3.2.1 Bernoulli distribution

Definition 3.6. X has the **Bernoulli distribution** if $\text{Range}(X) = \{0,\ 1\}$ and

$$P(X = x) = p^x(1-p)^{1-x} \quad x = 0,\ 1,$$

for some $p \in [0,\ 1]$. We denote this $X \sim \text{Bern}(p)$.

The Bernoulli random variable is the simplest possible example of a random variable, and it serves as our canonical model of a binary trial: yes/no, true/false, success/failure, heads/tails, win/lose, 1/0. The parameter p is the *probability of success*:

$$P(X = 1) = p^1(1-p)^{1-1} = p^1(1-p)^0 = p.$$

And so $1 - p$ is the probability of failure by the complement rule. To compute the expected value of this distribution, we simply apply the definition:

$$E(X) = \sum_{i=0}^{1} iP(X = i) = 0(1-p) + 1p = p.$$

In most cases of interest, $0 < p < 1$, and so we see that $E(X) \notin \text{Range}(X) = \{0,\ 1\}$. So it is a strange property of the expectation that this "typical value" need not actually be one of the values that X could take on when the random phenomenon is finally realized. Curious!

Example 3.2. Here is a simple construction that shows how the Bernoulli distribution can arise from first principles. Recall the formal construction of a random variable:

- (**base space**) an underlying probability space $(S,\ \mathcal{A} \subseteq S,\ P_0)$;
- (**random variable**) a function $X : S \to \mathbb{R}$ that takes outcomes s and returns real numbers $X(s)$;
- (**pushforward space**) a new probability space $(\mathbb{R},\ \mathcal{B} \subseteq \mathbb{R},\ P)$ induced by the function X "pushing forward" the randomness of the base space to the real numbers. The new probability measure P is called the probability distribution of X.

Consider some fixed event of interest $A \subseteq \mathbb{R}$. We can define the *indicator function*:

$$I_A(s) = \begin{cases} 0 & s \notin A \\ 1 & s \in A. \end{cases} \tag{3.5}$$

So this is the function that *indicates* whether or not the event A occurs. $I_A : S \to \mathbb{R}$ is an **indicator random variable** with

- (**range**) $\text{Range}(I_A) = \{0,\ 1\}$
- (**distribution**) $P(I_A = 1) = P_0(A)$ and $P(I_A = 0) = P_0(A^c) = 1 - P_0(A)$.

So we see that $I_A \sim \text{Bern}(p = P_0(A))$.

3.2.2 Binomial distribution

Definition 3.7. X has the **binomial distribution** if Range$(X) = \{0, 1, 2, ..., n\}$ and

$$P(X = k) = \binom{n}{k} p^k(1-p)^{n-k} \quad k = 0, 1, 2, ..., n,$$

for some $p \in [0, 1]$. We denote this $X \sim \text{Binom}(n, p)$.

The binomial distribution arises when we define a random variable X that counts the number of successes that occur in a sequence of independent binary trials, each with the same probability of success $0 \leq p \leq 1$. We can use $X \sim \text{Binom}(n, p)$ to model various phenomena:

- the number of heads you observe in $n = 100$ flips of a fair ($p = 1/2$) coin;
- an assembly line manufactures $n = 200$ light bulbs. How many actually work?
- you swipe right on $n = 1000$ dating profiles. How many swipe right on you?
- you text a survey to $n = 50$ eligible voters. How many actually respond?
- you attempt $n = 40$ soufflés. How many come out of the oven light, risen, fluffy, and delectable?
- so in general, in n independent attempts, each with probability of success p, how many attempts turn out successful?

This is the sort of random variable that we build up from first principles. The underlying sample space S is the set of all length-n binary strings. Each digit is 0 or 1 indicating whether or not the trial was a failure or a success. Given a string, X is essentially counting the number of ones: $X(011010) = 3$. To see where the pmf of the binomial comes from, consider the case of $n = 3$. We start by enumerating the underlying sample space, and then seeing how the underlying probabilities get pushed forward:

P_0	S	$X = k$	$P(X = k)$
$(1-p)^3$	000	0	$(1-p)^3$
$p(1-p)^2$	100	1	$3p(1-p)^2$
$p(1-p)^2$	010		
$p(1-p)^2$	001		
$p^2(1-p)$	110	2	$3p^2(1-p)$
$p^2(1-p)$	101		
$p^2(1-p)$	011		
p^3	111	3	p^3

