STA261 Lecture 7 — 2017-07-31

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unbiased estimators with smallest variance

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In today's tutorial you found an unbiased estimator where the bound was not acheived. So perhaps there is a better unbiased estimator (in fact there is not) for λ from a Gamma(α_0, λ) distribution.

sufficiency

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Note 7.0: Say $n = 20$ and $k = 10$. The conditional distribution of $(X_1, X_2, \ldots, X_{20})$ given $T_{10} = t$ depends on the value of p.

T_{10} leaves behind some information about p in the sample.

sufficiency

Note 7.1: But if $n = 20$ and $k = 20$, the conditional distribution of $(X_1, X_2, \ldots, X_{20})$ given $T_{20} = t$ does not depend on the value of p.

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Definition: A statistic $T(X_1, \ldots, X_n)$ is *sufficient* for θ if the conditional distribution of $(X_1, X_2, \ldots, X_n)$ given $T = t$ does not depend on θ.

Note: the definition works for vector-valued T and vector θ.

sufficiency via factorization of pmf or pdf

The definition doesn't help so much for actually finding sufficient statistics.
Proposition 7.2: $T(X_1, \ldots, X_n)$ is sufficient for θ if and only if there are two functions g and h such that:

$$L(\theta; \mathbf{x}) = g(T(\mathbf{x}), \theta) h(\mathbf{x}).$$

factorization examples, and notes

Example 7.3: Sample is from Uniform$(0,\theta)$. $X_{(n)}$ is sufficient for θ.

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Example 7.6: Sample is from $N(\mu, \sigma^2)$. The vector $(\sum X_i, \sum X_i^2)$ is sufficient for $\theta = (\mu, \sigma^2)$.

recall "conditional expectation"

For random variables X and Y, the expected value of Y given a particular value $X = x$ is (continuous case):

$$E(Y|X = x) = \int yf(y|x)\, dy$$

where $f(y|x) = f(x, y)/f_X(x)$ is the conditional density of Y given X *with that particular value of* x *plugged in.*

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The *random variable* $g(X)$ is called the "conditional expectation of Y given X", or $E(Y|X)$.

properties of conditional expectation | “conditional variance”

$$E(g(X)) = E(E(Y|X)) = E(Y)$$

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Fact:

$$\mathrm{Var}(Y) = \mathrm{Var}(E(Y|X)) + E(\mathrm{Var}(Y|X))$$

Rao-Blackwellization

Under certain conditions, if you find an unbiased estimator that is a function of a sufficient statistic, *it will be the best unbiased estimator*.

The reason is mostly because of what is called the Rao-Blackwell theorem.

Proposition 7.7: Suppose W is unbiased for θ, and suppose T is sufficient for θ. Then the random variable $E(W|T)$ is unbiased with smaller variance than W.

examples of where the “certain conditions” hold

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Example 7.9: Gamma(α, λ)

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Example 7.11: Uniform$(0,\theta)$ (not in the exponential family)