- Sample space $S$ - collection of outcomes
  - used at beginning
  - arbitrary and difficult at times.

$\{H, T\}$ $\quad$ $[0,1]$ interval

$\{A♠, A♣, A♦, A♥\}$ $\quad$ $\{H, TH, TTH, TTTH, \dots\}$

1. $\{HH, HT, TH, HH\}$
2. $\{\{H,H\}, \{H,T\}, \{T,T\}\}$

} arbitrary, or context sensitive.

- Event $A, B, C$: (suitable) subset of $S$.
  - when $S$ is finite or countable, all subsets are "suitable"
- Algebra of events include $A^c$, $A \subset B$, $A \setminus B = A \cap B^c$, $A \cup B$, $A \cap B$

  Definition of "disjoint"

  Assoc.
  Dist.
  Comm.
- "Probability": LTRF interp.
  "Bayesian" subjective interp
  Others... philosophy topic.

Our approach is purely axiomatic

- A "Probability measure" is a function $P : \mathcal{A} \to [0,1]$
  Domain: a collection of suitable events $\mathcal{A}$
  Range: $[0,1]$
  Properties:
  1. $P(S) = 1$ (normalized to 1)
  2. $P(A) \geq 0$ (redundant)
  3. If $A_1, A_2, A_3, \dots$ are disjoint then $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

Different from book but not that important

Q1 Are these axioms "consistent"? Yes.

eg. $S = \{H, T\}$
$\mathcal{A} = \{\phi, \{H\}, \{T\}, S\}$

Set $P(\{H\}) = P(\{T\}) = \frac{1}{2}$

$P(\emptyset) = 0$

$P(S) = 1$ . With Prop ① and ② below, done.

Q2. (Advanced) When $S = [0,1]$ can we just use $\mathcal{Q} = 2^S$ (all subsets)?

NO then ①, ②, ③ are inconsistent.

So $\mathcal{Q}$ has to be restricted. ① unions ② complements!

Prop ⓪ $P(\emptyset) = 0$. Proof Let $A_1 = A_2 = \dots = \emptyset$. Then...

Prop ① (ILLUSTRATION OF AXIOMATIC APPROACH)

If $A_1 \cap A_2 = \emptyset$ then

$$P(A_1 \cup A_2) = P(A_1 \cup A_2) \quad //$$

Proof: Let $A_3, A_4, \dots$ all be $\emptyset$.

Then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_i P(A_i) =$

$$P(A_1 \cup A_2) = P(\bigcup A_i) = \sum P(A_i) = P(A_1) + P(A_2)$$

//

Prop ③ $P(A^c) = 1 - P(A)$

Proof: $A^c \cap A = \emptyset$ and $A^c \cup A = S$

So $1 = P(A^c \cup A) = P(A^c) + P(A)$

//

Prop ④ If $A \subset B$ then $P(A) \leq P(B)$

Proof:

$$B = A \cup (B \cap A^c)$$

$$P(B) = P(A) + P(B \cap A^c)$$

So $P(A) \leq P(B)$ since $P(\cdot) \geq 0$

Prop (4a) $P(A \setminus B) = P(A) - P(A \cap B)$

and if $B \subseteq A$ then $= P(A) - P(B)$

**Prop 5** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

**Proof** $A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$

So: $P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$

$$P(A) = P(A \cap B^c) + P(A \cap B)$$

$$P(B) = P(A \cap B) + P(A^c \cap B)$$

$$P(A) + P(B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B) + P(A \cap B)$$

$$P(A) + P(B) - P(A \cap B) = P(A \cup B) \quad \blacksquare$$

**EXAMPLE** $S = [0,1]$, $P((a,b)) = b - a$ "based on"

Recall "continuous function"

One version if $x_1 \le x_2 \le x_3 \le \dots \to x$

then $f(x_i) \to f(x)$ ("left continuous")

(similarly, "right cont.")

If left and right, then continuous.

Let's consider "left continuous" for $P(\cdot)$

We need a notion of $A_n \nearrow A$. We define this notion as follows:

$$A_i \subseteq A_{i+1} \quad \text{and} \quad \bigcup_{i=1}^{\infty} A_i = A$$

Example: $\quad S = [0,1]$

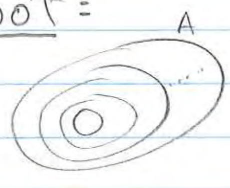
$$A_1 = \left[0, \tfrac{1}{2}\right] = \left[0, 1 - \tfrac{1}{2^1}\right]$$
$$A_2 = \left[0, \tfrac{3}{4}\right] = \left[0, 1 - \tfrac{1}{2^2}\right]$$
$$\vdots$$
$$A_n = \left[0, 1 - \tfrac{1}{2^n}\right]$$

then $A_i \subseteq A_{i+1}$ and $\bigcup_{i=1}^{\infty} A_i = [0,1) = A$

so

Theorem: If $A_n \nearrow A$ then $P(A_n) \to P(A)$

Proof:




$$B_1 = A_1$$
$$B_2 = A_2 \setminus A_1$$
$$B_3 = A_3 \setminus A_2$$
$$B_n = A_n \setminus A_{n-1}$$

Then $B_1, B_2, \ldots$ are disjoint.

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i = A$$

$$P(B_1) = P(A_1)$$
$$P(B_2) = P(A_2) - P(A_1)$$
$$P(B_3) = P(A_3) - P(A_2)$$

$$\sum_{i=1}^{n} P(B_i) = P(A_n)$$

$$\lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i)$$
$$= \sum_{i=1}^{\infty} P(B_i)$$
$$= P\left(\bigcup_{i=1}^{\infty} B_i\right)$$
$$= P(\cup A_i) = P(A)$$ "Left continuous"

Corollary: If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ and $\bigcap_{i=1}^{\infty} A_i = A$ then $P(A_n) \to P(A)$

"Right continuous"

We will use this corollary a few times.

Finite and countable $S$

Suppose $S = \{\omega_1, \omega_2, \dots \omega_n\}$ "finite"
or $\{\omega_1, \omega_2, \dots\}$ "countable"

A valid $P(\cdot)$ is defined by

$$P(\{\omega_i\}) = p_i \quad \text{with } 0 \le p_i \le 1 \text{ and } \sum p_i = 1$$

and $$P(A) = \sum_{\omega_i \in A} P(\{\omega_i\})$$

"theorem without proof"

Permutations: $_nP_k = \frac{n!}{(n-k)!}$

Combinations: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$= \frac{n\cdot(n-1)\cdot\dots\cdot(n-k+1)}{k!}$$

$$\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!}$$

Room has $k$ people. Chance of a shared birthday?

$S$ = all birthdays

$|S| = 365^n$

$A$ = "shared birthday"

$|A^c| = {}_{365}P_k$

$P(A) = 1 - \frac{_{365}P_k}{365^n}$

Exceeds 0.5 at $n=23$.

Lotto 6/49 $\binom{49}{6} \approx 14\,000\,000$

Lotto Max $\binom{49}{7}$, but 3 draws per ticket

$\approx 86\,000\,000$

---

Binomial theorem.

$\boxed{\text{Also}}$ Negative extension: $\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!}$

DEFINITION

$A = (0, \frac{1}{2})$
$B = (\frac{1}{2}, 0)$ $P(A \cap B)$

$A \subseteq D = \{1, 2, 5\}$

$(A \cup B)^c = \{\frac{1}{2}\}$
$P(A \cup B) = \frac{1}{2} + \frac{1}{2} - 1$

Conditional Probability

$S_B = B \quad P_B(A) = \frac{1}{3}$ $\quad P_B$ is a new P.

$S_C = C \quad P_C(A) = \frac{1}{2}$ $\quad$ "

LTRF

HIV Example $\quad$ Sens 99.3 - 99.7
$\quad$ Spec 99.91 - 99.97

$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

B : Is HIV+
A : Tests HIV+

$P(A|B) = 0.995$ (true positive)
$P(B) = 0.00212$ (prevalence)

$P(A \cap B) = \cancel{0.0398}$ $\quad$ "People who know they are HIV+"
$0.0021094$

$P(A) = 0.0021094 + 0.005(0.99788)$
$= 0.0070988$

Also:
$P(A \cap B \cap C) = P(A | B \cap C) P(B|C) P(C)$

and so on

Independence: If $P(A)=0$

$P(A\cap B)=0$

so $P(A\cap B)=P(A)P(B)$ no matter what

$S \perp S$. $\emptyset \perp \emptyset$

Don't confuse motivation with definition

"any" and "all"

$A_1 \ldots A_n$

$P(A)=P(A_1 \cup \ldots \cup A_n)$ Hard!

$P(A^c)=P(A_1^c \cap \ldots A_n^c) = (1-p)^n$

The Four Quantities

$$p^n \qquad 1-p^n$$

$$(1-p)^n \qquad 1-(1-p)^n$$

Pre - RV ⓪ Pay \$1

① Toss coin

(1) $H \to 1$ (Receive \$1 plus \$1)

$T \to -1$ (Lose entry fee)

(1a) $H \to 10$ (different fee)

$T \to -10$

Toss coin repeatedly until H appears.

