Shock and damage model parameter estimation and forecasting with an unobserved number of shocks

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Abstract We investigate the problem of parameter estimation and forecasting for a shock and damage model in which the number of shocks cannot be observed. Shock and damage models are widely studied by reliability theorists but there has been little published work on the statistical analysis of datasets related to such models. It is usually assumed that the number of shocks and the amounts of damage are observed to some extent. A company asked us to analyse a dataset in which only the total amount of damage (possibly zero) was observed. We proposed a Poisson process model for the number of shock events and estimated the rate parameter using the method of maximum likelihood, obtaining numerical estimates for the rate parameter and a standard error for the estimate. We then estimated the average amount of damage per shock event using a method of moments estimator. We used these parameter estimates to forecast the probability of one or more total failures among the entire population of all such fittings. The company was able to use our results and forecasts to make key asset management decisions relating to this class of fittings.

1. Introduction

Items of any kind can, and do, fail as a result of damage due to one or more sudden external forces that cause the item to exceed a threshold beyond which it is capable of operating. Often the occurrence of the external force or the amount of damage are unpredictable to the extent that they are modelled using random processes. So shock and damage models, and similar models under different names, have received plenty of attention in the academic literature in pure and applied probability, reliability, insurance, and others.

The basic idea is usually that the shocks occur as a homogeneous Poisson process $N(t)$ with rate λ. The cumulative shock times T_i therefore have Erlang distributions with parameters λ and i. The amount of damage X_i suffered at shock i is considered to be constant or modelled by some non-negative random variable. The total damage suffered up to time t is $Z(t) = \sum_{i=1}^{N(t)} X_i$, a doubly stochastic process. The total damage eventually causes the item to fail (which censors the underlying process).

This model is accessible to the novice reliability theorist. Extensions have been developed in all possible probabilistic directions, considering any variety of shock models, damage models, various dependencies among all the processes, and the effects all such variations might have on the first passage time properties that might characterize the actual moment of failure. From all such models a wide variety of maintenance policies might be developed. The literature in this area is vast and we do not attempt to summarize it here. In the field of reliability we point the interested reader in the direction of Nakagawa (2007) and its extensive list of references as a place to start.

There is relatively little discussion of the problem of parameter estimation for all the proposed models. Kahle and Wendt (2010) list three possible data scenarios. They do so in the context of a model in which a frailty variable Y is included permitting the underlying shock process to vary by individual item, which we can ignore, reducing the number of data scenarios to two. In the first scenario, the shock and degradation history (T_i, X_i) is observed, and the underlying parameters for each process can be estimated separately. This scenario seems to be implicitly assumed in most practical work in shock and damage models, which might explain the lack of attention given to parameter estimation. In the second scenario only the failures are observed, a situation also treated in Murthy and Iskander (1991).

We encountered a third scenario when a company asked us to help them with a situation they faced in which they were able to determine, through destructive testing, only the total damage done to an asset class (of "fittings") but not the actual number of shocks (unless the number is 0). The assets in operation were buried underground and the shock mechanism was believed to be the freeze-thaw cycle of the soil which was not directly observed. The company had no insights that might inform a possible distribution for the amount of damage suffered during each shock. They knew how much total damage an item could accumulate before total failure, which as an added complication was a function of each item's initial state upon installation possibly many decades ago. This initial state is equivalent to an initial amount of "damage". No total failure had ever occurred. The company was seeking forecasts for the probability of one or more total failures occurring in the population of fittings. They had received forecasts from the testing firm hired to do the physical analysis of the items and wanted our review of their calculations.

In this paper we implement a very simple shock and damage model but in what appears to be a novel situation in which no shocks are observed, no failures occurred, and only the total amount of damage (possibly 0) per item is available in a sample. We use the model to make useful forecasts of failure probability for the population.

2. Data description, model specification, and parameter estimation

2.1. *Data*

The dataset consisted of data collected from n destructive tests undertaken on a sample of fittings that had been exhumed for the express purpose of assessing the health of the population of interest. The two variables of interest at the parameter estimation stage of the analysis are the raw total amounts of damage suffered $d^{\text{raw}} = \{d_1^{\text{raw}}, \dots, d_n^{\text{raw}}\}$ and the ages $t = \{t_1, \dots, t_n\}$. Another measurement was taken to estimate the item's initial state, but this variable is only needed for forecasting later on.

It is unknown how many shock events, if any, each item had endured. We only know if the item had 0 events, or 1 or more events. Define the variable d based on d^{raw} by letting $d_i = 0$ when $d_i^{\text{raw}} = 0$ and $d_i = 1$ otherwise.