First, we see that Range$(X) = \{0, 1, 2, 3\}$, because in three trials you could have anywhere from no successes to all successes. Next, we see that, to compute $P(X = 1)$ for instance, we recognize that the event "$X = 1$" only happens if the original sequence is 100, or 010, or 001. This is a disjoint union, so

$$P(X = 1) = P_0(100 \cup 010 \cup 001) = 3p(1-p)^2.$$

$p(1-p)^2$ is the probability of observing exactly one success and exactly two failures in a sequence of three *independent* trials, and so $P(X=1)$ is equal to this baseline probability multiplied by the total number of ways it can happen. In the general n case, the event $X=k$ only occurs is we observe exactly k successes and exactly $n-k$ failures in our n trials. The individual probability of any one such outcome is $p^k(1-p)^{n-k}$, and so the overall probability of $X=k$ is this number multiplied by the total number of ways we could construct a length-n binary string with exactly k ones in it. This is a “select k from n” counting problem, where we have k slots set aside for successful trials, and we are selecting which of the n trials to designate as the successes. We are drawing without replacement, and order does not matter, so the total number of ways our length-n string could have exactly k ones is $\binom{n}{k}$, thus giving the final probability

$$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k},\quad k=0,1,2,...,n.$$

Next, let us compute the expected value. For now, we will do it by proceeding directly from the definition, but later we will develop tools that allow us to perform the calculation with much less pain:

$$\begin{aligned}
E(X)&=\sum_{k=0}^{n}kP(X=k)\\
&=\sum_{k=0}^{n}k\binom{n}{k}p^k(1-p)^{n-k}\\
&=\sum_{k=1}^{n}k\binom{n}{k}p^k(1-p)^{n-k}\\
&=\sum_{k=1}^{n}k\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}\\
&=\sum_{k=1}^{n}\frac{n!}{(k-1)!(n-k)!}p^k(1-p)^{n-k}\\
&=np\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!(n-k)!}p^{k-1}(1-p)^{n-k}\\
&=np\sum_{k=1}^{n}\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k}\\
&=np\sum_{i=0}^{n-1}\binom{n-1}{i}p^{i}(1-p)^{n-1-i}\\
&=np[p+(1-p)]^{n-1}\\
&=np.
\end{aligned}$$

At the end of the computation, we invoked the *binomial theorem*: $(x+y)^m=\sum_{j=0}^{m}\binom{m}{j}x^jy^{m-j}$ with $m=n-1$, $x=p$, and $y=1-p$. Indeed, it is due to their role in this theorem that the binomial coefficients get their name, and that the distribution we study gets *its* name. In any case,

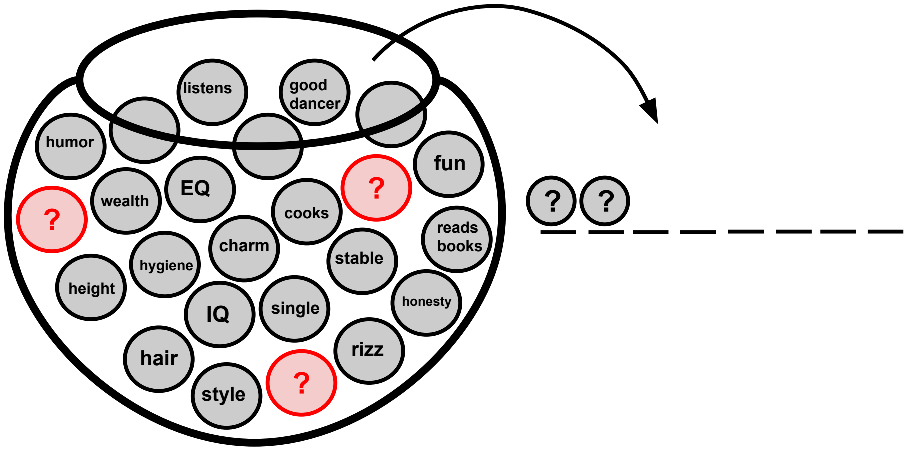


Figure 3.6: The silly model in Example 3.3 imagines that a given person has eight qualities that they value most in a romantic partner, which are randomly drawn from a set of 100 desirable qualities.

we see that this was a rather clumsy and inartful computation, and we will have cleaner methods for doing it later. The final result, though, is intuitive. It implies that the probability of success p has the interpretation as the proportion of the total number of trials that we expect to be a success: $p = E(X)/n$.

