STA221

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multiple regression

regression with more than one input variable

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The most important stastical model (in my opinion) is the linear regression model with more than one "x" variable. For example, with 3 input variables:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon$$

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*A variable cannot be a linear function of other variables in the model.

what is being accomplished in multiple regression?

R comes with some sample datasets. One is called `trees` and has variables Girth, Height, and Volume. Here's a peek at the data:

```
## # A tibble: 31 × 3
##   Girth Height Volume
##   <dbl>  <dbl>  <dbl>
## 1   8.3     70   10.3
## 2   8.6     65   10.3
## 3   8.8     63   10.2
## 4  10.5     72   16.4
## 5  10.7     81   18.8
## # ... with 26 more rows
```

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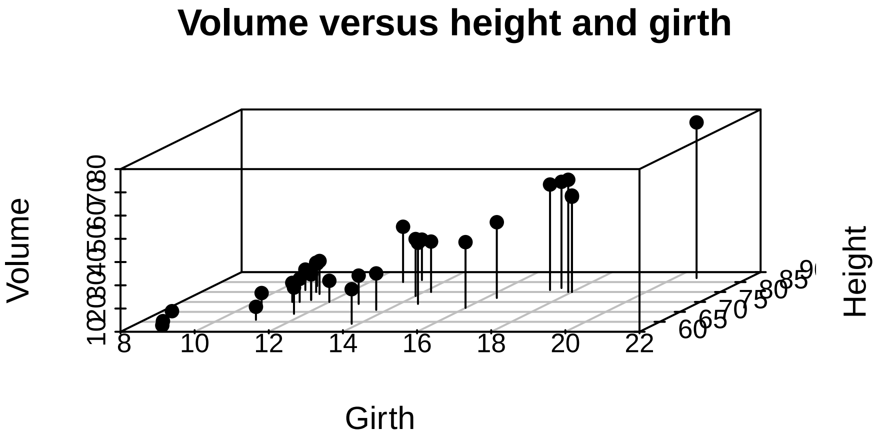
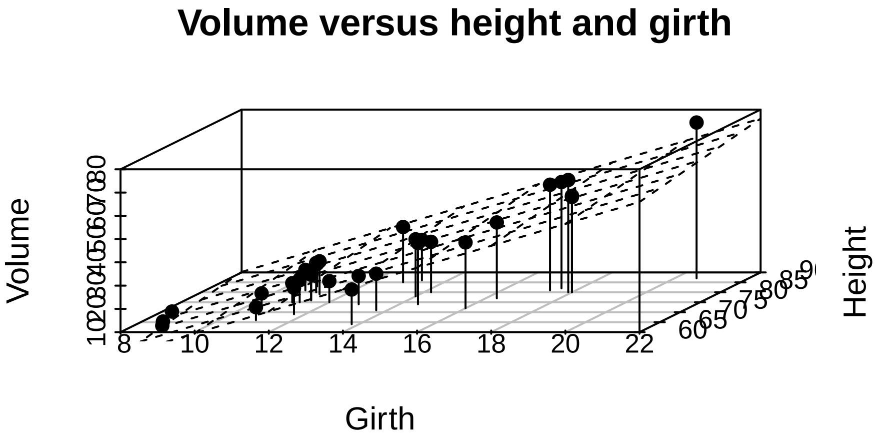


multiple regression fits a surface to the points



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  - "Multicollinearity" (highly correlated inputs)

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The multiple regression model:

$$y = \beta_0 + \beta_1 x_1 + \ldots \beta_k x_k + \varepsilon, \quad \varepsilon \sim N(0, \sigma)$$

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The β_i from $i \in \{1, \ldots, k\}$ are the slope parameters, and have a different interpretation than before.

slope parameter interpretation

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That bold, italic statement should echo in your mind any time you think of anything to do with β_i.

trees example

We might want to model $y =$`Volume` (the amount of wood) as a linear model of the input variables $x_1 =$`Girth` and $x_2 =$`Height`, as follows:

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The computer uses the method of "least squares", like before. A full treatment of the analysis requires matrix algebra.

fitted values | residuals

Here's the first row of the `trees` data:

Girth	Height	Volume
8.3	70	10.3

We could call these values y_1, x_{11}, and x_{21}

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For a dataset with n rows (the sample size), there is a fitted value and residual for each row.

trees data fitted model

Here's what R produces:

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -57.9877     8.6382  -6.713 2.75e-07
## Girth         4.7082     0.2643  17.816  < 2e-16
## Height        0.3393     0.1302   2.607   0.0145
##
## Residual standard error: 3.882 on 28 degrees of freedom
## Multiple R-squared:  0.948,  Adjusted R-squared:  0.9442
## F-statistic:   255 on 2 and 28 DF,  p-value: < 2.2e-16
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individual slope parameter hypothesis testing

The usual hypothesis test for a single parameter:

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the overall hypothesis test

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It is also possible to test any subset of these parameters, such as:

$$H_0 : \beta_1 = \beta_2 = 0$$

although at the moment it's not clear why this might be a good idea.

estimating σ

This works the same as with simple regression, in which we used $\sqrt{MSE}$ where:

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There was only one input variable, so another way to think of this was "sample size minus the number of input variables, then minus 1."

estimating σ

In multiple regression, nothing changes. Use $\sqrt{MSE}$, where:

$$MSE = \frac{\sum_{j=1}^{n} \left(y_j - \hat{y}_j\right)^2}{n - (k+1)}$$

hypothesis testing for β_i

The computer produces the estimate b_i, which has these properties:

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Just like before, we get:

$$\frac{b_i - \beta_i}{\sqrt{MSE}\sqrt{c_i}} \sim t_{n-k+1}$$

hypothesis testing for β_i in the trees example

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