STA286 Lecture 10

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extension to more than two random variables - a few illustrations

The concepts of joint and marginal distributions extend to n random variables at a time. Things get ugly fast. Here is a taste. . . .

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Joint pmf of X_1, X_2, X_3:

$$p(x_1, x_2, x_3) = P(X_1 = x_1, X_2 = x_2, X_3 = x_3)$$

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Joint density for Y_1, Y_2, Y_3:

$$P(a_1 < Y_1 < b_1, a_2 < Y_2 < b_2, a_3 < Y_3 < b_3) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(y_1, y_2, y_3)\, dy_3\, dy_2\, dy_1$$

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Important: Random variables $X_1, X_2, \ldots, X_n$ are *independent* if:

$$f(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \ldots f_{X_n}(x_n)$$

(continuous version with densities... discrete version is similar.)

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This is conceptually important as the basis for the mathematical model of the observations in a dataset.

the expected value operator

the mean of a distribution

Recall the sample average:

$$\overline{x} = \sum_{i=1}^{n} x_i \cdot \frac{1}{n}$$

which can be considered as a weighted sum with weights $w_i = 1/n$.

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BIG MONEY. We play a gambling game called BIG MONEY. Roll a die. This is your outcome after each play of BIG MONEY:

Roll	1, 2	3	4, 5	6
Outcome \$	-2	0	1	2

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BIG MONEY is a “fair game”

Your *expected* financial outcome is (theoretically):

$$-2\frac{2}{6}+0\frac{1}{6}+1\frac{2}{6}+2\frac{1}{6}=0$$

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The *expected value* of a discrete random variable X is:

$$E(X) = \sum_{x} xp(x)$$

if the sum exists (actually it has to converge absolutely.)

expected number of coin tosses to first H

Denote by X the number of coin tosses until the first H. The pmf is (new version!):

$$p(x) = \left(1 - \frac{1}{2}\right)^{x-1} \frac{1}{2}$$

for $x \in \{1, 2, 3, \ldots\}$. For a moment replace $\frac{1}{2}$ with p.

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E(X) &= \sum_{x=1}^{\infty} x(1-p)^{x-1} p \\
&= p \sum_{x=0}^{\infty} x(1-p)^{x-1} \\
&= p \sum_{x=0}^{\infty} \frac{d}{dp} \left(-(1-p)^x\right)
\end{aligned}$$

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&= p\sum_{x=0}^{\infty} x(1-p)^{x-1} \\
&= p\sum_{x=0}^{\infty} \frac{d}{dp}\left(-(1-p)^{x}\right) \\
&= p\frac{d}{dp}\left(-\sum_{x=0}^{\infty}(1-p)^{x}\right) = p\frac{d}{dp}\left(-\frac{1}{1-(1-p)}\right) = \frac{1}{p}
\end{aligned}$$

generalization of "tosses to first head"

A factory makes a defective item with probability p (per item) with $0 < p < 1$. What is the expected number of items until the first defective item?

Denote the number of items by X. The pmf of X will be:

$$p(x) = (1 - p)^{x-1}\, p$$

for $x \in \{1, 2, 3, \ldots\}$.

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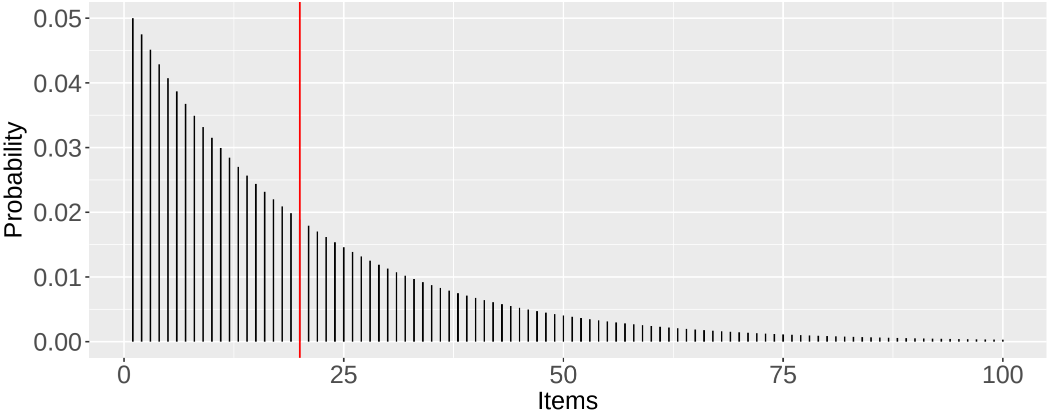
According to the previous slide, $E(X) = \frac{1}{p}$

graphical view of expected value

Suppose $p = 0.05$. Then $E(X) = 20$. The expected value is the "physical" balance point of the pmf (a.k.a. the *first moment*)



expected value non-example

Let X have the following pmf (!):

$$p(x) = \frac{6}{\pi^2 x^2}, x \in \{1, 2, 3, \ldots\}$$

X does not have an expected value, because $\sum_x xp(x)$ does not converge.

expected value - continuous version

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$$\int_0^{10} x \frac{1}{10}\, dx = x^2 \frac{1}{20}\bigg|_{x=0}^{10} = 5$$

another continuous example; plus a non-example

Consider $f(x) = 2x^{-3}$ for $x > 1$, and 0 otherwise. This is a valid density.

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But suppose Y has density $f(y) = y^{-2}$ on $y > 1$, and 0 otherwise. Then:

$$\int_1^\infty y\,y^{-2}\,dy = [\log y]_{y=1}^\infty$$

so $E(Y)$ does not exist.

expected values of functions of random variables

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BIG MONEY—*after a few schnapps version*.

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We adopt fake German accents and up the game.

Roll	1, 2	3	4, 5	6
Outcome \$	-200	0	100	200
Probability	2/6	1/6	2/6	1/6

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Roll	1, 2	3	4, 5	6
Outcome $	-200	0	100	200
Probability	2/6	1/6	2/6	1/6

Denote by Y your outcome after a play of this modified game. $E(Y)$ is also zero, using the definition:

$$E(Y) = -200\frac{2}{6} + 0\frac{1}{6} + 100\frac{2}{6} + 200\frac{1}{6} = 0$$

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Theorem: For a random variable X and a function $g : \mathbb{R} \to \mathbb{R}$:

$$E(g(X)) = \sum_{x} g(x)p(x) \qquad \text{discrete} \qquad E(g(X)) = \int_{\infty}^{\infty} g(x)f(x)\,dx \qquad \text{continuous}$$

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For example, the outcome Y of BIG MONEY (SCHNAPPS VERSION) is related the the outcome X of BIG MONEY by:

$$Y = 100X$$

So the theorem says $E(Y) = E(100X)$ and the calculation is (technically):

$$E(Y) = [100 \cdot (-2)]\frac{2}{6} + [100 \cdot (0)]\frac{1}{6} + [100 \cdot (1)]\frac{2}{6} + [100 \cdot (2)]\frac{1}{6} = 0$$

$E(\cdot)$ rules

The theorem lets us develop some basic rules.

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\begin{aligned}
E(a + bX) &= \sum_x (a + bx)p(x) \\
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(Continuous "version"proof is the same.)