Example 3.3. Hopefully young people in college do not feel the need to use dating apps. But for us decrepit old hags that were foolish enough to defer adulthood to the autumn of their thirties, they are a regrettable part of the toolkit. So how do these work? Whelp, I make a profile, someone else on the app reviews it for no more than three seconds, and if I meet enough of their criteria, they swipe right (like my profile). If I fail to meet their criteria, they swipe left. If we both like each other's profiles, it's a match, and awkwardness ensues.

Let's model this. When the other person reviews my profile, they are deciding if I possess qualities they value. What might those qualities be? Charm, humor, wealth, height, education, hygiene, style, a deep and abiding appreciation for the films of Luis Buñuel, etc. Figure 3.6 imagines that the set of potentially desirable qualities are balls in the proverbial urn. For the purposes of discussion, let's say there are a total of one hundred qualities a person could potentially value (100 balls in the jar). Every human has a list of 8 that they care most about. Everyone's different, so we will model a given person's preferences as a random selection (without replacement!) of 8 balls from this jar of 100.

So, given a person's preferences, will they like my profile? Well, let's say that I possess exactly three of the one hundred desirable qualities. I won't tell you what those three are, but suffice it to say, we have yet to list them in Figure 3.6. We will treat my three qualities as three red balls in the jar, and the remaining 97 are black. Next, let A be the event that at least *two* of my qualities are among the eight that a person values. If this happens, then they will swipe right on my profile. Based on this, define an indicator random variable I_A:

$$I_A = \text{“do they like me?”} = \begin{cases} 0 \text{ (no)} & \text{if } A^c \text{ happens} \\ 1 \text{ (yes)} & \text{if } A \text{ happens.} \end{cases}$$

Given this setup, what is the probability that a random person on the app swipes right on my profile? This will be a job for our basic counting methods:

$$\begin{aligned} P(I_A = 1) = P(A) &= 1 - P(A^c) \\ &= 1 - P(\text{they value none or only one of my qualities}) \\ &= 1 - P(\text{none}) - P(\text{only one}) \\ &= 1 - \frac{\binom{97}{8}}{\binom{100}{8}} - \frac{\binom{3}{1}\binom{97}{7}}{\binom{100}{8}} \\ &\approx 1 - 0.777 - 0.207 \\ &= 0.016. \end{aligned}$$

Yikes! Time to consider the priesthood. But my prospects aside, we see that $I_A \sim \text{Bernoulli}(p = 0.016)$.

So that's what it looks like for one person. But as we all know, dating apps are a numbers game. One strategy is to go on there, indiscriminately swipe right on everyone until you run out of people in your area, wait for the matches to roll in, and then filter through only those folks that have already expressed interest in you. The dating app creators know this is a possibility though, and so they cap the number of right swipes you can send in a 24 hour period (unless you buy a premium subscription). Let's say that number is $n = 25$, as I believe it is on Bumble. So on a given day, I send 25 likes out into the world. How many of those $n = 25$ people will like me back, resulting in a match? Introduce a binary random variable I_i indicating whether or not person $i = 1, 2, ..., 25$ likes my profile. As we have seen, each indicator has $I_i \sim \text{Bernoulli}(0.016)$, and we will now assume that these are *independent*. We have yet to formally define this, but in context it means that the swiping decisions among these 25 people are not affecting one another. So none of them are sitting next to each other discussing my profile, in other words.

In this way, we see that each person I liked constitutes an independent binary trial (they like me back or they don't) with the same pitifully low probability of success, so if we define a new random variable X that counts the total number of matches I get, we have that

$$X = \sum_{i=1}^{25} I_i \sim \text{Binomial}(n = 25,\ p = 0.016).$$

My expected number of matches is thus a bleak $E(X) = np = 25 \cdot 0.016 \approx 0.4$, and the probability that I get at least one match is

$$
\begin{aligned}
P(X \geq 1) &= \sum_{k=1}^{25} \binom{25}{k} 0.016^k 0.984^{25-k} \\
&= 1 - P(X < 1) \\
&= 1 - P(X = 0) \\
&= 1 - \binom{25}{0} 0.016^0 0.984^{25-0} \\
&= 1 - 0.984^{25} \\
&\approx 0.33.
\end{aligned}
$$

Hit it, Barbra.

