STA286 Lecture 06

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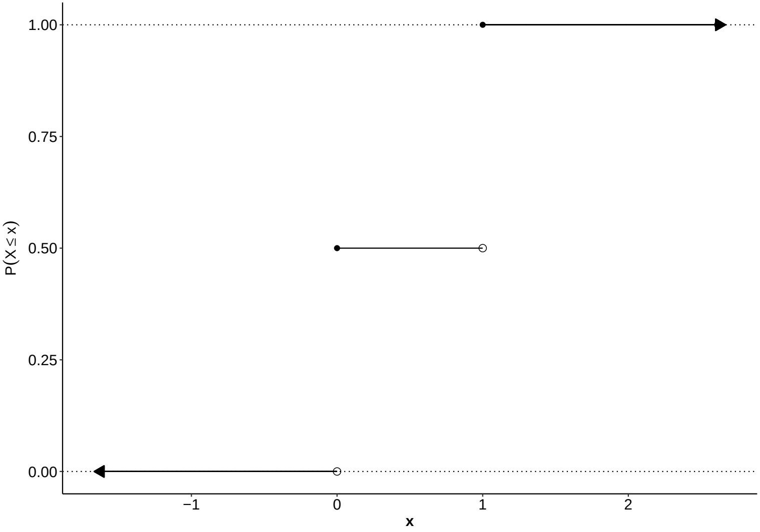

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Equality vs. inequality not really important in a "continuous" probability model.

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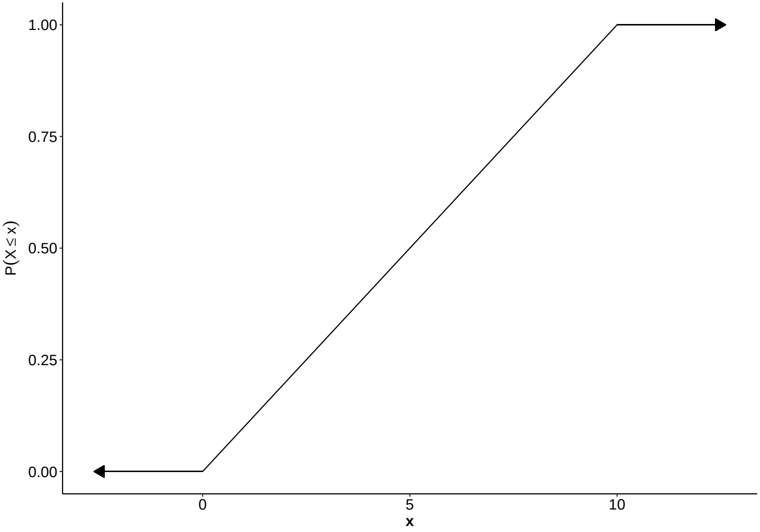

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As an example of the latter, consider the time-to-failure of an electronic component that:

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As an example of the latter, consider the time-to-failure of an electronic component that:

- fails immediately the first time you try it, with probability 0.01
- works immediately with probability 0.99 and subsequently fails according to some continuous probability model TBA.

discrete random variables

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$$p(x) = f(x) = P(X = x) = \left(\frac{1}{2}\right)^x, \quad x \in \{1, 2, 3, \dots\}$$

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- "probability distribution" (name already being used for a fundamental concept!)

more pmf examples

See if a product is defective: A factory makes a defective item with probability p. Select an item at random from a factory. Let $X = 1$ if the item is defective, and let $X = 0$ otherwise.

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More compact version: $p(x) = p^x (1 - p)^{1-x}$

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A function $p(x)$ is a pmf if and only if:

1. $$p(x) \geqslant 0$$

2. $$\sum_{\{x \,|\, P(X=x)>0\}} p(x) = 1$$

checking if a function is a valid pmf

I said this function is a pmf. Is it?

$$p(x) = f(x) = P(X = x) = \left(\frac{1}{2}\right)^x, \quad x \in \{1, 2, 3, \ldots\}$$

Verify:

1. $p(x) \geqslant 0$

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Verify:

1. $p(x) \geqslant 0$
2. Fact: $\sum\limits_{x=0}^{\infty} ar^x = \frac{a}{1-r}$ for $0 < r < 1$. So:

$$\sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^x = \sum_{x=0}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)^x = 1$$

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A discrete random variable has a pmf. Does the pmf characterize the distribtuion?

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Yes, because you can compute a cdf from a pdf and vice versa. "Obviously:"

$$F(x) = \sum_{y \leqslant x} p(y)$$

For the reverse direction you take the jump points of the cdf and determine the magnitude of the jump.

possibly easier to see than to understand the formal statement