Count # of tosses

(2)

$H \to 1$

$TH \to 2$

$TTH \to 3$

$TTTH \to 4$

$TTTTH \to 5$

$f(\{H\}) = 1$

$f(\{TTTH\}) = 3$

(3) Pick a real number $\omega$ between 0 and 1 uniformly

(Identity function on S)

$f(\omega) = \omega$

(3a) Pick a real number $\omega$ between 0 and 1

$f(\omega) = -\log \omega$

## Probabilities of values

$$\{H, T\} \xrightarrow{X} \{1, -1\}$$

$$P(\{\omega : X(\omega) = 1\})$$
$$= P(\{H\})$$
$$= 1/2$$

IS USUALLY WRITTEN

$$P(X=1) = 1/2$$

Notice: $P(\{\omega : X(\omega) \le 0.42\})$

$$= P(\{X \le 0.42\}) = P(X \le 0.42) = 1/2$$

$$P(\{\omega : X(\omega) > -1.86\}$$
$$= P(X > -1.86) = 1$$

$$P(\{\omega : X(\omega) > 0\} \cap \{\omega : X(\omega) < 2\})$$
$$= P(0 < X < 2) = \frac{1}{2}$$

$$P(X \le 42) = 1$$

and so on.

We can assign probabilities to the events $\{X \in A\}$ where $A \subseteq \mathbb{R}$ is interval-like: $X \le 0.42$

$0 \le X \le 2$

etc.

as well as $X = 1$ $\quad P(X=1) = 1/2$

$X = -1$ $\quad P(X=-1) = 1/2$

$X = 0.42$ $\quad P(X=.42) = 0$

and so on.

Example (2) $X$ is # of tosses to first H.

$$P(X=1) = 1/2$$
$$P(X=2) = 1/2 \cdot 1/2 = \frac{1}{4}$$
$$P(X=k) = \frac{1}{2^k}$$
$$P(X \le 3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$
$$P(X < 3) = \frac{1}{2} + \frac{1}{4}$$

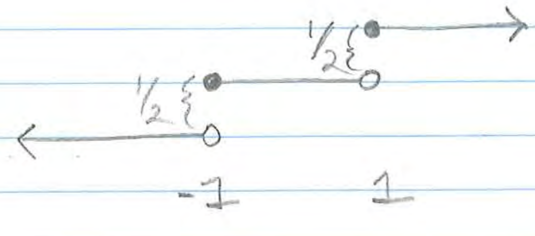
Example (3) $X$ is selected from $(0,1)$ uniformly.

$$P(X \le 1/2) = 1/2$$
$$P(X = 1/2) = 0$$
$$P(X \le 1/2) = P(\{X < 1/2\} \cup \{X = 1/2\})$$
$$= P(X < 1/2) + P(X = 1/2)$$
$$1/2 = P(X < 1/2) + 0$$

So $P(X \le 1/2) = P(X < 1/2)$ !

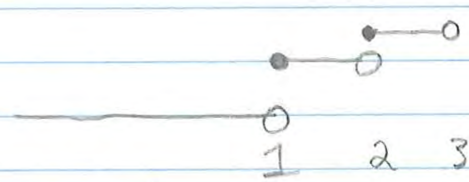
## Example ①

$$F(x) = \begin{cases} 0 & : x < -1 \\ 1/2 & : -1 \le x < 1 \\ 1 & : x \ge 1 \end{cases}$$




## Example ②

$$F(x) = \begin{cases} 0 & : x < 1 \\ \sum_{i=1}^{k} \frac{1}{2^i} & : k \le x < k+1 \\ & \text{for } k \in \{1, 2, \dots\} \end{cases}$$




**Theorem** $\lim_{x \to \infty} F(x) = 1$

**Proof:** We will use the sequence def.$^{n}$ of function limit, which is:

$$\lim_{x \to a} f(x) = c \quad \Leftrightarrow \quad \lim_{n \to \infty} f(x_n) = c$$

for all $x_n \to x$

Consider any $x_n \nearrow \infty$. Define $A_n = (-\infty, x_n]$

$A_i \subseteq A_{i+1}$ and $\bigcup_{i=1}^{\infty} A_i = (-\infty, \infty)$

So $\lim_{n\to\infty} P(A_n) = P(\bigcup_{i=1}^{\infty} A_i) = 1$ ∎

P.M.F. examples

1. $p(-1) = \frac{1}{2}$

   $p(1) = \frac{1}{2}$

   $\left( p(x) = 0 \quad \text{otherwise} \right)$

2. $p(k) = P(X=k) = \frac{1}{2^k}$

   for $k \in \{1, 2, 3, \dots\}$

Proof

Given $p(x)$, $F(x) = \sum_{x \le y} p(y)$

Given $F(x)$, $p(x) = P(X=x)$

$= P(\{X \le x\} \setminus \{X < x\})$

$= P(X \le x) - P(X < x)$

"size of jump" $= F(x) - \lim_{y \to x-} F(y)$

Lemma $P(X < x) = \lim_{y \to x-} F(y)$

Let $y_n \le y_{n+1} \nearrow x$ and $A_n = (-\infty\ y_n)$

$\bigcup A_n = (-\infty, x)$ so $F(y_n) \nearrow P(\bigcup A_n)$

$= P(X < x)$

Outcome $A$. $X = I_A$

$\mathcal{Q}_1 = \{ A_1 , A_1^c \}$

$\mathcal{Q}_2 = \{ A_2 , A_2^c \}$

$\vdots$

The classes of events are independent

Binomial p.m.f.: say $n = 4$

How could $X = 2$?

$0\,0\,1\,1 \quad p^2(1-p)^2$

$1\,0\,1\,0 \quad p^2(1-p)^2$

$\vdots$

$\binom{n}{k}$ ways.

Binomial Theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$$1 = (p + (1-p))^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$\boxed{\sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}}$$ Geometric "Key Fact"

Binomial etc.

$$\boxed{\begin{aligned} P(X=0) &= (1-p)^n \\ P(X=n) &= p^n \end{aligned}}$$

$$\sum_{k=1}^{\infty}(1-p)^{k-1}p$$

$$= p\sum_{k=1}^{\infty}(1-p)^{k-1}$$

$$= p\cdot\frac{1}{1-(1-p)}$$

$$= 1$$

Geometric is "memoryless"

$$P(X > (n-1)+k \mid X > n-1)$$

$$= \frac{P(\{X > (n-1)+k \mid X > n-1)}{P(X > n-1)}$$

$$= \frac{P(X > (n-1)+k)}{P(X > n-1)}$$

$$=$$

Neg Bin

$$\underbrace{0\ 0\ 1\ 0\ 0\ 0\ 1\ 0}_{\substack{k-1 \\ r-1}}\ 1$$

$\uparrow$ success on $k$

$$P(X=k) = \binom{k-1}{r-1}p^{k-1}(1-p)^{k-r}\cdot p$$

Identity:

$$(1-a)^{-w} = \sum_{w=r}^{\infty}\binom{w-1}{r-1}a^{w-r} \quad \text{for } |a|<1$$

Memoryless

$$P(X > (n-1)+k \mid X > n-1)$$

$$= \frac{P(\{X > (n-1)+k\} \cap \{X > n-1\})}{P(X > n-1)}$$

$$= \frac{P(X > (n-1)+k)}{P(X > n-1)}$$

$$= \frac{(1-p)^{(n-1)+k}}{(1-p)^{n-1}}$$

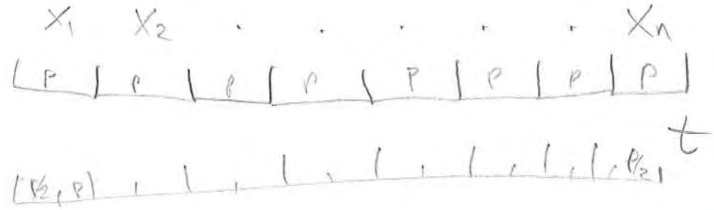
$$= (1-p)^k = P(X > k)$$

Note: $P(X \leq k) = 1 - P(X > k)$

$$= 1 - (1-p)^k$$

$\binom{r}{\cancel{k}}\Big(^{n-\cancel{m}}$

$$\binom{r}{k}\binom{n-r}{m-k} \Big/ \binom{n}{m} \approx \binom{r}{k} p^k (1-p)^{m-k}$$



$$\lim_{n\to\infty} P(X_n = k) = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k}$$

$$= \lim_{n\to\infty} \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k}$$

$$= \frac{(\lambda t)^k}{k!} \lim_{n\to\infty} \frac{n!}{(n-k)!} \frac{1}{n^k} \left(1 - \frac{\lambda t}{n}\right)^n \left(1 - \frac{\lambda t}{n}\right)^{-k}$$

$$= \frac{(\lambda t)^k}{k!} \lim_{n\to\infty} \underbrace{\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{n \cdot n \cdot n \qquad n}}_{k \text{ terms}} \left(1 - \frac{\lambda t}{n}\right)^n \left(1 - \frac{\lambda t}{n}\right)^{-k}$$

$$= \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

- Convergence is ~~f~~ fast

$$\lim_{n\to\infty} P(X_n \in A) = P(Y \in A) \quad \text{for all } A$$

Actually $\left| P(X_n \in A) - P(Y \in A) \right| \leq \frac{\lambda^2}{n}$

(other better bounds)

$$\binom{k-1}{r-1} p^r (1-p)^{k-r} \qquad k \in \{r, r+1, \dots\}$$

$$1 = \sum_{k=r}^{\infty} \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

$$p^{-r} = \sum_{k=r}^{\infty} \binom{k-1}{r-1} (1-p)^{k-r}$$

$$(1-(1-p))^{-r} = \sum_{k=r}^{\infty} \binom{k-1}{r-1} (1-p)^{k-r}$$

$$(1-a)^{-r} = \sum_{k=r}^{\infty} \binom{k-1}{r-1} a^{k-r}$$

$$\binom{k-1}{r-1} = \binom{k-1}{k-r}$$

$r-1$ $\quad 1+r-1$

$1$

$\boxed{1549}$ 20 years or less

15 years at turnaround.

started 5 years ago

few changes

2 3 5

7.1.1. $P(N(2) > 3)$

$= 1 - P(N(2) \leq 2)$

$= 1 - \left[ \frac{2^0 e^{-1}}{0!} + \frac{2^1 e^{-1}}{1!} + \frac{2^2 e^{-1}}{2!} \right]$

$= 0.323$

$P( N(1) > 1, N(2) - N(1) > 1 )$

$= P(N(1) > 1) \, P(N(2) - N(1) > 1)$

$= (0.264)(0.264)$

<u>Theorem</u>: Density characterizes dist$^{\underline{n}}$.