Remark 3.1. Figure 3.7 displays the pmf of the binomial distribution for various choices of $n \in \mathbb{N}$ and $p \in (0, 1)$. We see that, regardless what p is, the pmf resembles a bell-shaped curve more and more as n increases. This is an example of the **central limit theorem**, which we will study later.

3.2.3 Geometric distribution

Definition 3.8. X has the **geometric distribution** if $\text{Range}(X) = \{1, 2, 3, ...\}$ and

$$P(X = k) = (1 - p)^{k-1} p \quad k = 1, 2, 3, ...,$$

for some $p \in (0, 1)$. We denote this $X \sim \text{Geom}(p)$.

The geometric distribution arises when we define a random variable X that counts the number of independent binary trials we must sit through until we observe the first success. The first success could occur on the first attempt, or the second, or the third, or the billionth, so $\text{Range}(X) = \{1, 2, 3, ...\}$. The event that $X = k$ is equivalent to the event that the first $k - 1$ trials are failures (each occurs with probability $1 - p$) and the kth trial is a success (occurs with probability p). Since the trials are independent, the probability of $k - 1$ failures followed by a success is the product of these probabilities, like so:

k	Trial 1	Trial 2	Trial 3	Trial 4	$\cdots$	Trial k	$\cdots$	$P(X = k)$
1	1							p
2	0	1						$(1-p)^1 p$
3	0	0	1					$(1-p)^2 p$
4	0	0	0	1				$(1-p)^3 p$
$\vdots$								$\vdots$
k	0	0	0	0	$\cdots$	1		$(1-p)^{k-1} p$
$\vdots$								$\vdots$

