

# STA286 Lecture 05

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## cumulative distribution functions

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$$F(x) = \left\{ \begin{array}{ll} 0 & \text{if } x < 0 \\ 0.5 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{array} \right.$$

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Equality vs. inequality not really important in a “continuous” probability model.

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As an example of the latter, consider the time-to-failure of an electronic component that:

- ▶ fails immediately the first time you try it, with probability 0.01
- ▶ works immediately with probability 0.99 and subsequently fails according to some continuous probability model TBA.

## discrete random variables

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- ▶ “probability distribution” (name already being used for a fundamental concept!)

## more pmf examples

**See if a product is defective:** A factory makes a defective item with probability  $p$ . Select an item at random from a factory. Let  $X = 1$  if the item is defective, and let  $X = 0$  otherwise.

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More compact version:  $p(x) = p^x(1 - p)^{1-x}$

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2.

$$\sum_{\{x \mid P(X=x) > 0\}} p(x) = 1$$

## checking if a function is a valid pmf

I said this function is a pmf. Is it?

$$p(x) = f(x) = P(X = x) = \left(\frac{1}{2}\right)^x, \quad x \in \{1, 2, 3, \dots\}$$

Verify:

1.  $p(x) \geq 0$



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Verify:

1.  $p(x) \geq 0$
2. Fact:  $\sum_{x=0}^{\infty} ar^x = \frac{a}{1-r}$  for  $0 < r < 1$ . So:

$$\sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^x = \sum_{x=0}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)^x = 1$$

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Yes, because you can compute a cdf from a pmf and vice versa. Obviously:

$$F(x) = \sum_{y \leq x} p(y)$$

For the reverse direction you take the jump points of the cdf and determine the magnitude of the jump.