STA286 Lecture 15

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Consider the cumulative sums $X(t)$ of this Bernoulli process:

$$
\begin{aligned}
X(1) &= X_1 \\
X(2) &= X_1 + X_2 \\
\vdots \quad & \quad \vdots \\
X(n) &= \sum_{i=1}^{n} X_i \\
\vdots \quad & \quad \vdots
\end{aligned}
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- $E(X(t)) = np = \frac{t}{\Delta}\lambda\Delta = \lambda t$

pass to the limit

Fix a time t. Keep $E(X(t)) = np = \lambda t$ constant, so that $p = \lambda t / n$.

The goal is to find the limit of $P(X(t) = k)$ as $n \to \infty$ for any $k \geqslant 0$.

For any fixed n, $X(t) \sim \text{Binomial}\left(n, \frac{\lambda t}{n}\right)$

pass to the limit

$$\lim_{n\to\infty} P(X(t)=k) = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda t}{n}\right)^{k} \left(1-\frac{\lambda t}{n}\right)^{n-k}$$

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&= \frac{(\lambda t)^k}{k!} e^{-\lambda t}
\end{aligned}
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Poisson process/distribution

Nothing to do with fish. Named after some French guy.

A Poisson process $N(t)$ is a counting process which counts the number of "events" that happen inside the time interval $[0, t]$, subject to the following:

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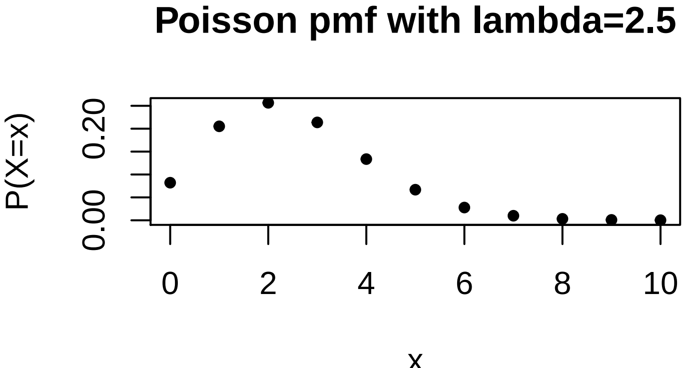
The Poisson process is a common model for events that happen "completely randomly" in time (to be further discussed)

Poisson distribution

There is also the closely related Poisson distribution, in which the t is not explicitly used. We say $X \sim \text{Poisson}(\lambda)$ if it has pmf:

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{for } k \in \{0, 1, 2, \ldots\}$$



Poisson examples

Customers enter a store at a rate of 1 per minute. Find the probabilities that:

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Why or why not might a Poisson process model be suitable here?

other Poisson distribution properties

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Then it's easy to show $E(X) = \lambda$ and $\text{Var}(X) = \lambda$.

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The time when the event happened is a continuous random variable, which we can call X. What is it's distribution?

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We can derive its cumulative disribution function from first principles. We know $P(X \leqslant x | N(t) = 1) = 0$ for $x \leqslant 0$ and $P(X \leqslant x | N(t) = 1) = 1$ for $x > t$.

cdf of "the time when that one thing happened" - cont'

Between 0 and t we have:

$$P(X \leqslant x | N(t) = 1) = 1 - P(X > x | N(t) = 1)$$

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Between 0 and t we have:

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&= 1 - \frac{t - x}{t} = \frac{x}{t}
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density of "the time when that one thing happened"

Putting it all together, the cdf of the time X when the event occurred given $N(t) = 1$ is:

$$F_X(x) = \begin{cases} 0 & : x \leqslant 0 \\ \frac{x}{t} & : 0 < x \leqslant t \\ 1 & : x > t \end{cases}$$

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The density is flat between 0 and t. We call this a "uniform distribution" between 0 and t.

the uniform distributions

The random variable X, the result of "picking a real number at random between a and b", is modeled using a flat density:

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mgf is easy to determine, but not really useful for anything.