## STA261 Lecture 3 — 2017-07-12

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So we plan to gather a sample  $X_1, \ldots, X_n$  i.i.d. from the underlying distribution. Then what?

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- e.g. Sample is from  $N(\mu, \sigma^2)$ . We want to estimate  $(\mu, \sigma)$ .

## open questions about estimators

What are some desirable properties that an estimator might have? How do I figure out which estimator to use, from first principles?



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**Large sample properties:** we would like to know how the estimator behaves when the sample size is large.

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**Example 3.2:** Suppose  $X_1, \ldots, X_n$  is i.i.d. Exponential with rate  $\lambda$ . The mean of an  $\text{Exp}(\lambda)$  is  $1/\lambda$ . What to do???

#### mean square error

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A common criteria used to say an estimator is the "best" is to say, among all unbiased estimators, it has the smallest variance. (Major goal of this course.)

#### unbiased with smaller variance

**Example 3.4:** Consider estimating  $\mu$  from a  $N(\mu, \sigma^2)$  population using a sample  $X_1, \ldots, X_n$ . These are some unbiased estimators for  $\mu$ :  $\overline{X}$ ,  $X_1$ ,  $(X_1 + X_n)/2$ . Compare their variances.

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It turns out  $\overline{X}$  has the smallest variance among all unbiased estimators. (To be proven.)

## consistency

This is a minimally good property to have.

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Recall (or welcome to...) Chebyshev's inequality:

$$P(|Y - E(Y)| > \varepsilon) \leqslant \frac{\mathsf{Var}(Y)}{\varepsilon^2}$$



**Theorem 3.5:** Given a sample  $X_1, \ldots, X_n$  i.i.d. from any distribution with mean  $\mu$  and variance  $\sigma^2$ ,  $\overline{X}_n$ , is a consistent estimator for  $\mu$ . In addition, if the distribution is normal,  $S_n^2$  is consistent for  $\sigma^2$ .

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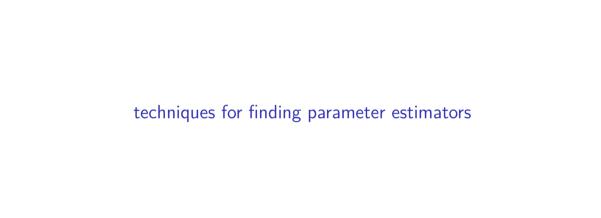
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**Non-example 3.8:** The stupid estimators for  $\mu$  from before,  $X_1$  and  $(X_1 + X_n)/2$ , are not consistent.



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That's the method of moments, in a nutshell.

The  $k^{th}$  moment of X (the "underlying population") is  $E(X^k)$  (if it exists).

**Definition:** the  $k^{th}$  sample moment of a sample  $X_1, \ldots, X_n$  i.i.d. with same distribution as X is:

$$\hat{\mu}_k = \frac{1}{n} \sum_{i=1}^n X_i^k$$

**The method of moments:** Express the parameter(s) of the distribution as function(s) of moment(s), invert the functions, and replace moments with sample moments.

method of moments examples

**Example 3.9:** Bernoulli(p) distribution.

**Example 3.10:** Estimate  $\mu$  and  $\sigma$  from a  $N(\mu, \sigma^2)$  distribution.

**Example 3.11:** Estimate  $\eta$  from a Weibull $(2, \eta)$  distribution.

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## value of method of moments as a technique

Method of moments estimators are useful because:

- 1. They are consistent under some mild conditions.
- 2. They might be the only estimators available (i.e. other techniques we haven't seen yet don't work.)