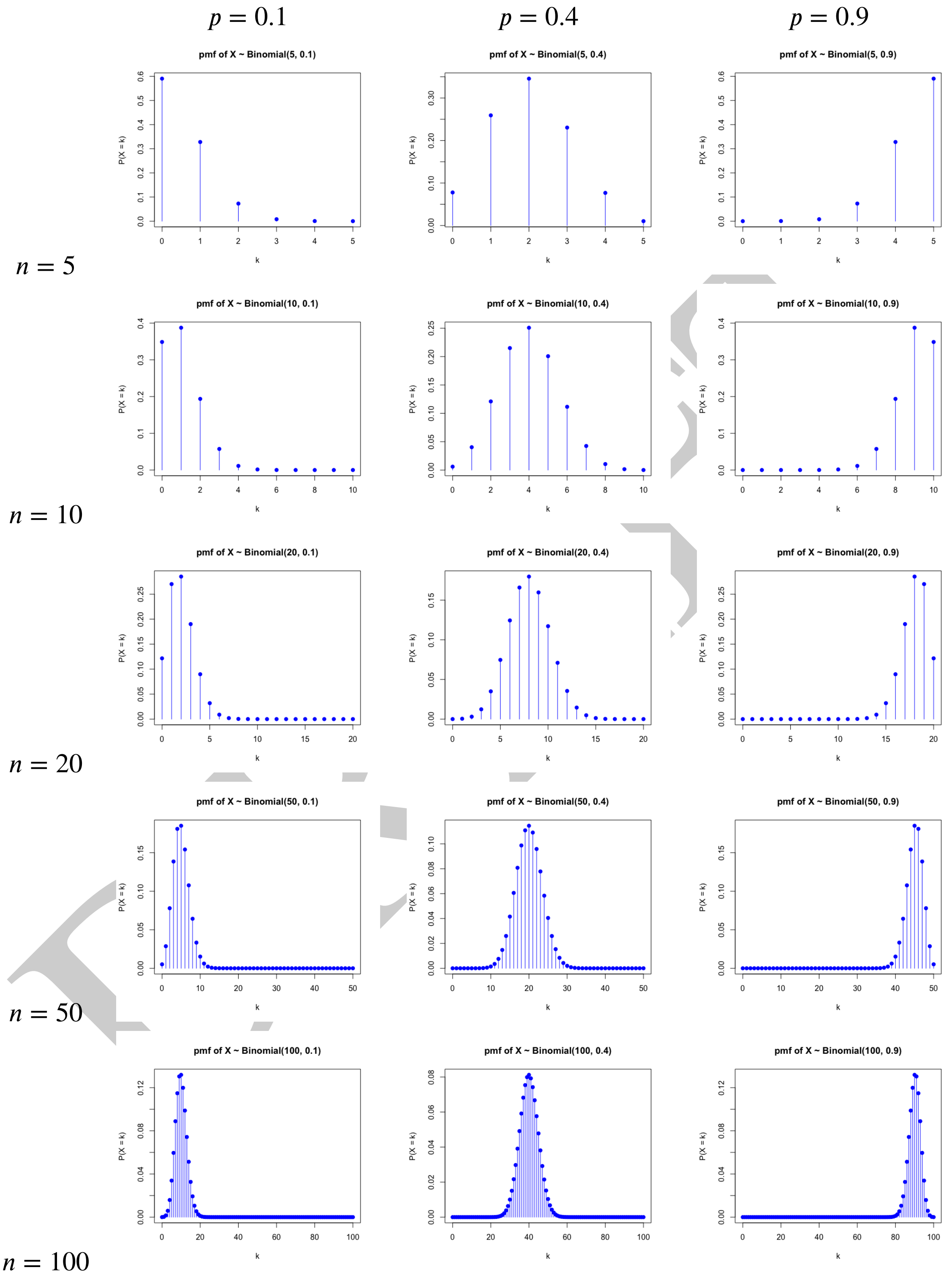


Figure 3.7: As n increases the pmf of Binom(n, p) looks more like a bell curve.

To see where this distribution gets its name, let us verify that the pmf sums to one:

$$\begin{aligned}
\sum_{k=1}^{\infty} P(X=k) &= \sum_{k=1}^{\infty}(1-p)^{k-1}p \\
&= p\sum_{k=1}^{\infty}(1-p)^{k-1} \\
&= p\sum_{i=0}^{\infty}(1-p)^{i} \\
&= p\frac{1}{1-(1-p)} && \text{geometric series, since } |1-p|<1 \\
&= p\frac{1}{p} \\
&= 1.
\end{aligned}$$

Next let us calculate the expected value:

$$\begin{aligned}
E(X) &= \sum_{k=1}^{\infty} kP(X=k) \\
&= \sum_{k=1}^{\infty} k(1-p)^{k-1}p \\
&= \sum_{k=1}^{\infty}(k+0)(1-p)^{k-1}p \\
&= \sum_{k=1}^{\infty}(k-1+1)(1-p)^{k-1}p \\
&= \sum_{k=1}^{\infty}[(k-1)(1-p)^{k-1}p+(1-p)^{k-1}p] \\
&= \sum_{k=1}^{\infty}(k-1)(1-p)^{k-1}p+\sum_{k=1}^{\infty}(1-p)^{k-1}p \\
&= \sum_{i=0}^{\infty} i(1-p)^{i}p+1 \\
&= 0+\sum_{i=1}^{\infty} i(1-p)^{i}p+1 \\
&= (1-p)\sum_{i=1}^{\infty} i(1-p)^{i-1}p+1 \\
&= (1-p)E(X)+1.
\end{aligned}$$

Solving $E(X)=(1-p)E(X)+1$, we have that $E(X)=1/p$.

Example 3.4. The first probability problem we encountered was “how many flips of a fair coin does it take on average until you flip the first head?” A sequence of fair coin flips is a sequence of independent binary trials with probability of success $p = 1/2$, and so the number of flips until the first head has $X \sim \text{Geom}(1/2)$. We see then that the expected number of flips until the first head is $E(X) = 1/(1/2) = 2$. So on average, the first head occurs on the second flip.

3.2.4 Poisson distribution

Definition 3.9. X has the **Poisson distribution** if $\text{Range}(X) = \mathbb{N} = \{0, 1, 2, 3, ...\}$ and

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!} \quad k \in \mathbb{N} \tag{3.6}$$

for some *rate* $\lambda > 0$. We denote this $X \sim \text{Poisson}(\lambda)$.

The Poisson distribution is often used to model the number of random *arrivals* in a given window of time: the number of emails you receive in an hour, the number of claims an insurance company receives in a month, the number of α-particles discharged from a radioactive material, etc.

