STA221

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trees data fitted model

Here's what R produces:

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -57.9877     8.6382  -6.713 2.75e-07
## Girth         4.7082     0.2643  17.816  < 2e-16
## Height        0.3393     0.1302   2.607   0.0145
##
## Residual standard error: 3.882 on 28 degrees of freedom
## Multiple R-squared:  0.948,  Adjusted R-squared:  0.9442
## F-statistic:   255 on 2 and 28 DF,  p-value: < 2.2e-16
```

individual slope parameter hypothesis testing

The usual hypothesis test for a single parameter:

$$
\begin{aligned}
H_0 &: \beta_i = 0 \\
H_a &: \beta_i \neq 0
\end{aligned}
$$

individual slope parameter hypothesis testing

The usual hypothesis test for a single parameter:

$$H_0 : \beta_i = 0$$
$$H_a : \beta_i \neq 0$$

If H_0 is true, it means the ith variable (x_i) is not significantly related to y

individual slope parameter hypothesis testing

The usual hypothesis test for a single parameter:

$$H_0 : \beta_i = 0$$
$$H_a : \beta_i \neq 0$$

If H_0 is true, it means the ith variable (x_i) is not significantly related to y ***given all the other x's in the model***

the overall hypothesis test

“Is there any linear relationship between y and the input variables?”

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Null hypothesis can be expressed as:

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_k = 0$$

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It is also possible to test any subset of these parameters, such as:

$$H_0 : \beta_1 = \beta_2 = 0$$

although at the moment it's not clear why this might be a good idea.

estimating σ

This works the same as with simple regression, in which we used $\sqrt{MSE}$ where:

$$MSE = \frac{\sum_{j=1}^{n} \left(y_j - \hat{y}_j\right)^2}{n-2}$$

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$n - 2$ was the sample size minus the number of parameters (two: β_0 and β_1) being estimated.

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$n-2$ was the sample size minus the number of parameters (two: β_0 and β_1) being estimated.

There was only one input variable, so another way to think of this was "sample size minus the number of input variables, then minus 1."

estimating σ

In multiple regression, nothing changes. Use $\sqrt{MSE}$, where:

$$MSE = \frac{\sum_{j=1}^{n} \left(y_j - \hat{y}_j\right)^2}{n - (k+1)}$$

hypothesis testing for β_i

The computer produces the estimate b_i, which has these properties:

$$E(b_i) = \beta_i$$
$$\text{Var}(b_i) = \sigma^2 \cdot c_i$$

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Just like before, we get:

$$\frac{b_i - \beta_i}{\sqrt{MSE}\sqrt{c_i}} \sim t_{n-(k+1)}$$

hypothesis testing for β_i in the trees example

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```

the overall F test

"Is there any linear relationship between y and the input variables?"

Based on the same, original SS decomposition.

$$\text{variation in the } y \; = \; \text{variation due to the model} \; + \; \text{variation due to error}$$

$$\sum (y_i - \overline{y})^2 = \qquad\qquad\qquad +$$

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variation in the y = variation due to the model + variation due to error

$$\sum (y_i - \overline{y})^2 = \sum (\hat{y}_i - \overline{y})^2 + \sum (y_i - \hat{y}_i)^2$$
$$SS_{Total} = SS_{Regression} + SS_{Error}$$

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$$\begin{aligned}
\sum(y_i - \overline{y})^2 &= \sum(\hat{y}_i - \overline{y})^2 + \sum(y_i - \hat{y}_i)^2 \\
SS_{Total} &= SS_{Regression} + SS_{Error} \\
\chi^2_{n-1} &= \chi^2_k + \chi^2
\end{aligned}$$

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The p-value then comes from **CORRECTED 2017-04-08**:

$$\frac{SS_{Regression}/k}{SS_{Error}/(n-k-1)} = \frac{MSR}{MSE} \sim F_{k,n-k-1}$$

the overall *F* test - trees example

The information is in the usual `R` output:

```
##
## Residual standard error: 3.882 on 28 degrees of freedom
## Multiple R-squared:  0.948,  Adjusted R-squared:  0.9442
## F-statistic:   255 on 2 and 28 DF,  p-value: < 2.2e-16
```

One can obtain an “ANOVA” table from this information:

```
## Warning: package 'broom' was built under R version 3.3.3
```

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Regression					
Error					

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Regression	2				
Error	28				

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Regression	2			254.97	1.07×10^{-18}
Error	28				

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MSE = Square of the ‘Residual standard error’

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One can obtain an “ANOVA” table from this information:

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	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Regression	2	7684.16	3842.08	254.97	1.07×10^{-18}
Error	28	421.92	15.07		

model assumptions and calculation requirements

Model:

$$y = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k + \varepsilon, \qquad \varepsilon \sim N(0, \sigma)$$

Pretty much the same as with simple regression.

model assumptions and calculation requirements

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Pretty much the same as with simple regression.

First, there's the independence assumption, which can't really be verified without knowledge of the data collection itself (common violation - repeated measures.)

The main ones to worry about are:

1. The linear model is appropriate (fatal if violated).

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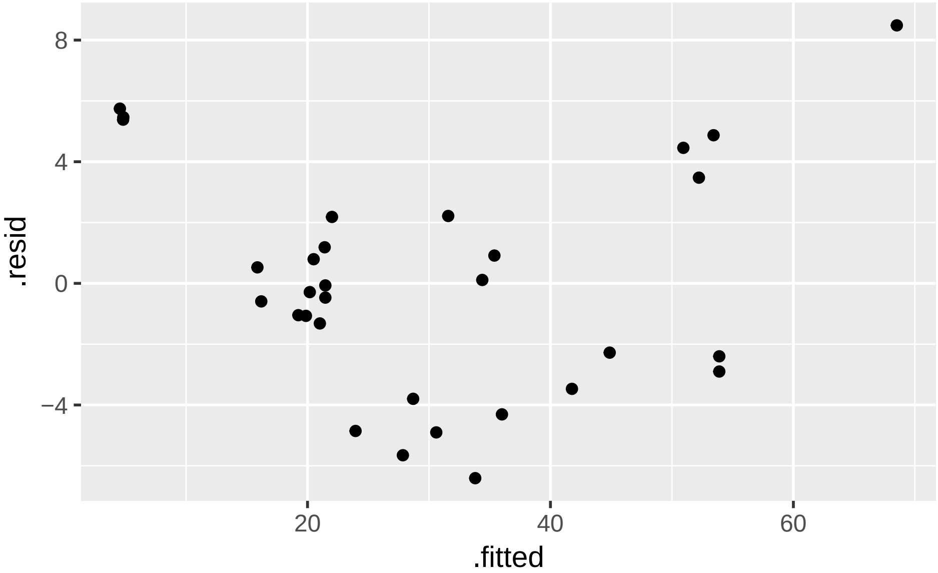
1. and 2. are verified with a plot of residuals versus fitted values, and 3. is verified with a normal quantile plot of the residuals.