2.2. *Shock and damage model*

To model the shock process we use a homogeneous Poisson process model with parameter λ for the (mostly) unobserved shock history and estimate λ using maximum likelihood as follows. The number of shocks endured by item i at time t is denoted by $N_i(t)$. So the probabilities of item i having endured either 0 or 1 or more shocks during its lifetime t_i are given by:

$$P(N_i(t_i) = 0) = e^{-\lambda t_i}, \tag{1}$$

and

$$P(N_i(t_i) > 0) = 1 - e^{-\lambda t_i}. \tag{2}$$

The likelihood for λ is therefore:

$$L(\lambda) = \prod_{i=1}^{n} \left(e^{-\lambda t_i}\right)^{1-d_i} \left(1 - e^{-\lambda t_i}\right)^{d_i}, \tag{3}$$

with log-likelihood given by:

$$l(\lambda) = -\lambda \sum_{i=1}^{n} t_i(1 - d_i) + \sum_{i=1}^{n} d_i \log\left(1 - e^{-\lambda t_i}\right). \tag{4}$$

There is no closed-form solution to find the maximum of this function, but it is straightforward to accomplish using widely available routines. We use the *bbmle* package in R (Bolker et al. 2014) which produces estimates and standard errors. As a first approximation to $\hat{\lambda}$, required by the numerical routine, we use $\sum_{i=1}^{n} d_i \big/ \sum_{i=1}^{n} t_i$, in the spirit of a total-time-on-test style of estimator.

The company had no insight into the possible distribution of the damage suffered at a shock event. We choose to treat the damage as a constant, acknowledging that this tends to result in conservative projections of failure probabilities. The constant amount of damage a nevertheless needs to be estimated, which we do indirectly by considering the product λa, which is the average amount of damage per unit time. We use a method of moments approach to the estimation of λa by simply dividing the total amounts of damage by the total item ages to obtain the following estimator:

$$\widehat{\lambda a} = \frac{\sum_{i=1}^{n} d_i^{\text{raw}}}{\sum_{i=1}^{n} t_i}. \tag{5}$$

To estimate a itself we compute the maximum likelihood estimator $\hat{\lambda}$ for λ and solve for a in equation (5) to obtain:

$$\hat{a} = \frac{\widehat{\lambda a}}{\hat{\lambda}}. \tag{6}$$

The company's destructive testing subcontractor had itself attempted to estimate λ and a. They used an interesting but ultimately doomed ad-hoc method, beginning with a rather than λ. To estimate a the used the average of the non-zero damages d_i^{raw}. This is a sensible first approximation that might be accurate if shocks are so rare that most items would suffer only 0 or 1 events, but given the variation in the d_i^{raw} in the data this was unlikely to be the case, certainly if the damage is to be treated as a constant as we do in this paper. We believe the subcontractor's estimate for a was therefore probably too large. Their estimate for λ involved imputing the number of shock events for each item from there average damage estimate and using these imputed values as "data" to plug into the standard MLE for λ from a Poisson process. Since their a was probably too large their λ was probably too small.

To compare our estimates with the subcontractor's estimates we performed a variety of simulations which showed ours to be as accurate as the maximum likelihood theory gives. The subcontractor's estimates tended to overestimate a and underestimate λ unless λ was small enough so that very few cases of 2 or more shocks occurred, as we suspected.

3. Forecasting failure probabilities

An item will fail once it accumulates sufficient total damage. In general this threshold might be fixed or random. In this case study the threshold is fixed on a per-item basis but will vary depending on an initial condition which depends on how the item was installed. The n items exhumed for the sample each had this initial condition measured, and so the amount of damage each could have suffered can be calculated. Since we are treating the per-shock damage a as constant, the probability of item i failing by time t is simply:

$$P_i(t) = 1 - \sum_{j=0}^{k_i-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!}, \tag{7}$$

where k_i is the number of shocks required to cause item i to fail, which is a function of a and item i's initial condition. The probability of having one or more failures from the sampled items before time t, had they been left in service, is given by:

$$P^{(n)}(t) = 1 - \prod_{i=1}^{n} (1 - P_i(t)). \tag{8}$$

The company was interested in forecasting the probability of one or more failures in the entire population of N items. This probability depends in part on λ, a, and the distributions of ages of items in the population. The ages are known and λ and a are estimated from the sample. It also depends on the distribution of initial conditions which determine the number of shocks a particular item can suffer before failure, which is not known. If we assume the initial values within the sample are representative of the population we can estimate the probability of one or more failures in the population by $1 - (1 - P^{(n)}(t))^{N/n}$. This effectively "scales up" the sample size to the population size using the sample's average failure probability for every item in the population.

4. Exploring the data and numerical results

The age and damage amounts from the original data have been rescaled to obscure the true values, with even the rescaled values omitted from plots, but the overall properties of the sample remain very similar. The sample size was $n = 212$, out of which $\sum_{i=1}^{n}(1 - d_i) = 136$ items had suffered no damage. Figures 1 shows the distrubution of ages in years of items in the sample, which is bimodal due to the nature of the way the company had expanded and acquired smaller companies over several decades. Figure 2 shows the distribution of the amount of damage suffered by those 76 items that had suffered non-zero damage.

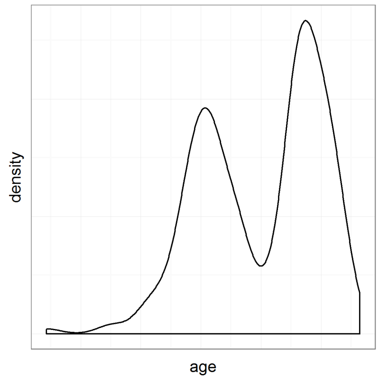

Figure 1. Density plot of ages in years of sampled items.