<u>Proof</u>: For any $x$

$$F(x) = F(x) - 0$$

$$= \int_{-\infty}^{x} f(u)\,du$$

So c.d.f. can be calculated from p.d.f.

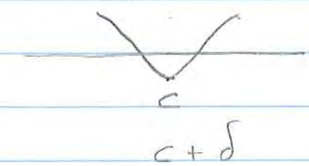
A "Fundamental Theorem of Calculus" says

$$F'(x) = f(x)$$

(anywhere $f$ is continuous)

Corollaries: (1) $F$ is continuous
(true by FTC) ∎

(2) $f \geq 0$. Suppose $f(c) < 0$ for some $c$, and is continuous at $c$.




$$\int_{c-\delta}^{c+\delta} f(x)\,dx < 0$$

$$F(c+\delta) < F(c-\delta)$$

∎

(3) $$\int_{-\infty}^{\infty} f(x)\,dx = F(\infty) - F(-\infty) = 1 - 0$$

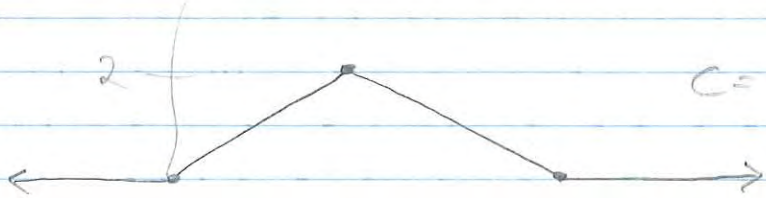
$X \sim \text{Uniform}[-1,2]$

$$P(0 < X < 3/2) = \int_0^{3/2} f(x)\,dx = \int_0^{3/2} \frac{1}{3}\,dx = \frac{1}{2}$$

$$P(-3 < X \leq 0) = \int_{-3}^{0} f(x)\,dx = \int_{-3}^{-1} 0\,dx + \int_{-1}^{0} \frac{1}{3}\,dx$$

$$f(x) = \begin{cases} cx & : \quad 0 < x \le 1 \\ c(2-x) & : \quad 1 < x \le 2 \\ 0 & : \quad \text{otherwise} \end{cases}$$

Picture




$c = 1$ !!!

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 cx\,dx + \int_1^2 c(2-x)\,dx$$

$$= \frac{cx^2}{2}\Big|_0^1 - \frac{c(2-x)^2}{2}\Big|_1^2$$

$$= \frac{c}{2} + \frac{c}{2} = c$$

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

$$= \begin{cases} 0 & : \quad x < 0 \\ \int_0^x u\,du = \frac{x^2}{2} & : \quad 0 \le x < 1 \\ \frac{1}{2} + \int_1^x (2-u)\,du & \\ \quad = \frac{1}{2} + \left[-\frac{(2-u)^2}{2}\right]_1^x & \\ \quad = \frac{1}{2} + \left[-\frac{(2-x)^2}{2} + \frac{1}{2}\right] & \\ \quad = 1 - \frac{(2-x)^2}{2} & : \quad 1 \le x \le 2 \\ 1 & : \quad \text{otherwise.} \end{cases}$$

$$P(0.75 < X < 1.5)$$

$$= \int_{0.75}^{1.5} f(x)\,dx$$

$$= \int_{0.75}^{1} x\,dx + \int_{1}^{1.5} (2-x)\,dx$$

$$= 0.59375$$

Poisson waiting time

$$P(X \le x) = 1 - P(X > x)$$

$$= 1 - P(N(x) = 0)$$

$$= 1 - \frac{(\lambda x)^0 e^{-\lambda x}}{0!}$$

$$= 1 - e^{-\lambda x}$$

Memoryless

$$P(X > t+s \mid X > s)$$

$$= \frac{P(X > t+s,\ X > s)}{P(X > s)}$$

$$= \frac{P(X > t+s)}{P(X > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda(s)}}$$

$$= e^{-\lambda t} = P(X > t)$$

Unique

$$S(t+s) = S(t)\,S(s)$$

Only solutions: $e^{ax}$, $0$, $1$

Exponential (5)

$$P(X>3) = \int_3^\infty \frac{1}{5} e^{-x/5}\,dx$$

$$= -e^{-x/5}\Big|_3^\infty$$

$$= e^{-3/5}$$

$$= 0.549$$

$$Y \sim \text{Binomial}(20, \overset{p}{0.549})$$

$$P(Y>10) = \sum_{k=11}^{20} \binom{20}{k} p^k (1-p)^{20-k}$$

$$\approx 0.587$$

Gamma

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u}\, dx$$

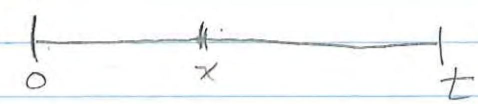
Let $u = \lambda x$

$du = \lambda\, dx$

$$\Gamma(\alpha) = \int_0^\infty (\lambda x)^{\alpha-1} e^{-\lambda x} \lambda\, dx$$

$$1 = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx$$

Poisson - Uniform




Between 0 and $t$:

$$P(X \le x) = 1 - P(X > x)$$

$$= 1 - P(N(x) = 0 \mid N(t) = 1)$$

$$= 1 - \frac{P(N(x) = 0, N(t) = 1)}{P(N(t) = 1)}$$

$$= 1 - \frac{P(N(x) = 0,\ N(t) - N(x) = 1)}{P(N(t) = 1)}$$

$$= 1 - \frac{P(N(x)=0)\,P(N(t)-N(x)=1)}{P(N(t)=1)}$$

$$= 1 - \frac{e^{-\lambda x}\,\lambda(t-x)\,e^{-\lambda(t-x)}}{\lambda t e^{-\lambda t}}$$

$$= 1 - \left(1 - \frac{x}{t}\right)$$

$$= \frac{x}{t}$$

Density:

$$f(x) = F'(x) = \begin{cases} \frac{1}{t} & : 0 \le x \le t \\ 0 & : \text{otherwise.} \end{cases}$$

So $X \sim \text{Uniform}(0, t)$

//

Poisson - Binomial.

At fixed time $t$ ...

$$P(X=k) = P(N(s)=k \mid N(t)=n)$$

$$= \frac{P(N(s)=k,\ N(t)=n)}{P(N(t)=n)}$$

$$= \frac{P(N(s)=k,\ N(t)-N(s)=n-k)}{P(N(t)=n)}$$

$$= \frac{\frac{(\lambda s)^k e^{-\lambda s}}{k!} \frac{(\lambda(t-s))^{n-k} e^{-\lambda(t-s)}}{(n-k)!}}{\frac{(\lambda t)^n e^{-\lambda t}}{n!}}$$

$$= \frac{n!}{k!(n-k)!} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$$

Normal Densities

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$0 = \frac{d}{dx} f(x) = -\frac{1}{\sigma^2}(x-\mu) f(x) \quad (\text{at } x = \mu)$$

$$0 = \frac{d^2}{dx^2} f(x) = -\frac{1}{(\sigma)^2} f(x) + \frac{1}{\sigma^2}(x-\mu)^2 f(x)$$

$$\text{at} \begin{pmatrix} x = \mu + \sigma \\ x = \mu - \sigma \end{pmatrix}$$

Symmetry

$$e^{-(\mu - a - \mu)^2} = e^{-(\mu + a - \mu)^2}$$

Because of Symmetry:

$$P(X-\mu \leq -a) = P(X-\mu \geq a)$$

$$P(-a \leq X-\mu \leq a) = 2P(X \leq a)$$

etc.

CDF of $Y = a + bX$

$$F_Y(y) = P(Y \leq y)$$

$$= P(g(X) \leq y)$$

$$= P(a+bX \leq y)$$

$$= \begin{cases} P\left(X \leq \frac{y-a}{b}\right) & : \quad b>0 \\ P\left(X \geq \frac{y-a}{b}\right) & : \quad b<0 \end{cases}$$

$X \sim N(\mu, \sigma^2) \Rightarrow Z$

$$g(x) = \underbrace{-\frac{\mu}{\sigma}}_{a} + \underbrace{\frac{1}{\sigma}}_{b} x$$

$$f_Z(z) = f_X\left(\frac{z+\frac{\mu}{\sigma}}{1/\sigma}\right) \cdot \sigma$$

$$= \sigma f_X(\sigma z + \mu)$$

$$= \sigma \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\sigma z + \mu - \mu)^2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

## SAT example

$$X \sim N(1020, 194)$$

$$P(X > 1250)$$

$$= P\left(\frac{X-1020}{194} \leq \frac{1250-1020}{194}\right)$$

Let $Z = \frac{X-1020}{194}$. $\quad Z \sim N(0,1)$

$$= P(Z > 1.19)$$

$$= 1 - P(Z \leq 1.19) = 1 - 0.8830$$

$$= 0.117$$

## Uniform example

$$f_X(x) = \begin{cases} 1 & : \quad x \in [0,1] \\ 0 & : \quad \text{otherwise.} \end{cases}$$

$$f_Y(y) = \frac{1}{|b|} f_X\left(\frac{y-a}{b}\right)$$

Case (1) $b>0$ $\quad = \begin{cases} \frac{1}{b} & : \quad y \in [a, a+b] \\ 0 & : \quad \text{otherwise.} \end{cases}$

Case (2) $b<0$ $\quad = \begin{cases} \frac{1}{b} & : \quad y \in [a+b, a] \\ 0 & : \quad \text{otherwise.} \end{cases}$

cdf of $X^2$

$$F_Y(y) = P(Y \le y)$$
$$= P(X^2 \le y)$$
$$= P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \left(-\frac{1}{2\sqrt{y}}\right)$$
$$\left[ = \frac{1}{\sqrt{y}} f_X(\sqrt{y}) \quad \text{(when symmetric)} \right]$$

example $\quad Y = Z^2$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-1/2(\sqrt{y})^2}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$= \frac{{1/2}^{1/2}}{\sqrt{\pi}} y^{1/2 - 1} e^{-y/2}$$

This <u>must</u> integrate to 1.

