0.3247 1.000 0.769   0.728 0.728 0.708
## Chest      0.7007  0.891  0.2236 0.769 1.000   0.910 0.910 0.825
## Abdomen    0.8237  0.874  0.1867 0.728 0.910   1.000 1.000 0.861
## waist      0.8237  0.874  0.1867 0.728 0.910   1.000 1.000 0.861
## Hip        0.6327  0.933  0.3967 0.708 0.825   0.861 0.861 1.000
```

Neck versus Chest



correlation analysis

```
##
##  Pearson's product-moment correlation
##
## data:  Neck and Chest
## t = 20, df = 200, p-value <2e-16
## alternative hypothesis: true correlation is not equal to 0
## 95 percent confidence interval:
##  0.713 0.815
## sample estimates:
##   cor
## 0.769
```

another example: `Pct BF` versus `Height`

Recall:

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  25.5808    14.1540    1.81    0.072
## Height       -0.0932     0.2012   -0.46    0.644
##
## Residual standard error: 8.31 on 248 degrees of freedom
## Multiple R-squared:  0.000864,   Adjusted R-squared:  -0.00317
## F-statistic: 0.214 on 1 and 248 DF,  p-value: 0.644
```

compare p-value of 0.644 for $H_0 : \beta_1 = 0$

Now the correlation analysis:

```
##
##  Pearson's product-moment correlation
##
## data:  Pct BF and Height
## t = -0.463, df = 248, p-value = 0.644
## alternative hypothesis: true correlation is not equal to 0
## 95 percent confidence interval:
##  -0.1528991  0.0950239
## sample estimates:
##        cor
## -0.0293896
```

Not a coincidence! The conclusion must be identical.

the analysis of designed experiments

Formal definitions: factor, level

A *factor* is a controllable experimental condition.

Formal definitions: factor, level

A *factor* is a controllable experimental condition.

A factor can take on two or more *levels*.

Formal definitions: factor, level

A *factor* is a controllable experimental condition.

A factor can take on two or more *levels*.

E.g., in a study of haul trucks "oil brand" could be a factor, with levels "Castrol", "Volvo", "Komatsu".

Formal definitions: factor, level

A *factor* is a controllable experimental condition.

A factor can take on two or more *levels*.

E.g., in a study of haul trucks "oil brand" could be a factor, with levels "Castrol", "Volvo", "Komatsu".

When experimental units are randomly assigned to levels of a factor and some output measure is observed, this is called a *designed experiment*. The formal model is typically written as (more on this later):

$$Y_{ij} = \mu_i + \varepsilon_{ij}$$

Formal definitions: factor, level

A *factor* is a controllable experimental condition.

A factor can take on two or more *levels*.

E.g., in a study of haul trucks "oil brand" could be a factor, with levels "Castrol", "Volvo", "Komatsu".

When experimental units are randomly assigned to levels of a factor and some output measure is observed, this is called a *designed experiment*. The formal model is typically written as (more on this later):

$$Y_{ij} = \mu_i + \varepsilon_{ij}$$

You've seen the case of $i \in \{1, 2\}$—such an experiment would be analyzed using a two-sample t procedure.

Formal definitions: factor, level

A *factor* is a controllable experimental condition.

A factor can take on two or more *levels*.

E.g., in a study of haul trucks "oil brand" could be a factor, with levels "Castrol", "Volvo", "Komatsu".

When experimental units are randomly assigned to levels of a factor and some output measure is observed, this is called a *designed experiment*. The formal model is typically written as (more on this later):

$$Y_{ij} = \mu_i + \varepsilon_{ij}$$

You've seen the case of $i \in \{1, 2\}$—such an experiment would be analyzed using a two-sample t procedure.

In reality any dataset with one categorical "input" variable and one numerical "output" variable will be analysed the same as a formally designed experiment.

Typical dataset...

Truck.ID	Oil	Viscosity
HT 265	Volvo	25.5
HT 372	Castrol	25.7
HT 572	Komatsu	25.6
HT 908	Volvo	24.7
HT 201	Castrol	26.5
HT 898	Komatsu	25.4
HT 944	Volvo	24.4
HT 660	Castrol	22.8
HT 629	Komatsu	26.1
HT 61	Volvo	25.0
HT 205	Castrol	25.0
HT 176	Komatsu	25.9

One factor notation, models

"Balanced" case with equal sample size n for each of k levels for $N = nk$ total.

Levels:	1	2	...	i	...	k
	y_{11}	y_{21}	...	y_{i1}	...	y_{k1}
	y_{12}	y_{22}	...	y_{i2}	...	y_{k2}
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
	y_{1n}	y_{2n}	...	y_{in}	...	y_{kn}
Sample average:	$\overline{y}_1$	$\overline{y}_2$	...	$\overline{y}_i$	...	$\overline{y}_k$

Grand overall average: $\overline{\overline{y}}$

Models:

$$y_{ij} = \mu_i + \varepsilon_{ij}, \qquad \varepsilon_{ij} \text{ i.i.d. } N(0, \sigma^2)$$

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \qquad \sum \alpha_i = 0 \qquad \varepsilon_{ij} \text{ i.i.d. } N(0, \sigma^2)$$

The main question

The main question is $H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$ versus the negation (equivalently: all the $\alpha_i = 0$.)

In other words "is the variation among all the y_{ij} due to the factor variable, or just due to random chance?". The analysis even follows this logic.

The variation among the y_{ij} is quantified as (as usual?):

$$(N-1) \cdot s_y^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} \left(y_{ij} - \overline{\overline{y}}\right)^2$$

We will split this up into the "factor" part and the "random chance" part (like done in regression).