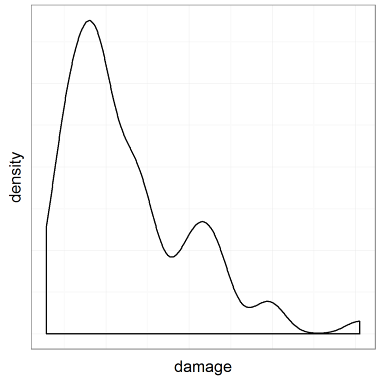

Figure 2. Density plot of total damage suffered by sampled items that had suffered any damage at all.

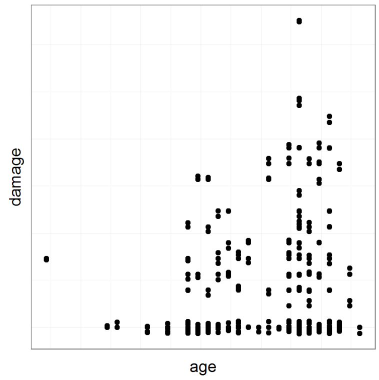

Figure 3. Scatterplot of age versus amount of damage.

Figure 3 shows the relationship between age and amount of damage suffered in items from the sample, with some vertical jitter added to points to deal with some overplotting due to the physical limits on the accuracy of per-item damage estimation in the destructive testing procedure.

4.1. *Parameter estimation*

The rate parameter maximum likelihood estimate is $\hat{\lambda} = 0.0090068$ with standard error 0.0010431. The estimate of the damage per unit time is $\widehat{\lambda a} = 0.0358$, giving an estimate for the amount of damage per shock of $\hat{a} = 3.98$.

These values can be compared to the subcontractor's ad hoc estimates of 5.0 for the damage per shock and 0.00764 for the shock rate. The product of these two subcontractor estimates, which estimates the damage per unit time, is 0.0382. These over-estimations of a and λa compared with our more accurate calculations results in substantial over-estimation of the probabilities of one or more failures in the entire population.

Table 1. Probability of one or more failures in population within five year increments, with parameter estimates given.

Years	Low	Middle	High
$\hat{\lambda}$	*0.011093*	*0.009007*	*0.006921*
$\hat{a}$	*3.23170*	*3.98023*	*5.18004*
5	0.000000	0.000001	0.000028
10	0.000001	0.000022	0.000433
15	0.000012	0.000162	0.002144
20	0.000067	0.000661	0.006627
25	0.000244	0.001956	0.015789
30	0.000701	0.004715	0.031841
35	0.001698	0.009864	0.057097
40	0.003633	0.018579	0.093697
45	0.007067	0.032262	0.143248
50	0.012744	0.052467	0.206445
55	0.021601	0.080779	0.282706
60	0.034760	0.118643	0.369948
65	0.053485	0.167138	0.464584
70	0.079116	0.226719	0.561829
75	0.112951	0.296970	0.656310
80	0.156085	0.376416	0.742877
85	0.209206	0.462473	0.817447
90	0.272382	0.551576	0.877624
95	0.344846	0.639535	0.922945
100	0.424871	0.722069	0.954666

4.2. *Failure probability forecasts*

The company was ultimately interested in long-term forecasting of the probabilities of one or more items in the population suffering a total failure. If the probability is too high for the company's tolerance for risk, the likely outcome would be to consider an expensive project to replace the current population with new fittings more resistant to stress and damage. We use a population size of $N = 100000$, which is within an order of magnitude of the true value.

We use the parameter estimates from 4.1 in equations (7) and (8) to make the probability forecasts. We also use $\hat{\lambda} \pm 2 \cdot \text{std.err.}\left(\hat{\lambda}\right)$ to propose simple lower and upper values on plausible forecasts (without going so far as to calling these limits "confidence intervals"). Table 1 shows probabilities of one or more failures in the population. The estimate of the damage a is very important for these projections. The "High" probability estimates actually correspond to the lower bound on the λ, which results in a higher value for $\hat{a}$ and a higher forecasted probability.

5. Discussion and future work

This is a paper about dealing with partial information. We developed maximum likelihood and method of moment estimators for the parameters of a simple shock and damage model but for the novel situation in which no shock history was available and no failures had been observed—only the total amount of damage on a sample of items was available. Had a small number of failures occurred our approach can still be used since the occurrence of a failure would indicate one more shocks had been suffered. The majority of items had no damage and it is interesting to note that having items with no damage is required for our method to be useful.

We used these estimates to produce long range probability forecasts for one or more failures in the population of assets. Our approach is capable of forecasting probabilities for any number of failures, which could be of interest for other applications in which failures are more tolerable and expected. The company who asked us to analyse the problem was able to use these forecasts to inform an important and possibly expensive asset management decision.

It would be interesting extend our approach to improve upon estimation of a, to relax the constant damage assumption, and to consider the relationship between age and damage.

References

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