Therefore $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ !!

## Change of variables

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$
$$= P(X \le g^{-1}(y))$$
$$= F_X(g^{-1}(y))$$
$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

## Change of variables II

$$\int_c^d f_X(x)\,dx$$

$$y = g(x)$$
$$x = g^{-1}(y) \qquad dx = \frac{d}{dy} g^{-1}(y)\,dy$$

$$= \int_{g^{-1}(c)}^{g^{-1}(d)} f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)\,dy$$

$g^{-1} \nearrow$ or $\searrow$ etc.

Example: $X \sim \text{Unif}[0,1]$ $\quad Y = a + bX$, $b < 0$

$$\int_c^d 1\,dx \; ; \quad \begin{matrix} y = a + bx \\ dy = b\,dx \end{matrix} \quad = \int_{a+b\cdot c}^{a+b\cdot d} \frac{1}{b}\,dy = \int_{a+bd}^{a+bc} \frac{1}{|b|}\,dy$$

# Change of variables III

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$
$$= P(X \le g^{-1}(y))$$
$$= \int_{-\infty}^{g^{-1}(y)} f_X(x)\, dx$$

So $f_Y(y) = f_X(g^{-1}(y)) \left| \underbrace{\frac{d}{dy} g^{-1}(y)}_{\text{chain rule.}} \right|$

Prob. Int. Trans. $\text{Exp}(\lambda)$

$P(Y \le y) = 0$ for $y < 0$ and $1$ for $y > 1$

$$P(Y \le y) = P(1 - e^{-\lambda X} \le y)$$
$$= P(e^{-\lambda X} \ge 1 - y)$$
$$= P(X \le -\log(1-y)/\lambda)$$
$$= 1 - e^{-\lambda(-\log(1-y)/\lambda)}$$
$$= 1 - (1-y) = y$$

So $Y \sim \text{Unif}[0,1]$

$$P(Y \le y) = P(F_X(X) \le y)$$

$$= P(F_X^{-1}(F_X(X)) \le F_X^{-1}(y))$$

$$= P(X \le F^{-1}(y))$$

$$= F_X(F_X^{-1}(y)) = y$$

for $y \in [0,1]$

$U[0,1]$ to $X$ via $F^{-1}(U)$

$$Y = F_X^{-1}(U)$$

$$P(Y \le y) = P(F_X^{-1}(U) \le y)$$

$$= P(U \le F_X(y))$$

$$= F_X(y)$$

so $Y \sim X$

# Inverse Gamma

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0$$

$$F_Y(y) = \begin{cases} 0 & : \quad y < 0 \\ P(Y \le y) & : \quad y \ge 0 \end{cases}$$

$$= P(1/X \le y)$$

$$= P\left(X \ge \frac{1}{y}\right)$$

$$= 1 - P\left(X \le \frac{1}{y}\right)$$

$$f_Y(y) = f_X\left(\frac{1}{y}\right)\frac{1}{y^2}$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \left(\frac{1}{y}\right)^{\alpha-1} e^{-\lambda/y} \frac{1}{y^2}$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{-\alpha-1} e^{-\lambda/y}$$

On $y > 0$, $0$ otherwise.

Applications: "Bayesian Statistics"

Inverse Gaussian.

$$X \sim N(\mu, \sigma^2)$$

$$g(x) = \frac{1}{|x|}, \quad Y = g(X)$$

$$P(Y \le y) = P\left(\frac{1}{|X|} \le y\right)$$

$$= P\left(|X| \ge \frac{1}{y}\right)$$

$$= P(X \le -y) + P(X \ge y)$$

$$= P(X \le -y) + 1 - P(X \le y)$$

$$f_Y(y) = f_X(-y)(-1)$$

$$g(x) = |x|$$

$$P(|X| \le y)$$

$$= P(-y \le X \le y)$$

$$= P(X \le y) - P(X \le -y)$$

$$2 f_X(y)$$

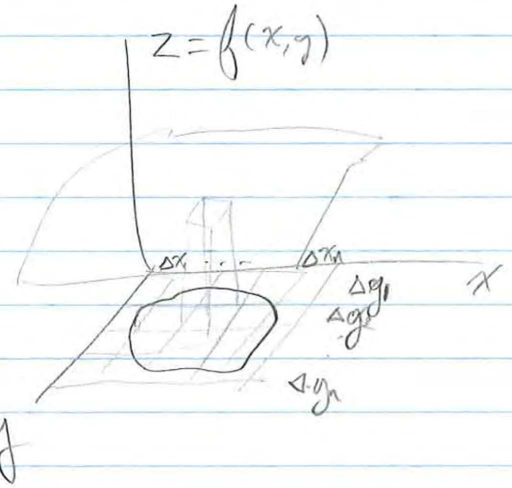
| | | X | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| Y | 0 | 1/36 | 0 | 1/36 | 0 | 1/36 | 0 | 1/36 | 0 | 1/36 | 0 | 1/36 |
| | 1 | 0 | 2/36 | 0 | 2/36 | 0 | 2/36 | 0 | 2/36 | 0 | 2/36 | 0 |
| | 2 | 0 | 0 | 2/36 | 0 | 2/36 | 0 | 2/36 | 0 | 2/36 | 0 | 0 |
| | 3 | 0 | 0 | 0 | 2/36 | 0 | 2/36 | 0 | 2/36 | 0 | 0 | 0 |
| | 4 | 0 | 0 | 0 | 0 | 2/36 | 0 | 2/36 | 0 | 0 | 0 | 0 |
| | 5 | 0 | 0 | 0 | 0 | 0 | 2/36 | 0 | 0 | 0 | 0 | 0 |

Volumes under Surfaces

$P(4 \leq X_1 \leq 8,\ 2 \leq X_2 \leq 5)$

How about $P(4 \leq X_1 \leq X_2 + 2,\ 2 \leq X_2 \leq 5)$

Systematically one variable at a time.




$$\sum_i \sum_j g(x_i, y_i)\, \Delta x_i\, \Delta y_i$$

$$\int_0^2 \int_1^3 xy^2 \, dx dy$$

$$= \int_0^2 \left[ \int_1^3 xy^2 \, dx \right] dy$$

$$= \int_0^2 \left[ \frac{x^2 y^2}{2} \right]_1^3 dy \qquad y^2 \left[ \frac{9}{2} - \frac{1}{2} \right]$$

$$= \int_0^2 4y^2 \, dy$$

$$= \frac{4}{3} y^3 \Big|_0^2$$

$$= \frac{32}{3}$$

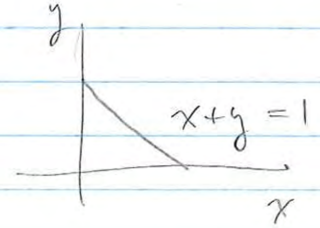
$$\int_1^3 \int_0^2 xy^2 \, dy \, dx$$

$$= \int_1^3 \left[ x \frac{y^3}{3} \right]_0^2 dx$$

$$= \int_1^3 \frac{8}{3} x \, dx$$

$$= \frac{8}{6} x^2 \Big|_1^3$$

$$= \frac{64}{6} = \frac{32}{3}$$




$$\int_0^1 \int_0^{1-y} xy^2 \, dx \, dy = \int_0^1 \int_0^{1-x} xy^2 \, dy \, dx$$

$$= \int_0^1 \left[ \frac{x^2}{2} y \right]_0^{1-y} dy$$

$$= \int_0^1 \frac{(1-y)^2 y}{2} \, dy = \frac{1}{24}$$

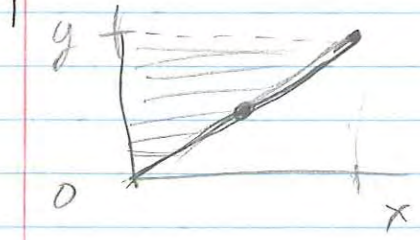
---

Joint Uniform

$$\iint_{\mathbb{R}^2} f(x,y) \, dx \, dy = \int_0^1 \int_0^1 1 \, dx \, dy$$

$$= \int_0^1 1 \, dy = 1$$

$$\int_0^{1/2} \int_0^{1/2} 1 \, dx \, dy = \int_0^{1/2} \frac{1}{2} \, dy = \frac{1}{4}$$




$$\int_0^1 \int_0^y 1 \, dx \, dy$$

$$= \int_0^1 y \, dy = \frac{1}{2}$$

$$1 = \int_0^1\int_0^2 cxy \, dy\,dx = \int_0^1 \frac{cxy^2}{2}\Big|_{y=0}^{2} dx$$

