STA261 Lecture 2 — 2017-07-10

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t distributions

Theorem 1.6 Suppose $Z \sim N(0,1)$ and $U \sim \chi^2_\nu$. Define $T = Z/\sqrt{U/\nu}$. The density of T is:

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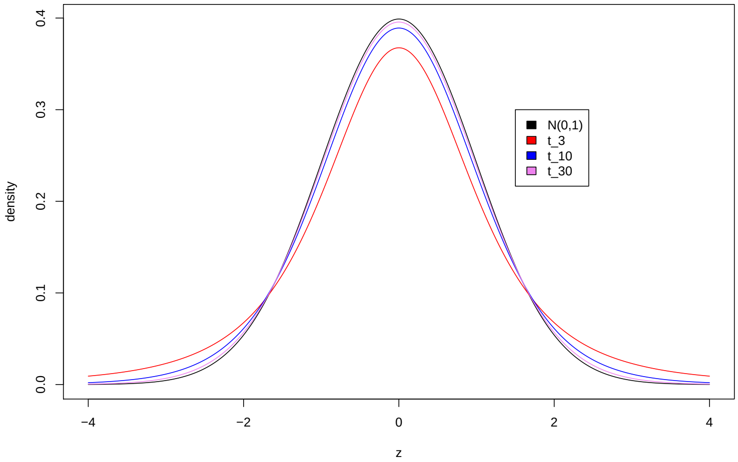
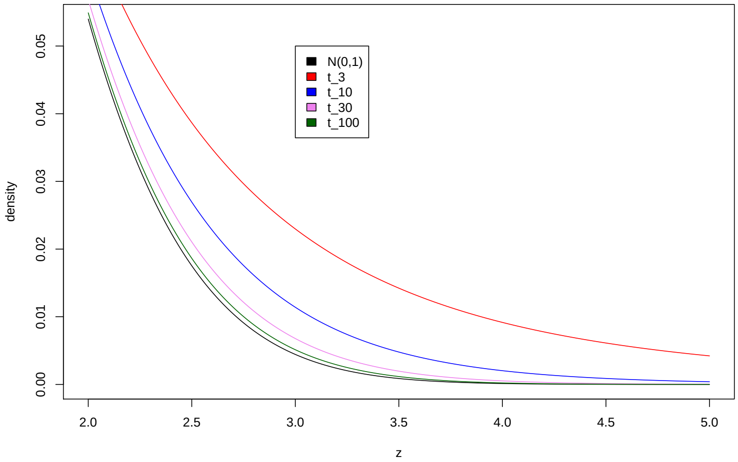
In the theorem, there was nothing about ν being an integer. It only has to be bigger than 0.

pictures of t densities (overall)



detail of the tails



properties of t_ν

The density is:

$$f_T(t) = \frac{\Gamma[(\nu+1)/2]}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2}$$

Note 2.0: T distributions are symmetric around 0.

Proposition 2.1: If $T \sim t_\nu$ and $\nu > 1$, then $E(T) = 0$, and if $0 < \nu \leqslant 1$ $E(T)$ does not exist.

Proposition 2.2: $\mathrm{Var}(T) = \frac{\nu}{\nu-2}$ when $\nu > 2$ and does not exist when $0 < \nu \leqslant 2$.

recall “convergence in distribution”

A sequence $X_1, X_2, X_3, \ldots$ is said to *converge in distribution* to X (notation: $X_n \Rightarrow X$)

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

for all x where F_X is continuous.

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It is the mode of convergence in the Central Limit Theorem (proved in STA257 using moment generating functions.)

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Fact 2.3: If continuous random variables $X_1, X_2, X_3, \dots$ have densities $f_1(x), f_2(x), \dots$ which converge to a density $f(x)$ for X, at all x, then $X_n \Rightarrow X$

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Proposition 2.4: If $T_n \sim t_n$, then $T_n \Rightarrow Z$ where $Z \sim N(0,1)$.

