STA286 Lecture 15

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Consider the cumulative sums $X(t)$ of this Bernoulli process:

$$
\begin{aligned}
X(1) &= X_1 \\
X(2) &= X_1 + X_2 \\
\vdots \quad & \quad \vdots \\
X(n) &= \sum_{i=1}^{n} X_i \\
\vdots \quad & \quad \vdots
\end{aligned}
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- $E(X(t)) = np = \frac{t}{\Delta}\lambda\Delta = \lambda t$

pass to the limit

Fix a time t. Keep $E(X(t)) = np = \lambda t$ constant, so that $p = \lambda t/n$.

The goal is to find the limit of $P(X(t) = k)$ as $n \to \infty$ for any $k \geqslant 0$.

For any fixed n, $X(t) \sim \text{Binomial}\left(n, \frac{\lambda t}{n}\right)$

pass to the limit

$$\lim_{n\to\infty} P(X(t)=k) = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1-\frac{\lambda t}{n}\right)^{n-k}$$

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&= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{\lambda t}{n}\right)^{k} \left(1-\frac{\lambda t}{n}\right)^{n-k}
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&= \frac{(\lambda t)^k}{k!} e^{-\lambda t}
\end{aligned}
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Poisson process/distribution

Nothing to do with fish. Named after some French guy.

A Poisson process $N(t)$ is a counting process which counts the number of "events" that happen inside the time interval $[0, t]$, subject to the following:

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- $P(N(t) - N(s) = k) = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}$ for $0 \leqslant s < t$

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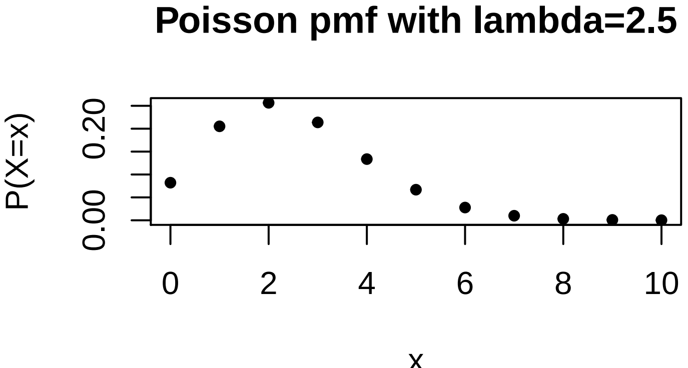
The Poisson process is a common model for events that happen "completely randomly" in time (to be further discussed)

Poisson distribution

There is also the closely related Poisson distribution, in which the t is not explicitly used. We say $X \sim \text{Poisson}(\lambda)$ if it has pmf:

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{for } k \in \{0, 1, 2, \ldots\}$$



Poisson examples

Customers enter a store at a rate of 1 per minute. Find the probabilities that:

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Why or why not might a Poisson process model be suitable here?