$$= \int_0^1 c\frac{x^2}{2}y\Big|_0^2 d \qquad = \int_0^1 c2x \, dx$$

$$= \int_0^1 2cy \, dy \qquad = cx^2\Big|_0^1$$

$$= cy^2\Big|_0^1 \qquad = c$$

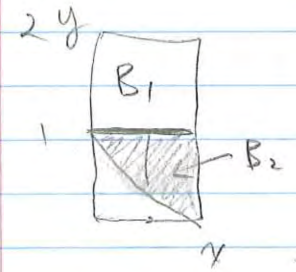
$$= c$$

$$\int_0^1\int_{0.5}^1 xy \, dx\,dy$$

$$= \int_0^1 \frac{x^2}{2}y\Big|_{1/2}^1 dy$$

$$= \int_0^1 \frac{3}{8}y \, dy$$

$$= \frac{3}{16}y^2\Big|_0^1 = 3/16$$




$$P(Y > X) = \iint_{B_1 \cup B_2} f(x,y)\,dx\,dy$$

$$= \int_0^1\int_1^2 xy \, dy\,dx + \int_0^1\int_x^1 xy \, dy\,dx$$

$B_1$ $\qquad\qquad$ $B_2$

$$= \int_0^1 x\frac{y^2}{2}\Big|_1^2 \, dx \qquad = \int_0^1 x\frac{y^2}{2}\Big|_x^1 \, dx$$

$$= \int_0^1 \frac{3}{2}x \, dx \qquad = \int_0^1 \frac{x}{2} - \frac{x^3}{2} \, dx$$

$$= \frac{3}{4}x^2\Big|_0^1 \qquad = \left[\frac{x^2}{4} - \frac{x^4}{8}\right]_0^1$$

$$= \frac{3}{4} \qquad = \frac{1}{8}$$

$$P(Y > X) = \frac{7}{8}$$

---

$$\frac{\partial^2}{\partial x \partial y} F(x,y) = \frac{\partial^2}{\partial x \partial y} \int_{-\infty}^{y} \int_{-\infty}^{x} f(u,v) \, du \, dv$$

$$= \frac{\partial}{\partial y} \int_{-\infty}^{y} \frac{\partial}{\partial x} \int_{-\infty}^{x} f(u,v) \, du \, dv$$

$$= \frac{\partial}{\partial y} \int_{-\infty}^{y} f(x,v) \, dv = f(x,y)$$

---

F

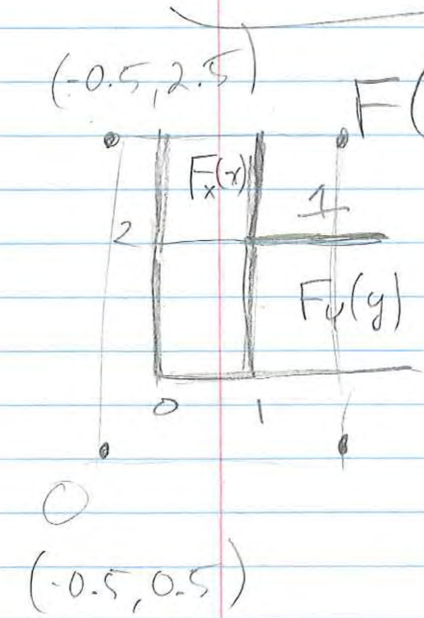
## Marginal CDF

$$F_X(x) = P(X \le x)$$

$$= \lim_{y \to \infty} P(X \le y,\ Y \le y)$$

$$= \lim_{y \to \infty} F(x,y)$$

$$\boxed{= \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(u,v)\, dv\, du} = \int_{-\infty}^{x} f(u)\, du$$




$$F(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ 1 & x > 1 \text{ and } y > 2 \\ F_X(x) & 0 < x < 1 \quad y > 0 \\ F_Y(y) & x > 1 \text{ and } 0 < y < 2 \end{cases}$$

$$\int_0^x \int_0^y uv\, dv\, du \qquad 0 < x < 1 \qquad 0 < y < 2$$

$$= \int_0^x \frac{uv^2}{2} \Big|_0^y du$$

$$= \int_0^x \frac{uy^2}{2}\, du = \frac{u^2 y^2}{4} \Big|_0^x = \frac{x^2 y^2}{4}$$

Example $\quad f(x,y) = xy \qquad 0 < x < 1$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < y < 2$

$$F_X(x) = \begin{cases} \int_{-\infty}^{x}\int_{-\infty}^{\infty} uv\,dv\,du & * \\ 0 & x < 0 \\ 1 & x > 1 \end{cases}$$

Not Needed ~~just~~ use $x \to \infty$ $y \to \infty$

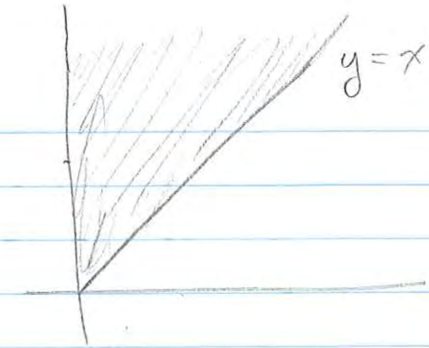
$$* \quad \int_0^x \int_0^2 uv\,dv\,du$$

$$= \int_0^x \frac{uv^2}{2}\Big|_0^2 du$$

$$= \int_0^x 2u\,du$$

$$= u^2\Big|_0^x = x^2$$

$$F_Y(y) = \begin{cases} 0 & : \quad y < 0 \\ \int_0^y \int_0^1 uv\,du\,dv = \int_0^y \frac{v}{2}\,dv = \frac{y^2}{4} \\ 1 & : \quad y > 2 \end{cases}$$




$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{x}^{\infty} \lambda^2 e^{-\lambda y}\,dy$$

$$= -\lambda e^{-\lambda y}\Big|_x^{\infty}$$

$$= \lambda e^{-\lambda x}$$

So $X \sim \text{Exponential}(\lambda)$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^y \lambda^2 e^{-\lambda y}\,dx$$

$$= \lambda^2 x e^{-\lambda y}\Big|_0^y$$

$$= \lambda^2 y e^{-\lambda y}$$

$$= \frac{\lambda^2 y^{2-1} e^{-\lambda y}}{\Gamma(2)}$$

So $Y \sim \text{Gamma}(2, \lambda)$

**Thm** Product of cdf ...

Let $A_x = (-\infty, x]$ and $B_y = (-\infty, y]$ for any $x, y \in \mathbb{R}$.

$$
\begin{aligned}
F(x,y) &= P(X \le x,\ Y \le y) \\
&= P(X \in (-\infty, x],\ Y \in (-\infty, y]) \\
&= P(X \in (-\infty, x])\,P(Y \in (-\infty, y]) \\
&= F_X(x)\,F_Y(y)
\end{aligned}
$$

**Example**

$$
f(x,y) = \begin{cases} xy & : \ 0 < x < 1,\ 0 < y < 2 \\ 0 & : \ \text{otherwise.} \end{cases}
$$

$$
I_X(x) = I_{\{x \in (0,1)\}} = \begin{cases} 1 & : \ x \in (0,1) \\ 0 & : \ \text{otherwise} \end{cases}
$$

$$
I_Y(y) = I_{\{y \in (0,2)\}}(y) = \begin{cases} 1 & : \ y \in (0,2) \\ 0 & : \ \text{otherwise} \end{cases}
$$

$$
I_X(x)\,I_Y(y) = \begin{cases} 1 & : \ x \in (0,1),\ y \in (0,2) \\ 0 & : \ \text{otherwise.} \end{cases}
$$

$$
\begin{aligned}
f(x,y) &= xy\, I_X(x)\, I_Y(y) \\
&= [x I_X(x)][y I_Y(y)]
\end{aligned}
$$

example $f(x,y) = \lambda^2 e^{-\lambda y} \quad 0<x<y<\infty$

Density factors but not a rectangle!

example $f(x,y) = \dfrac{\lambda^3 y e^{-\lambda(x+y)}}{2} \quad \begin{matrix} x>0 \\ y>0 \end{matrix}$

$$= \left[\lambda e^{-\lambda x}\right]\left[\frac{\lambda^2}{2} y e^{-\lambda(x+y)}\right]$$

---

Conditional Distribution.

Fix $x_1 = 5$

$$P(X_2 = 1 \mid X_1 = 5) = \frac{2}{36} \Big/ \frac{4}{36} = \frac{1}{2}$$

$$P(X_2 = 3 \mid X_1 = 5) = \frac{2}{36} \Big/ \frac{4}{36} = \frac{1}{2}$$

$$P(X_2 = k \mid X_1 = 5) = 0 \quad \text{otherwise.}$$

Let X be # times I answer

$$p_{X|Y=n}(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

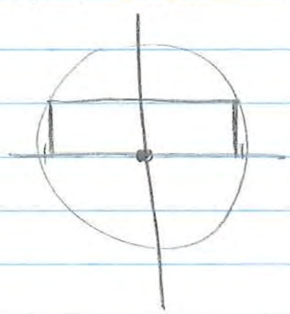
$$p_Y(n) = \frac{\lambda^n e^{-\lambda}}{n!}$$

$$p_X(x) = \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{(\lambda p)^k e^{-\lambda p}}{k!} \quad \text{(see book)}$$

---

Uniform on unit circle.



