STA286 Lecture 09

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Last edited: 2017-01-31 13:39

conditional distributions

Consider again the gas pipe joint (and marginal) distributions:

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Y	1	1.5	1.75	Marginal
0.5	0.075	0.100	0.150	0.325
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$$P(X = 1.5|Y = 2) = \frac{0.14}{0.395} \qquad P(X = 1.75|Y = 2) = \frac{0.095}{0.395}$$

conditional pmf and conditional density

Given the joint pmf $p(x, y)$ for X and Y, the conditional pmf for X given $Y = y$ is:

$$p(x|y) = \frac{p(x, y)}{p_Y(y)},$$

provided $P(Y = y) > 0$.

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Given the joint density $f(x, y)$ for X and Y, the conditional density for X given $Y = y$ is:

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conditional density example "uniform on a triangle"

Consider again X and Y with the following density.

$$f(x, y) = \begin{cases} 2 & : 0 < x < 1,\ 0 < y < 1,\ x + y < 1 \\ 0 & : \text{otherwise} \end{cases}$$

The marginal density for X was found to be $f_X(x) = 2(1 - x)$ for $0 < x < 1$.

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The conditional density of Y given $X = 0.8$ will be 0 when y is outside $(0, 0.2)$. Otherwise it will be:

$$f(y|x = 0.8) = \frac{2}{2 \cdot (1 - 0.8)} = 5$$

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Unlike marginal densities, conditional densities can sometimes be visualized. They are just "slices" of the joint density (normalized to 1.)

conditional density example

Reconsider the two electronic components example:

$$f(x,y)\begin{cases} 2e^{-x-2y} & : x>0, y>0 \\ 0 & : \text{otherwise} \end{cases}$$

The marginal density for Y is $2e^{-2y}$ on $y>0$.

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Heck, the conditional density for X given Y equals *anything* greater than 0 is still always e^{-x} on $x > 0$.

That's because X and Y have a special relationship...to be revisited.

independence

In the "two electronic component" example, the conditional density for X given $Y = y$ (no matter what $y > 0$) never changes:

$$f(x|y) = f(x)$$

Knowledge of the outcome of Y doesn't tell you anything about the distribution of X.

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Knowledge of the outcome of Y doesn't tell you anything about the distribution of X.

In the "uniform on a triangle" example, the conditional density for Y given $X = x$ is always going to be (for $0 < x < 1$):

$$f(y|x) = \frac{2}{2(1-x)} = \frac{1}{1-x}$$

for $0 < y < x$ and 0 otherwise. So knowledge of the outcome of X tells something about the distribution of Y.

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Definition: X and Y are *independent* (or $X \perp Y$) if:

$$p(x, y) = p_X(x)p_Y(y) \qquad \text{(discrete)} \qquad\qquad f(x, y) = f_X(x)f_Y(y) \qquad \text{(continuous)}$$

gas pipe diameters and pressures “made” independent

Suppose the marginal distributions are the same. What would the joint pmf have to be?

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relationship with independence of events

For an event A from a sample space S, define the *indicator random variable*:

$$I_A(\omega) = \begin{cases} 1 & : \omega \in A \\ 0 & : \omega \notin A \end{cases}$$

This is the general "Bernoulli trial" and one of the most important random variables there is.

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I_B	0	1	Marginal
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The formal definition of independence is:

$$P(X \in C, Y \in D) = P(X \in C)P(Y \in D)$$

for any $C, D \subset \mathbb{R}$.

notes on verifying independence

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Examples:

- $f(x, y) = x + y$ on $0 < x, y < 1$ and 0 otherwise.
- $f(x, y) = 24xy$ on $x > 0, y > 0, 0 < x + y < 1$ and 0 otherwise.