residuals versus fitted values - trees example (fatal)



not surprising, since the model was obviously wrong

If you really wanted to model the $y =$`Volume` of wood using $x_1 =$`Girth` and $x_2 =$`Height`, you need to include the square of `Girth`, because of the volume-of-a-cylinder formula $V = \pi r^2 h$.

So let's fit the model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1^2 + \varepsilon$$

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A few comments:

1. Order of input variables doesn't matter. It can be nice to "add" variables at the end, so that when comparing this model with

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

the original β's are at least conceptually similar.

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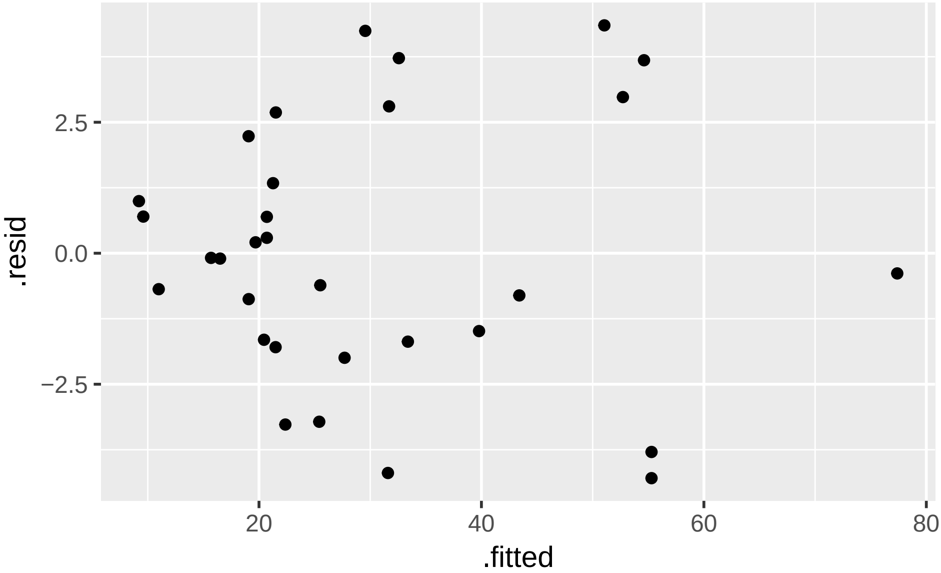
   $$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

   the original β's are at least conceptually similar.
2. When adding squares of variables (etc.), usually best to keep the original in the model as well.

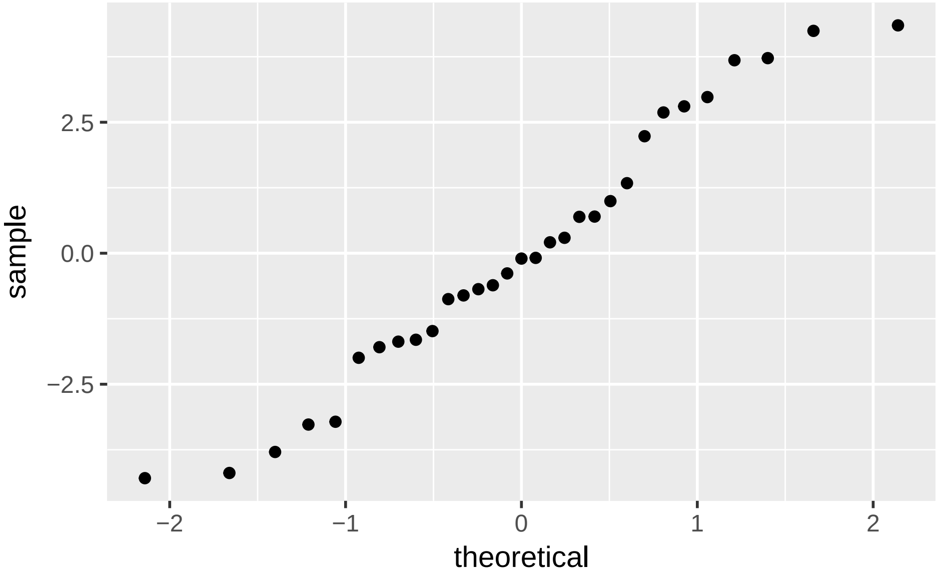
new trees model fit

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  -9.9204    10.0791   -0.98  0.33373
## Girth        -2.8851     1.3099   -2.20  0.03634
## I(Girth^2)    0.2686     0.0459    5.85  3.1e-06
## Height        0.3764     0.0882    4.27  0.00022
##
## Residual standard error: 2.6 on 27 degrees of freedom
## Multiple R-squared:  0.977,  Adjusted R-squared:  0.975
## F-statistic:  383 on 3 and 27 DF,  p-value: <2e-16
```

new trees model resids v. fits



normal quantile plot of residuals



towards an “adjusted” R^2

R^2 comes from dividing SS_{Total} through the SS decomposition:

$$SS_{Total} = SS_{Regression} + SS_{Error}$$

The definition $R^2 = SSR/SST = 1 - SSE/SST$ is the same no matter how many input variables there are.