$y = 0.5$

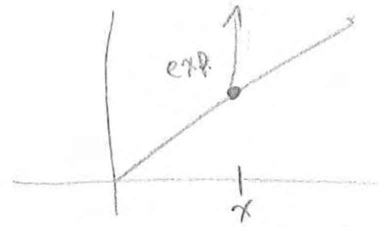
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1+y^2}} \frac{1}{\pi}\, dy$$

$$= \frac{2}{\pi} \sqrt{1-y^2}$$

So $f_{X|Y}(x|y) = \dfrac{\frac{1}{\pi}}{\frac{2}{\pi}\sqrt{1-y^2}}$

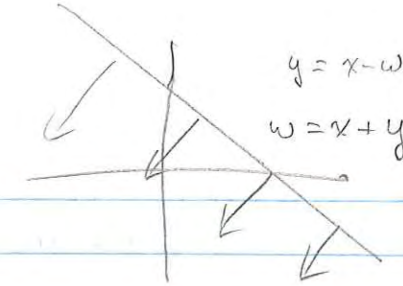
$$= \frac{1}{2\sqrt{1-y^2}} \quad -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$$




Recall: $f_X(x) = \lambda e^{-\lambda x}$

So $f_{Y|X=x} = \dfrac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}}$

$= \lambda e^{-\lambda(y-x)}$ on $y \geq x$



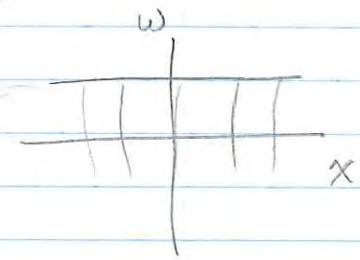

Start with cdf

$$F_{W_1}(w) = P(W_1 \le w) = P(X+Y \le w)$$

$$= \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{w-x} f(x,y)\,dy}_{\text{hold } x \text{ fixed}}\,dx$$

$$= \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{x+w-x\,=\,w} f(x,u-x)\,du\,dx}_{①}$$

$u = x+y$
$y = u-x$
$du = dy$




$$F_{W_1}(w) = \int_{-\infty}^{w} \underbrace{\int_{-\infty}^{\infty} f(x,u-x)\,dx}_{\text{function of } u}\,du$$

$$\boxed{f_{W_1}(w) = \int_{-\infty}^{\infty} f(x, w-x)\,dx}$$

<u>Example</u> $X \perp Y \quad N(0,1) \qquad W_1 = X + Y$

$-\infty < W_1 < \infty$

$$f_{W_1}(w) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-(w-x)^2/2} \, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(-\frac{1}{2}\left(x^2 + w^2 - 2wx + x^2\right)\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(-\frac{1}{2}\left(2x^2 - 2wx + \frac{w^2}{2} + \frac{w^2}{2}\right)\right)$$

$$= e^{-\frac{1}{2}\left(\frac{w^2}{2}\right)} \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(-\frac{1}{2}\left(\sqrt{2}x - \frac{w}{\sqrt{2}}\right)^2\right) dx$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \; e^{-\frac{1}{2}\left(\frac{w^2}{2}\right)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\left(\frac{1}{2}\right)} \exp\left(-\frac{1}{2}\left(\frac{x - \frac{w}{2}}{1/\sqrt{2}}\right)^2\right)$$

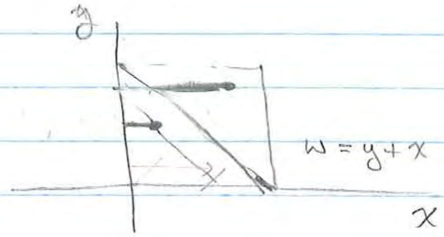
4 2 1 2 4 3

$$= \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2} w^2/2} \cdot \underline{1}$$

So $W_1 \sim N(0,2)$

<u>Example</u> $\quad X \perp Y \quad \text{Unif}(0,1)$

$$W_1 = X + Y$$




$$0 < W_1 < 2$$

$$f_{W_1}(w) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx$$

Notice: $\quad f_X(x) f_Y(w-x) = 1 \quad$ when:

$$\boxed{0 < x < 1}$$

<u>and</u> $\quad 0 < w - x < 1 \iff \boxed{w-1 < x < w}$

$$\boxed{w - 1 < \boxed{x}} < w$$

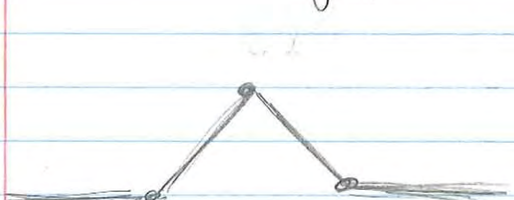
relevant when $w > 1$ $\qquad$ relevant when $w < 1$

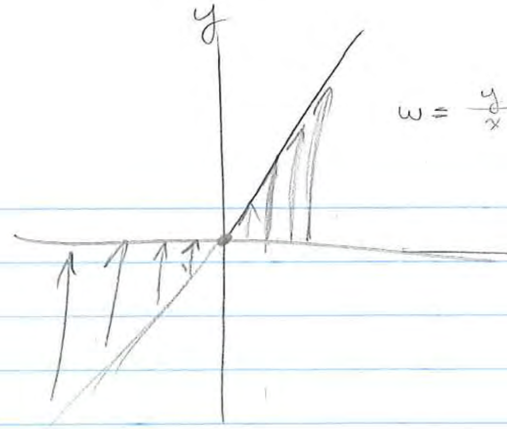
So:

$$f_{W_1}(w) = \int_0^w 1\, dx = w \quad : \quad 0 < w < 1$$

$$f_{W_1}(w) = \int_{w-1}^{1} 1\, dx = 1 - w + 1$$

$$= 2 - w \quad : \quad 1 \le w < 2$$






$W_2 = X/Y$

Start with cdf:

$$F_{W_2}(w) = P(W_2 \le w)$$

$$= P\left(\frac{Y}{X} \le w\right)$$

$$= P(Y \ge wX, X<0) + P(Y \le wX, X>0)$$

$$F_{W_2}(w) = \int_{-\infty}^{0}\int_{wx}^{\infty} f(x,y)\,dy\,dx + \int_{0}^{\infty}\int_{-\infty}^{wx} f(x,y)\,dy\,dx$$

$$y = xu \qquad dy = x\,du$$

$$= \int_{-\infty}^{0}\int_{w}^{-\infty} f(x,xu)\,x\,du\,dx + \int_{0}^{\infty}\int_{-\infty}^{w} f(x,xu)\,x\,du\,d$$

$$= \int_{-\infty}^{0}\int_{-\infty}^{w} f(x,xu)(-x)\,du\,dx + \int_{0}^{\infty}\int_{-\infty}^{w} f(x,xu)\,x\,du\,d$$

$$= \int_{-\infty}^{w}\int_{-\infty}^{\infty} f(x,xu)\,|x|\,dx\,du$$

$$f_{W_2}(w) = \int_{-\infty}^{\infty} f(x,xw)\,|x|\,dx$$

Example $W_2 = \frac{Y}{X}$ $\quad Y \perp X \sim Exp(\lambda)$

$0 < W_2 < \infty \qquad f(x,y) = \lambda e^{-\lambda x} \lambda e^{-\lambda y}$

$$f_{W_2}(w) = \int_{-\infty}^{\infty} f(x, xw)|x| dx$$

$$= \int_0^{\infty} f_X(x) f_Y(xw)\, x\, dx$$

$$= \int_0^{\infty} \lambda^2 e^{-\lambda x - \lambda(xw)}\, x\, dx$$

$$= \int_0^{\infty} \lambda^2 x e^{-\lambda x(1+w)}\, dx$$

$u = \lambda x \qquad du = \lambda dx$

$dv = \lambda e^{-\lambda(1+w)x} dx \qquad v = -\frac{e^{-\lambda(1+w)x}}{1+w}$

$$= -\frac{\lambda x e^{-\lambda(1+w)x}}{1+w} + \int_0^{\infty} \frac{e^{-\lambda(1+w)x}}{1+w} dx$$

$$= 0 + \left[ -\frac{e^{-\lambda(1+w)x}}{(1+w)^2} \right]_0^{\infty}$$

$$= \frac{1}{(1+w)^2}$$

$X_1, \ldots, X_n$ iid. $\text{Exp}(\lambda)$

$$P(X_{(1)} \le t) = 1 - P(X_{(1)} \ge t) = 1 - P(X_1 \ge t, \ldots, X_n \ge t)$$
$$= 1 - P(X_1 \ge t) \cdots P(X_n \ge t)$$
$$= 1 - P(X_i \ge t)^n$$
$$= 1 - (e^{-\lambda t})^n$$
$$= 1 - e^{-n\lambda t}$$

$$f_{X_{(1)}}(t) = n\lambda e^{-n\lambda t}$$

$X \sim \text{Bernoulli}(p)$

$$p(x) = \begin{cases} p & : x = 1 \\ 1-p & : x = 0 \end{cases} = p^x (1-p)^{1-x}$$

$$E(X) = 1 \cdot p + 0 \cdot (1-p) = p$$

Note: Our "game" had $Y = 200X - 100$

$$E(Y) = 0$$
$$200E(X) - 100 = 100 - 100 = 0$$

(This will be important.