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I'll only sketch the demonstration starting with:

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Note that the convergence is *really fast.*

F distributions

Proposition 2.5: Suppose U and V are independent with $U \sim \chi_n^2$ and $V \sim \chi_m^2$. Then the density of $F = \frac{U/n}{V/m}$ is:

$$f_F(x) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{m}{n}\right)^{n/2}\frac{x^{n/2-1}}{\left(1+(n/m)\,x\right)^{(n+m)/2}}, \quad x > 0$$

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This uses **Factoid 2.6:** if X has density $f_X(x)$, then the density of X/a for $a > 0$ is $af_X(ax)$.

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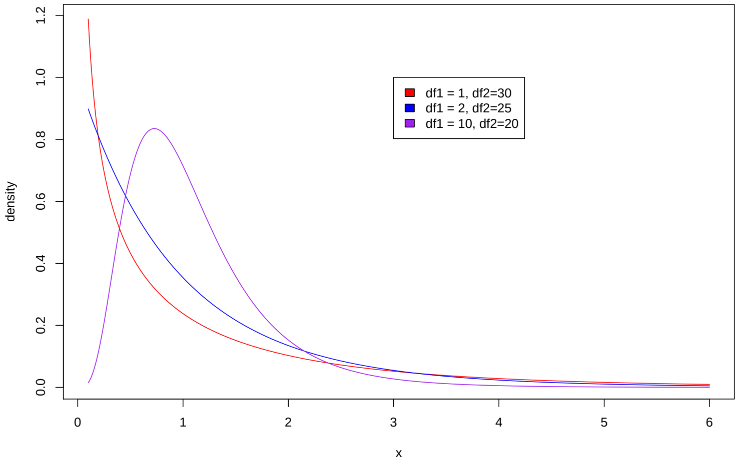
In words: "...has an F distribution with n and m degrees of freedom" or $F_{n,m}$.

pictures of F distributions



the square of a t_ν distribution

Proposition 2.7: If $T \sim t_\nu$, then $T^2 \sim F_{1,\nu}$

sample mean and sample variance from a $N(\mu, \sigma^2)$ sample

$\overline{X}$ and S^2

Suppose $X_1, \ldots, X_n$ is a *sample* from a population $X \sim N(\mu, \sigma^2)$. Define the *sample mean*:

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n},$$

and the *sample variance*:

$$S^2 = \sum_{i=1}^{n} \frac{\left(X_i - \overline{X}\right)^2}{n-1}.$$

distribution of $\overline{X}$

From last class $\overline{X} \sim N(\mu, \sigma^2/n)$.

This fact needs to flow in your blood.

the independence of $\overline{X}$ and S^2 when $n = 2$

An example from last class showed that when X_1 and X_2 are i.i.d. $N(0, 1)$, then $X_1 + X_2$ and $X_1 - X_2$ are independent.

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Theorem 2.8: When X_1 and X_2 are i.i.d. $N(\mu, \sigma^2)$, $\overline{X}$ and S^2 are indepedent.

the independence of $\overline{X}$ and S^2 in general

Proposition 2.9: S^2 can be expressed in terms of $\left(X_2 - \overline{X}, \ldots, X_n - \overline{X}\right)$, using the fact that $\sum_{i=1}^{n}(X_i - \overline{X}) = 0$.

Proposition 2.10: When $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$, $\overline{X} \perp S^2$

the distribution of S^2

Proposition 2.11: If $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma)$, then $\frac{n-1}{\sigma^2} S^2 \sim \chi^2_{n-1}$.

Proposition 2.12: $E(S^2) = \sigma^2$ and $\text{Var}(S^2) = \frac{2(\sigma^2)^2}{n-1}$

what's the point of all this?

Revisit my desire to know the average height of students in the class.

The distribution of heights might be $N(\mu, \sigma^2)$. I'm interested in μ.I am going to sample n people from the list, to obtain a sample $X_1, \ldots, X_n$.

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But we don't know σ. What do we do?

the distribution of $\frac{\overline{X}-\mu}{s/\sqrt{n}}$

Proposition 2.13: When $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$,

$$\frac{\overline{X} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$