STA286 Lecture 07

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Last edited: 2017-01-24 20:48

cumulative distribution functions

Recall the cdf:

$$F(x) = P(X \leqslant x),$$

which completely decribes the distribution of X.

discrete random variables

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- "probability function" (name already taken by P!)
- "probability distribution" (name already being used for a fundamental concept!)

more pmf examples

See if a product is defective: A factory makes a defective item with probability p. Select an item at random from a factory. Let $X = 1$ if the item is defective, and let $X = 0$ otherwise.

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More compact version: $p(x) = p^x(1 - p)^{1-x}$

defining properties of pmf

A function $p(x)$ is a pmf if and only if:

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1. $$p(x) \geqslant 0$$
2. $$\sum_{\{x \mid P(X=x)>0\}} p(x) = 1$$

checking if a function is a valid pmf

I said this function is a pmf. Is it?

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Verify:

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Verify:

1. $p(x) \geqslant 0$
2. Fact: $\sum_{x=0}^{\infty} ar^x = \frac{a}{1-r}$ for $0 < r < 1$. So:

$$\sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^x = \sum_{x=0}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)^x = 1$$

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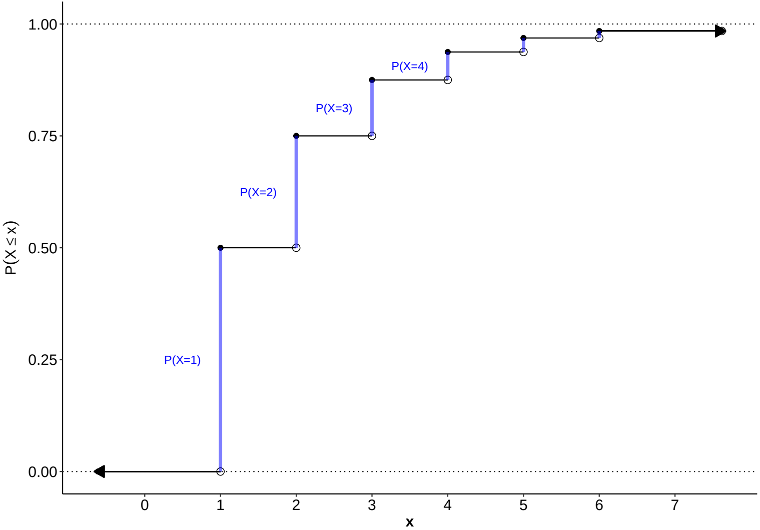
Yes, because you can compute a cdf from a pdf and vice versa. "Obviously:"

$$F(x) = \sum_{y \leqslant x} p(y)$$

For the reverse direction you take the jump points of the cdf and determine the magnitude of the jump.

possibly easier to see than to understand the formal statement

The cdf of X = "toss to first H", with pmf values in blue:



continuous random variables

For a random process taking on values in real intervals, we saw it made sense for $P(X = x) = 0$ for any particular value of x (e.g. bus stop example.)

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If there is a ("Riemann integrable") function f such that:

$$P(a < X \leqslant b) = F(b) - F(a) = \int_a^b f(x)\,dx$$

then we say X is "(absolutely) continuous" and has f as its *probability density function* (or pdf, or just density).

Note: a and b can be $-\infty$ or ∞.

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This density gives us all the probabilities such as:

$$P(2 < X \leqslant 4) = \int_2^4 \frac{1}{10}\, dx = \frac{2}{10} \qquad P(X = 2) = \int_2^2 \frac{1}{10}\, dx = 0$$

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$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

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Defining characteristics: A function f is a density as long as $f \geqslant 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

another density example

Consider:

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Suppose X has this density. Calculate $P(X > 1)$ and determine the cdf of X...

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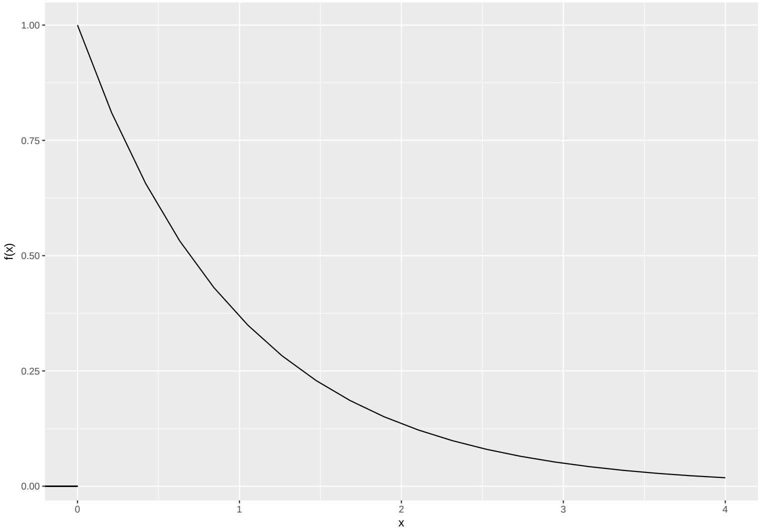
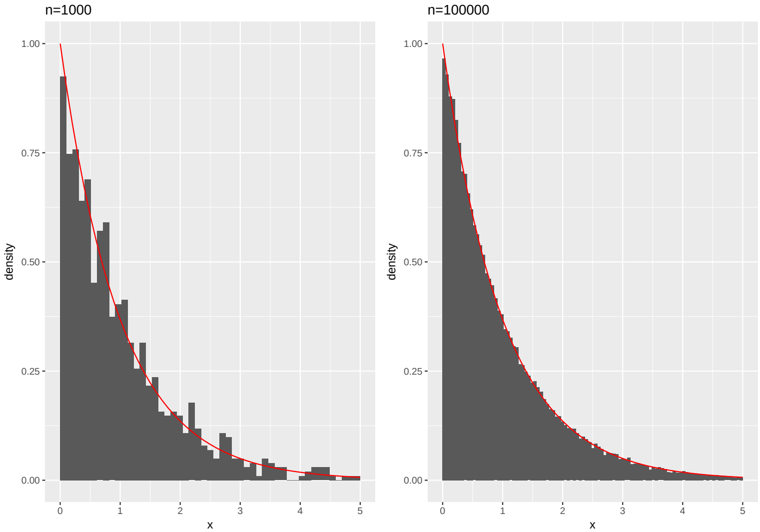
Pictures of densities can be useful, to show relative differences in probabilities.

illustration using e^{-x} density



histogram as “density estimator”

A density can be thought of as the “limit of histograms”.



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Define:

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X_1 and X_2 are not the same functions. But the have the same p.m.f.:

$$p(x) = \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{1-x}, \; x \in \{0, 1\}.$$

We say X_1 and X_2 are *identically distributed*.