$X \sim \text{Binomial}(n,p)$

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x \in \{0, \ldots, n\}$$

$$E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x}$$

$$= np \underbrace{\sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!((n-1)-(x-1))!} p^{x-1} (1-p)^{(n-1)-(x-1)}}_{\text{Binomial}(n-1,\,p) \text{ pmf}}$$

$$= np$$

$X \sim \text{Geometric}(p)$

$$p(x) = (1-p)^{x-1} p \qquad x \in \{1, 2, 3, \ldots\}$$

Fact: $\frac{d}{dr} \sum_{x=0}^{\infty} r^x = \sum_{x=1}^{\infty} x r^{x-1}, \quad |r| < 1$

$$E(X) = \sum_{i=1}^{\infty} x (1-p)^{x-1} p$$

$$= p \sum_{i=1}^{\infty} x (1-p)^{x-1}$$

$$= p \left[ \frac{d}{dp} \sum_{x=0}^{\infty} (1-p)^x \right]$$

$$= p \frac{d}{dp} \frac{1}{1-(1-p)} = \frac{1}{p}$$

Torture

1 $$\frac{6}{\pi^2}\sum_{x=1}^{\infty} x\frac{1}{x^2} = \infty$$

So $E(X)$ does not exist

2 $$\sum_{x\neq 0} \frac{3}{\pi^2} x\frac{1}{x^2} = 0$$

but $$\sum_{x\neq 0}^{\infty} \frac{3}{\pi^2} |x| \frac{1}{x^2} = \infty$$

So $E(X)$ does not exist.

3 $E(X) = aP(X=a) = a$

$Z \sim N(0,1)$

Method 0 Direct (Exercise)

Method 1 Even / Odd functions

Method 2 Symmetric argument (around $b$)

$f(b-x) = f(b+$

$$E(X) = \int_{-\infty}^{\infty} (x-b+b) f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-b) f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{b} (x-b) f(x)\,dx + \int_{b}^{\infty} (x-b) f(x)\,dx + b$$

$y = b - x$ $y = x - b$

$$= \int_{-\infty}^{0} y f(b-y)\,dy + \int_{0}^{\infty} y f(b+y)\,dy + b$$

$= b$

Unif $[a,b]$ $E(u) = \int_{-\infty}^{\infty} u f_u(u)\,du$

$$= \int_a^b u \frac{1}{b-a}\,du = \frac{1}{b-a} \frac{u^2}{2}\Big|_a^b$$

$$= \frac{1}{b-a} \frac{b^2-a^2}{2}$$

$$= \frac{b+a}{2}$$

Cauchy:

$$\int_{-\infty}^{\infty} x \frac{1}{\pi} \frac{1}{1+x^2}\,dx$$

does not exist.

Although symmetric!

Gamma:

$$\int_0^{\infty} x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\,dx$$

$$= \frac{\alpha}{\lambda} \int_0^{\infty} \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-\lambda x}\,dx$$

$$= \frac{\alpha}{\lambda}$$

Linearity

$$E(g(X)) = \int_{-\infty}^{\infty} x f_X\left(\frac{x-a}{b}\right) \frac{1}{|b|}\,dx$$

$$= \int_{-\infty}^{\infty} (a+by) f_X(y) \cdot \frac{1}{|b|} |b|\,dx$$

$$= a + b \int_{-\infty}^{\infty} y f_X(y)\,dy$$

$$= a + b E(X)$$

## Volume of Sphere

$$E(V) = \int_0^\infty \frac{4}{3}\pi r^3 e^{-r}\, dr$$

$$= \frac{4\pi}{3} 3! \int_0^\infty \frac{1^4 r^3 e^{-r}}{\Gamma(4)}\, dr$$

$$= 8\pi$$

$$E(X_1 X_2) = \iint x_1 x_2 f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int x_1 f_{X_1}(x_1)\, dx_1 \int x_2 f_{X_2}(x_2)\, dx_2$$

$$= E(X_1) E(X_2)$$

Converse is false

Example 1

| | | $X_1$: $-1$ | $0$ | $1$ |
|---|---|---|---|---|
| $X_2$ | $-1$ | $0$ | $\frac{1}{3}$ | $0$ |
| | $1$ | $\frac{1}{3}$ | $0$ | $\frac{1}{3}$ |

$$E(X_1 X_2)$$

$$= \sum\sum x_1 x_2\, p(x_1, x_2)$$

$$= -1\tfrac{1}{3} + 1\tfrac{1}{3}$$

$$= 0$$

But $X_1 \not\perp X_2$

Example ②  $X \sim N(0,1)$
$Y = X^2$

$$E(XY) = E(X^3) = 0$$

$$E(X)E(Y) = 0 \cdot 1 = 0$$

But $X \not\perp Y$

$$E(\overline{X}) = \int \cdots \int \left(\frac{x_1 + \cdots + x_n}{n}\right) f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

$$= \frac{1}{n}\left[\int x_1 f_{X_1}(x_1)\, dx_1 + \cdots + \int x_n f(x_n)\, dx_n\right]$$

$$= \frac{1}{n}\left[E(X_1) + \cdots + E(X_n)\right]$$

$$= \frac{1}{n}\left[n\mu\right] = \mu$$

Negative Binomial

I argue: $X = Y_1 + Y_2 + \cdots + Y_r$

with $Y_i$ i.i.d. Geometric$(p)$

$$\underbrace{0\ 0\ 0\ 1}_{Y_1}\ \underbrace{0\ 1}_{Y_2}\ \underbrace{1}_{Y_3}\ \underbrace{0\ 0\ 0\ 0\ 1}_{Y_4}$$

So $E(X) = \frac{r}{p}$

$E(X^2) = \int_{-\infty} x^2 f(x)\,dx$

$X \sim \text{Bernoulli}(p)$

$E(X) = p$

$Var(X) = (1-p)^2 \cdot p + (0-p)^2(1-p)$

1 $= (1-p)p[1-p+p]$

$= p(1-p)$

2 $E(X^2) = 1^2 \cdot p + 0^2(1-p) = p$

$Var(x) = E(X^2) - E(X)^2$

$= p - p^2$

$= p(1-p)$

Takes maximum at $p = 0.5 \ldots$

$Z \sim N(0,1)$

$Var(Z) = E(Z^2) - 0^2$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2}\,d\cancel{x} = 1$

$\boxed{\text{Parts with } u = Z}$

$u = z \qquad dv = z e^{-z^2/2}\,dz$

$du = dz \qquad v = -e^{-z^2/2}$

$= \frac{1}{\sqrt{2\pi}}\left[-z \cdot e^{-z^2/2}\right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\,dz = 1$

$X \sim \text{Poisson}(\lambda)$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{Inconvenient!}$$

$$E(X(X-1)) = E(X^2) - E(X)$$

$$= \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= \lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2} e^{-\lambda}}{(k-2)!}$$

$$= \lambda^2$$

So $E(X^2) = \lambda^2 + \lambda$

$$Var(X) = E(X^2) - E(X)^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

"Same mean and variance"

$$Var(a+bX) = E\left[\left(a+bX - E(a+bX)\right)^2\right]$$

$$= E\left[b^2 \left(X - E(X)\right)^2\right]$$

$$= b^2 E\left[\left(X - E(X)\right)^2\right]$$

$$= b^2 Var(X)$$

$$X = \mu + \sigma Z \quad \text{with} \quad Z \sim N(0,1)$$

$$E(X) = \mu$$

$$Var(X) = \sigma^2$$

$$Var(X+Y) = E\left((X+Y)^2\right) - \left[E(X+Y)\right]^2$$

$$= \left[E(X^2) - (EX)^2\right] + \left[E(Y^2) - E(Y)^2\right]$$

$$+ 2E(XY) - 2E(X)E(Y)$$

$$= Var(X) + Var(Y)$$

$$Var(\overline{X}) = Var\left(\frac{1}{n}(X_1 + \cdots + X_n)\right)$$

$$= \frac{1}{n^2} Var(X_1 + \cdots + X_n)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i)$$

$$= \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

$$\frac{1}{12} - \frac{1}{9}$$

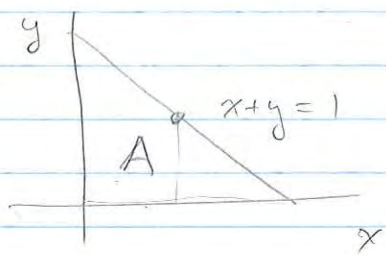
<u>Markov</u> $\quad t\,I(X \geq t) \leq X$

so $\quad P(X \geq t) \leq \dfrac{E(X)}{t}$ $\blacksquare$

<u>Chebyshev</u>

$$P\left( (X-\mu)^2 \geq t^2 \right) \leq \frac{Var(X)}{t^2}$$

$$P(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}$$




$y = 1-x$

$f(x,y) = 2$

$\int_0^1 (1-x)\,dx$

$$f_X(x) = \int_0^{1-x} 2 f(x,y)\,dy$$

$$= 2 - 2x \quad \text{on} \quad x \in [0,1]$$

$$E(X) = \int_0^1 x(2-2x)\,dx = \frac{1}{3} = E(Y)$$

$$E(XY) = \int_0^1 \int_0^{1-x} 2xy \;dy\,dx$$

$$= \int_0^1 xy^2 \Big|_0^{1-x} dx = \int_0^1 x(1-x)^2\,dx$$

$$= \frac{1}{12}$$

$E(XY) - E(X)E(Y)$

$= \frac{1}{12} - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) < 0$

$$P(X_2 = 0 \mid \underline{X_1 = 4}) = \frac{1/36}{1/36 + 2/36} = \frac{1}{3}$$

$$P(X_2 = 2 \mid \underline{X_1 = 4}) = \frac{2}{3}$$

| | | Prob |
|---|---|---|
| $E(X_2 \mid X_1 = 2) =$ | $0$ | $1/36$ |
| $E(X_2 \mid X_1 = 3) =$ | $1$ | $2/36$ |
| $E(X_2 \mid X_2 = 4) =$ | $\frac{4}{3}$ | $3/36$ |
| $E(X_2 \mid X_2 = 5) =$ | $2$ | $4/36$ |
| 6 | $12/5$ | $5/36$ |
| 7 | $3$ | $6/36$ |
| 8 | $12/5$ | $5/3$ |
| 9 | $2$ | 4 |
| 10 | $4/3$ | 3 |
| 11 | $1$ | 2 |
| 12 | $0$ | 1 |