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One use of R^2 is to compare two different regression models...

...but the problem is that R^2 always goes up when you add any new input variable to the model. This is because

$$SS_{Error}$$

always goes down with a new variable added.

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…but the problem is that R^2 always goes up when you add any new input variable to the model. This is because

$$SS_{Error}$$

always goes down with a new variable added.

For example, I can add a pure nonsense x_4 variable to the trees data and fit the “bigger” model.

trees vs. trees plus nonsense

The last best model we had:

```
##
## Residual standard error: 2.6 on 27 degrees of freedom
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```

trees vs. trees plus nonsense

The last best model we had:

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## Multiple R-squared:  0.977,  Adjusted R-squared:  0.975
## F-statistic:  383 on 3 and 27 DF,  p-value: <2e-16
```

With a `Nonsense` (randomly generated) variable added:

```
##
## Residual standard error: 2.5 on 26 degrees of freedom
## Multiple R-squared:  0.979,  Adjusted R-squared:  0.976
## F-statistic:  305 on 4 and 26 DF,  p-value: <2e-16
```

adjusting R^2 for the number of input variables

A more fair (but still not perfect) single-number-summary of a multiple regression fit is:

$$R^2_{adj} = 1 - \frac{MS_{Error}}{MS_{Total}}$$

where MS_{Total} is just another name for the sample variance of the output y values:

$$MS_{Total} = \frac{SS_{Total}}{n-1} = \frac{\sum\limits_{i=1}^{n} (y_i - \overline{y})^2}{n-1}$$

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$$MS_{Total} = \frac{SS_{Total}}{n-1} = \frac{\sum\limits_{i=1}^{n} (y_i - \overline{y})^2}{n-1}$$

The adjustment works on the basis of this trade-off: while SS_{Error}\$ goes down, the error degrees of freedom also goes down.

R^2_{adj} will play more of a role in the next topic—model selection

model selection preview

Recall the Body Fat % dataset.

```
## # A tibble: 250 × 15
##   `Pct BF`   Age Weight Height  Neck Chest Abdomen waist   Hip Thigh
##      <dbl> <int>  <dbl>  <dbl> <dbl> <dbl>   <dbl> <dbl> <dbl> <dbl>
## 1     12.3    23    154     68    36    93      85    34    94    59
## 2      6.1    22    173     72    38    94      83    33    99    59
## 3     25.3    22    154     66    34    96      88    35    99    60
## 4     10.4    26    185     72    37   102      86    34   101    60
## 5     28.7    24    184     71    34    97     100    39   102    63
## # ... with 245 more rows, and 5 more variables: Knee <dbl>,
## #   Ankle <dbl>, Bicep <dbl>, Forearm <dbl>, Wrist <dbl>
```

model selection preview

We had considered these two simple regression models:

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -14.6931     2.7605   -5.32  2.3e-07
## Weight        0.1894     0.0153   12.36  < 2e-16
```

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  25.5808    14.1540    1.81    0.072
## Height       -0.0932     0.2012   -0.46    0.644
```

model selection preview

Model with both. Is this a contradiction?

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  76.7810    10.0412    7.65  4.6e-13
## Weight        0.2633     0.0154   17.14  < 2e-16
## Height       -1.4883     0.1587   -9.38  < 2e-16
##
## Residual standard error: 5.6 on 247 degrees of freedom
## Multiple R-squared:  0.544,  Adjusted R-squared:  0.54
## F-statistic:  147 on 2 and 247 DF,  p-value: <2e-16
```