We first check that the pmf is valid:

$$\sum_{k=0}^{\infty} P(X = k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1.$$

Next we compute the expected value:

$$\begin{aligned}
E(X) &= \sum_{n=0}^{\infty} nP(X = n) \\
&= \sum_{n=0}^{\infty} n\frac{\lambda^n}{n!}e^{-\lambda} \\
&= e^{-\lambda}\sum_{n=0}^{\infty} n\frac{\lambda^n}{n!} && \text{Pull out constant} \\
&= e^{-\lambda}\sum_{n=1}^{\infty} n\frac{\lambda^n}{n!} && n = 0 \text{ term is equal to } 0 \\
&= e^{-\lambda}\sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} && n \text{ cancels} \\
&= \lambda e^{-\lambda}\sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} && \text{Pull out } \lambda \\
&= \lambda e^{-\lambda}\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} && \text{Reindex} \\
&= \lambda e^{-\lambda}e^{\lambda} && \text{Recall Taylor series: } e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}\ \forall\, x \in \mathbb{R} \\
&= \lambda.
\end{aligned}$$

Example 3.5. Let's continue with the theme of Example 3.3. When you post your profile on a dating app, everyone in your area can see and swipe on it. If someone swipes right on you, the app will usually alert you to the fact that this happened, but they will charge you for the privilege of seeing who exactly it is. So let X be the number of "likes" you receive in a given twenty-four hour period. We could model this using the Poisson distribution and say that $X \sim \text{Poisson}(\lambda)$. If we wish to model the likes received by some disgraced undesirable like James Corden, we might pick $X \sim \text{Poisson}(0.01)$, which has $E(X) = 0.01$. Suffice it to say, not a tremendous number of likes rolling in for that guy, on average. If we wish to model the likes received by Taylor Swift, we might pick $X \sim \text{Poisson}(20,000,000)$. You get the idea. For a normal guy, perhaps $X \sim \text{Poisson}(5)$, so the probability of them beating their average for the day is

$$P(X \geq 5) = 1 - P(X < 5) = 1 - P(X \in \{0, 1, 2, 3, 4\}) = 1 - \sum_{k=0}^{4} e^{-5}\frac{5^k}{k!} \approx 0.56.$$

Remark 3.2. Figure 3.8 displays the pmf of the Poisson for different values of the rate parameter λ. As λ increases, we see that the pmf shifts rightward, which makes sense given that $E(X) = \lambda$. We also see that it gets wider, which we will make sense of later when we show that $\text{var}(X) = \lambda$ as well. Lastly, we see that the pmf looks more and more bell-like as λ grows, similar to what we observed with the binomial distribution in Figure 3.7. These of course are not coincidences. Both are instances of the same general phenomenon: the central limit theorem.

3.2.5 Discrete uniform distribution

Definition 3.10. X has the (**discrete**) **uniform distribution** if it has a finite range and all values are equiprobable. So $\text{Range}(X) = \{x_1, x_2, ..., x_n\} \subseteq \mathbb{R}$ for some $n \in \mathbb{N}$ and

$$P(X = x_i) = \frac{1}{n}, \quad \forall i = 1, 2, ..., n.$$

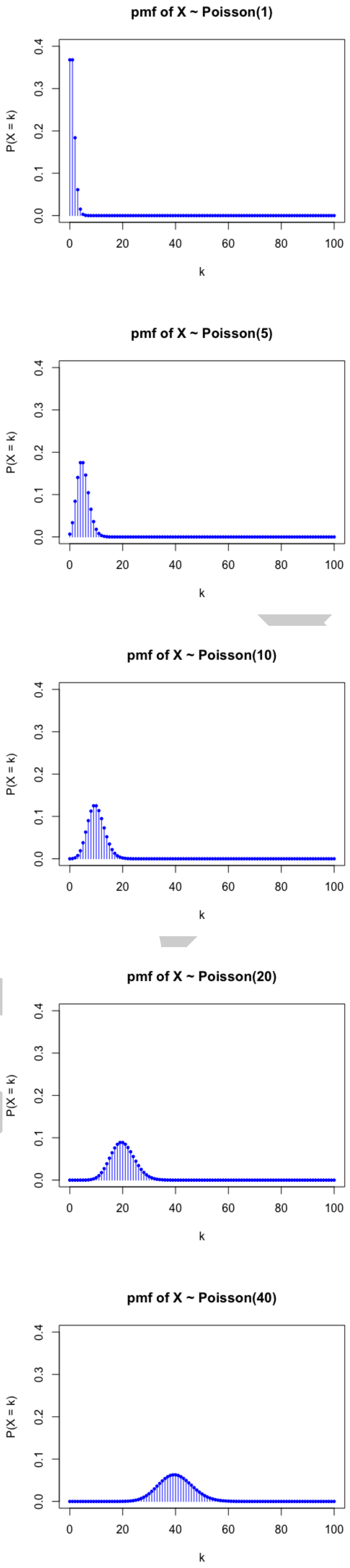
We denote this $X \sim \text{Unif}(x_1, x_2, ..., x_n)$.

