### STA286 Lecture 12

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$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

So,  $X \perp Y$  implies Cov(X,Y), in which case the variance of the sum is the sum of the variances.

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It is a measure of *linear* relationship, in the following sense(s):

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- ▶ with high probability, smaller values of *X* and larger values of *Y* happen at the same time, then:
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#### correlation coefficient

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Positive and negative correlation mean the same as positive and negative covariance; in addition, the correlation of different pairs of distributions can be compared. Also:

$$-1 \leqslant \rho \leqslant 1$$

		X		
Y	-1	0	1	Marginal
-1	0.30	0.02	0.01	0.33
0	0.02	0.30	0.02	0.34
1	0.01	0.02	0.30	0.33
Marginal	0.33	0.34	0.33	1.00

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Y	-1	0	1	Marginal
-1	0.30	0.02	0.01	0.33
0	0.02	0.30	0.02	0.34
1	0.01	0.02	0.30	0.33
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$$E(X) = E(Y) = 0$$
, so  $Cov(X, Y) = E(XY)$ .

		X		
Y	-1	0	1	Marginal
-1	0.30	0.02	0.01	0.33
0	0.02	0.30	0.02	0.34
1	0.01	0.02	0.30	0.33
Marginal	0.33	0.34	0.33	1.00

$$E(X) = E(Y) = 0$$
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In this case (tedious exercise)  $\rho=$  0.879.

		X		
Y	-1	0	1	Marginal
-1	0.30	0.02	0.01	0.33
0	0.02	0.30	0.02	0.34
1	0.01	0.02	0.30	0.33
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$$E(X) = E(Y) = 0$$
, so  $Cov(X, Y) = E(XY)$ .

In this case (tedious exercise)  $\rho = 0.879$ .

The probability is strongly concentrated along the "X = Y diagonal".

## discrete example - negative correlation close to ${\bf 1}$

		X		
Y	-1	0	1	Marginal
-1	0.01	0.02	0.30	0.33
0	0.02	0.30	0.02	0.34
1	0.30	0.02	0.01	0.33
Marginal	0.33	0.34	0.33	1.00

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Marginal	0.33	0.34	0.33	1.00

In this case (tedious exercise)  $\rho = -0.879$ .

The probability is strongly concentrated along the "X = -Y diagonal".

# discrete exmaple - $X \perp Y$

		^		
Y	-1	0	1	Marginal
-1	0.1089	0.1122	0.1089	0.33
0	0.1122	0.1156	0.1122	0.34
1	0.1089	0.1122	0.1089	0.33
Marginal	0.33	0.34	0.33	1.00

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X and Y are indepedent (can be tediously verified.)

Easy to show E(XY) = 0, so that  $\rho = 0$ .

# discrete example - $\rho = 0$ but very much not independent!

		X		
Y	-1	0	1	Marginal
-1	0.00	0.00	0.25	0.25
0	0.50	0.00	0.00	0.50
1	0.00	0.00	0.25	0.25
Marginal	0.50	0.00	0.50	1.00

# discrete example - $\rho = 0$ but very much not independent!

		X		
Y	-1	0	1	Marginal
-1	0.00	0.00	0.25	0.25
0	0.50	0.00	0.00	0.50
1	0.00	0.00	0.25	0.25
Marginal	0.50	0.00	0.50	1.00

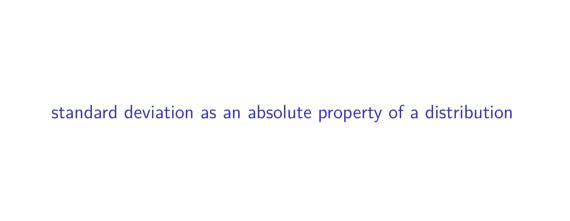
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Marginal	0.50	0.00	0.50	1.00

X and Y are not independent.

But E(XY) = 0, so  $\rho = 0$ .



## mean and SD aren't unique, but they do say something

E(X) and  $E(X^2)$  provide information about X that limit its values and probabilities to some extent. Two examples are Markov's and Chebyshev's inequalities.

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Theorem (Chebyshev): If  $Var(X) = \sigma^2$  and  $E(X) = \mu$ :

$$P(|X - \mu| \geqslant t) \leqslant \frac{\sigma^2}{t^2}$$

Equivalently:

$$P(\mu - k\sigma < X < \mu + k\sigma) \geqslant 1 - \frac{1}{k^2}$$

example - "uniform on (0,1)"

Suppose X has the density f(x) = 1 on 0 < x < 1 and 0 otherwise.

Then 
$$E(X) = \frac{1}{2}$$
 and  $Var(X) = \frac{1}{12}$ 

Various applications of Markov's and Chebyshev's inequality show how weak they really are — mainly useful in theory than in practice.