1.9444...

$$E(X|Y) = \int_{-\infty}^{\infty} y \, f_{X|Y}(x/y) \, dx$$

$$= \int_{-\infty}^{\infty} y \, \frac{f(x,y)}{f_Y(y)} \, dx$$

$$= \int_{-\infty}^{\infty} \int_{0}^{1} dy$$

$$E\Big(E(X|Y)\Big) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x \, f_{X|Y}(x/y) \, dx \right] f_Y(y) \, dy$$

$$= \int_{-\infty}^{\infty} x \int \frac{f(x,y)}{f_Y(y)} f_Y(y) \, dy \, dx$$

$$= \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

$$= E(X)$$

Conditional distr^n of

$$\boxed{X_2 \mid X_1 = 4}$$

| Values | Prob. |
| --- | --- |
| 0 | $P(X_2 = 0 \mid X_1 = 4) = \frac{P(X_2 = 0, X_1 = 4)}{P(X_1 = 4)} = \frac{1/36}{3/36} = 1/3$ |
| 2 | $\frac{2/36}{3/36} = 2/3$ |

4/3

$$\cancel{\frac{p(x,y)}{p_Y(y)} =}$$

$E(X^0) = 1$ always.

Moments

$X \sim \text{Bernoulli}(p)$

$$E(X^k) = 1^k p + (0)^k (1-p) = p \quad \text{for all } k.$$

$Z \sim N(0,1)$

$$E(Z^{2n+1}) = \int_{-\infty}^{\infty} z^{2n+1} f_Z(z)\,dz = 0 \quad \text{for } n \geq 1$$

$$E(Z^{2n}) = (2n-1)\,E(Z^{2n-2}) \quad \text{(integration by parts)}$$

so (e.g.) $E(Z^6) = 5 \cdot 3 \cdot 1$

etc.

$E(e^{tX})$ ?

$$\sum_{k=0}^{\infty} E(X^k) \frac{t^k}{k!}$$

$$= \sum_{k=0}^{\infty} E\left(\frac{(tX)^k}{k!}\right)$$

$$= E\left(\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right)$$

WHERE?

radius of convergence

$|t| < R$

$$= E\left(e^{tX}\right)$$

$X \sim \text{Bernoulli}(p)$

$$E(e^{tX}) = e^{t \cdot 1}(p) + e^{t0}(1-p)$$
$$= (1-p) + pe^{t}$$
$$= q + pe^{t} \qquad (\forall t)$$

$X \sim \text{Binomial}(n,p)$

$$E(e^{tX}) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} \underbrace{(pe^t)}_{a}^{k} \underbrace{q}_{b}^{n-k}$$

$$= \left[q + pe^t\right]^n \qquad (\forall t)$$

$Z \sim N(0,1)$

$$E(e^{tZ}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz} e^{-z^2/2} \, dz$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[z^2 - 2zt + t^2 - t^2\right]} dz$$
$$= e^{t^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2} \, dz$$
$$= e^{t^2/2}$$

$$\frac{d}{dt} E(e^{tX}) = E\left(\frac{d}{dt} e^{tX}\right)$$
$$= E\left(X e^{tX}\right)$$

Moments

Bernoulli: $\left.\frac{d^k}{dt^k}\left[q + pe^t\right]\right|_{t=0}$

$= p \quad \forall k$

Binomial: $\left.\frac{d}{dt}\left[q + pe^t\right]^n\right|_{t=0}$

$= \left. n\left[q + pe^t\right]^{n-1} \cdot pe^t \right|_{t=0}$

$E(X) = np$

$E(X^2)$ = product rule

$= np(q + np)$

Norma(0,1)

$$\frac{d}{dt} e^{t^2/2}$$

$$= t e^{t^2/2}$$

$$\frac{d^2}{dt^2} e^{t^2/2} = e^{t^2/2} + t^2 e^{t^2/2}$$

<u>Theorem</u> If $X_1, \dots, X_n$ i.i.d. Bernoulli(p)

then $W = \sum X_i \sim$ Binomial$(n,p)$

$$m_{X_i}(t) = q + pe^t$$

$$M_{\sum X_i}(t) = \left[q + pe^t\right]^n$$

$$M_{X_i}(t) = \sum_{k=\phi}^{\infty} e^{tk}(1-p)^{k-1} p = pe^t \sum_{k=1}^{\infty} \left((1-p)e^t\right)^{k-1}$$

$$= \frac{pe^t}{1-(1-p)e^t}$$

$$E\left(e^{t(a+bX)}\right) = e^{at} E\left(e^{btX}\right)$$

$$= e^{at} m_X(bt)$$

$$M_{a+bX}(t) = E\left(e^{t(a+bX)}\right) = e^{at}E\left(e^{tbX}\right)$$
$$= e^{at}M_X(tb)$$

$$X = \mu + \sigma Z$$

$$M_X(t) = e^{\mu t} e^{-(bt)^2/2}$$
$$= e^{\mu t + \sigma^2 t^2/2}$$

$$M_X(t) = \prod_{i=1}^{n} e^{\mu_i t + \sigma_i^2 t^2/2}$$
$$= e^{\left(\sum \mu_i\right)t + \left(\sum \sigma_i^2\right)t^2/2}$$

$$X \sim N\left(\sum \mu_i, \sum \sigma_i^2\right)$$

$$E\left(e^{tZ^2}\right) = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{2\pi} e^{-z^2/2}\, dz$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2(1-2t)}\, dz$$
$$= \frac{1}{(1-2t)^{1/2}} \int \frac{(1-2t)^{1/2}}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$
$$= \left[\frac{1}{(1-2t)}\right]^{\frac{1}{2}}$$
$$= \left[\frac{1/2}{\frac{1}{2} - t}\right]^{1/2}$$

$$P\left(\lim_{n\to\infty} \overline{X}_n = \mu\right) = 1 \quad \text{Strong LLN}$$

WLLN

$$E(\overline{X}_n) = \mu \qquad Var(\overline{X}_n) = \frac{\sigma^2}{n}$$

Chebyshev:

$$\lim_{n\to\infty} P\left(|\overline{X}_n - \mu| > \varepsilon\right) \le \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon} = 0$$

Convergence in Dist

$$\lim_{n\to\infty} P(X_n = k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \text{ etc...}$$

$$X_n \sim \text{Binomial}\left(n, \frac{\lambda}{n}\right)$$

$$\lim_{n\to\infty} M_{X_n}(t) = \lim_{n\to\infty}\left[\left(1-\frac{\lambda}{n}\right) + \frac{\lambda}{n}e^t\right]^n$$

$$= \lim_{n\to\infty}\left[1 + \frac{\lambda(e^t-1)}{n}\right]^n$$

$$= e^{\lambda(e^t-1)}$$

So $X_n \xrightarrow{D} X$ with $X \sim \text{Poisson}(\lambda)$

$$E(Y_n) = \frac{\sigma}{\sqrt{n}} E(\overline{X}_n - \mu) = \frac{\sigma}{\sqrt{n}}\left(E(\overline{X}_n) - \mu\right)$$

$$= \frac{\sigma}{\sqrt{n}}(\mu - \mu) = 0$$

$$Var(Y_n) = \frac{1}{\sigma^2/n} Var(\overline{X}_n - \mu) = \frac{1}{\sigma^2/\sqrt{n}} Var(\overline{X}_n) = 1$$

## CLT Proof:

Let $W_i = X_i - \mu$

$$E(W_i) = 0$$
$$Var(W_i) = \sigma^2$$

$$m_{\Sigma W_i/\sigma\sqrt{n}}(t) = m_{\Sigma W_i}\left(\frac{t}{\sigma\sqrt{n}}\right)$$

$$= \left[m_{W_i}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

Apply Taylor's theorem to $m_{W_i}(t)$ at $t=0$

$$m_{W_i}(t) = m_{W_i}(0) + t\,m_{W_i}'(0) + \frac{t^2}{2} m_{W_i}''(0) + o(t^2)$$

$$m_{W_i}(0) = 1$$

$$m_{W_i}'(0) = E(W_i) = 0$$

$$m_{W_i}''(0) = E(W_i^2) = E(W_i^2) - E(W_i)^2$$
$$= Var(W_i) = \sigma^2$$

$$m_{\Sigma W_i/\sigma\sqrt{n}}(t) = \left[1 + \frac{1}{2}\left(\frac{t}{\sigma\sqrt{n}}\right)^2 \sigma^2 + o(t^2)\right]^n$$

$$\lim_{n\to\infty} m_{\Sigma W_i/\sigma\sqrt{n}}(t) = \lim_{n\to\infty}\left[1 + \frac{t^2}{2n} + o(t^2)\right]^n$$

$$= e^{t^2/2}$$

$$E(X_i) = \frac{1}{2}$$

$$Var(X_i) = \frac{1}{12}$$

$$\sum X_i \sim^{approx} N\left(10, \frac{20}{12}\right) \sim X$$

$$P(\sum X_i > 11) \approx P(X > 11)$$

$$= P\left(\frac{X-10}{\sqrt{20/12}} > \frac{11-10}{\sqrt{20/12}}\right)$$

$$= P(Z > 0.775)$$

$$= 0.219$$