Figure 3.9 displays some examples of what the pmf might look like. Because all of the probabilities are the same, it has none of the peaks and valleys and tails we usually expect from a distribution plot. It's just *uniform* across the range. Furthermore, note that this is our first discrete random variable that is supported on a set of arbitrary real numbers. The Bernoulli, binomial, geometric, and Poisson distributions are all supported on a subset of $\mathbb{N}$, but this is not a requirement for a discrete random variable. Any finite (or countably infinite) set of real numbers will serve, and here we have our first.

The expected value of this distribution is

$$E(X) = \sum_{i=1}^{n} x_i P(X = x_i) = \sum_{i=1}^{n} x_i \frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

This is the familiar **average**. So whenever we compute the average or *mean* of a set of n numbers, we are implicitly treating that set of numbers as the range of a discrete random variable with uniform $(1/n)$ probabilities, and then calculating the expected value.



$\lambda = 1$

$\lambda = 5$

$\lambda = 10$

$\lambda = 20$

$\lambda = 40$

Figure 3.8: As λ increases, the Poisson pmf shifts right, widens, and becomes more bell-like.

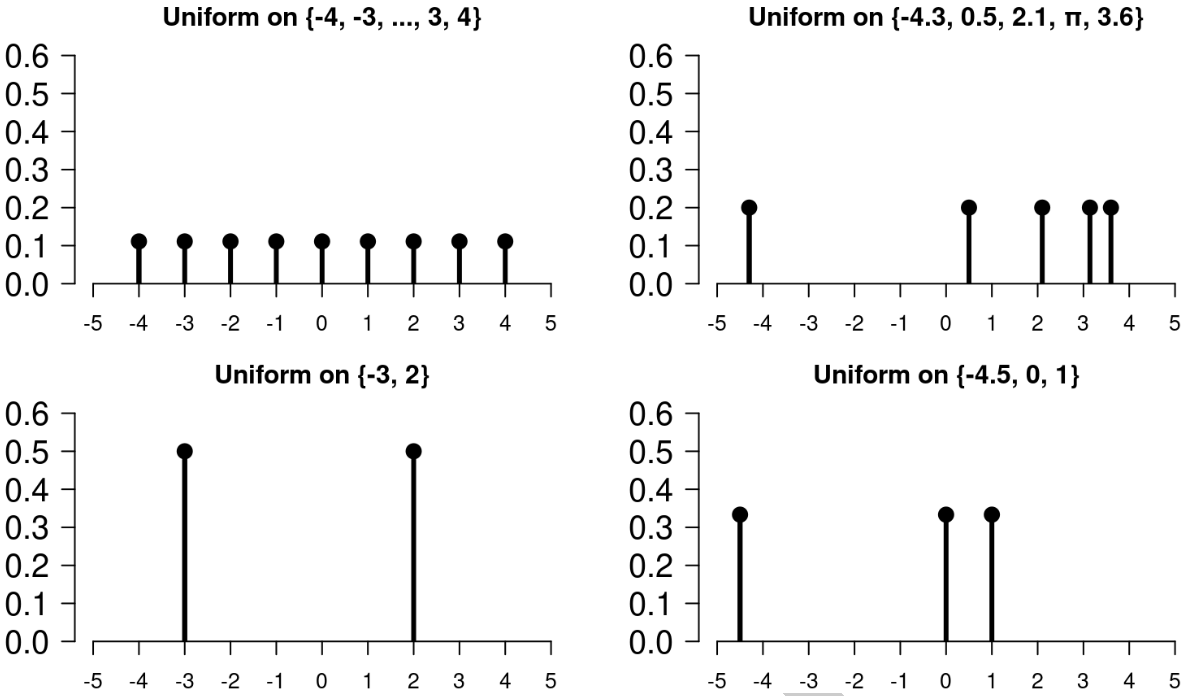


Figure 3.9: Uniform distributions on different sets of real numbers.