STA286 Lecture 15

Neil Montgomery

Last edited: 2017-03-01 11:54

other Poisson distribution properties

Suppose $X \sim \text{Poisson}(\lambda)$. Then (using s as the meaningless "dummy" variable):

other Poisson distribution properties

Suppose $X \sim \text{Poisson}(\lambda)$. Then (using s as the meaningless "dummy" variable):

$$M_X(s) = E(e^{sX}) = \sum_{k=0}^{\infty} e^{sk} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^s \lambda)^k}{k!}$$

other Poisson distribution properties

Suppose $X \sim \text{Poisson}(\lambda)$. Then (using s as the meaningless "dummy" variable):

$$M_X(s) = E(e^{sX}) = \sum_{k=0}^{\infty} e^{sk} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^s \lambda)^k}{k!} = e^{-\lambda} e^{e^s \lambda} = e^{\lambda(e^s - 1)}$$

other Poisson distribution properties

Suppose $X \sim \text{Poisson}(\lambda)$. Then (using s as the meaningless "dummy" variable):

$$M_X(s) = E(e^{sX}) = \sum_{k=0}^{\infty} e^{sk} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^s \lambda)^k}{k!} = e^{-\lambda} e^{e^s \lambda} = e^{\lambda(e^s - 1)}$$

Then it's easy to show $E(X) = \lambda$ and $\text{Var}(X) = \lambda$.

other Poisson distribution properties

Suppose $X \sim \text{Poisson}(\lambda)$. Then (using s as the meaningless "dummy" variable):

$$M_X(s) = E(e^{sX}) = \sum_{k=0}^{\infty} e^{sk} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^s \lambda)^k}{k!} = e^{-\lambda} e^{e^s \lambda} = e^{\lambda(e^s - 1)}$$

Then it's easy to show $E(X) = \lambda$ and $\text{Var}(X) = \lambda$.

Using full-blown Poisson process notation, we have that the expect value and the variance of the number of events in $[s, t]$ are both $\lambda(t - s)$.

Poisson - when did the event happen?

In a Poisson process we might have observed that $N(t) = 1$. (We will condition on this fact throughout.)

Poisson - when did the event happen?

In a Poisson process we might have observed that $N(t) = 1$. (We will condition on this fact throughout.)

When inside the interval $[0, t]$ did that single event occur?

Poisson - when did the event happen?

In a Poisson process we might have observed that $N(t) = 1$. (We will condition on this fact throughout.)

When inside the interval $[0, t]$ did that single event occur?

The time when the event happened is a continuous random variable, which we can call X. What is it's distribution?

Poisson - when did the event happen?

In a Poisson process we might have observed that $N(t) = 1$. (We will condition on this fact throughout.)

When inside the interval $[0, t]$ did that single event occur?

The time when the event happened is a continuous random variable, which we can call X. What is it's distribution?

We can derive its cumulative disribution function from first principles. We know $P(X \leqslant x | N(t) = 1) = 0$ for $x \leqslant 0$ and $P(X \leqslant x | N(t) = 1) = 1$ for $x > t$.

cdf of "the time when that one thing happened" - cont'

Between 0 and t we have:

$$P(X \leqslant x | N(t) = 1) = 1 - P(X > x | N(t) = 1)$$

cdf of "the time when that one thing happened" - cont'

Between 0 and t we have:

$$\begin{aligned} P(X \leqslant x | N(t) = 1) &= 1 - P(X > x | N(t) = 1) \\ &= 1 - P(N(x) = 0 | N(t) = 1) \end{aligned}$$

cdf of “the time when that one thing happened” - cont’

Between 0 and t we have:

$$
\begin{aligned}
P(X \leqslant x | N(t) = 1) &= 1 - P(X > x | N(t) = 1) \\
&= 1 - P(N(x) = 0 | N(t) = 1) \\
&= 1 - \frac{P(N(x) = 0, N(t) = 1)}{P(N(t) = 1)}
\end{aligned}
$$

cdf of "the time when that one thing happened" - cont'

Between 0 and t we have:

$$
\begin{aligned}
P(X \leqslant x | N(t) = 1) &= 1 - P(X > x | N(t) = 1) \\
&= 1 - P(N(x) = 0 | N(t) = 1) \\
&= 1 - \frac{P(N(x) = 0, N(t) = 1)}{P(N(t) = 1)} \\
&= 1 - \frac{P(N(x) = 0, N(t) - N(x) = 1)}{P(N(t) = 1)}
\end{aligned}
$$

cdf of "the time when that one thing happened" - cont'

Between 0 and t we have:

$$
\begin{aligned}
P(X \leqslant x | N(t) = 1) &= 1 - P(X > x | N(t) = 1) \\
&= 1 - P(N(x) = 0 | N(t) = 1) \\
&= 1 - \frac{P(N(x) = 0, N(t) = 1)}{P(N(t) = 1)} \\
&= 1 - \frac{P(N(x) = 0, N(t) - N(x) = 1)}{P(N(t) = 1)} \\
&= 1 - \frac{P(N(x) = 0)P(N(t) - N(x) = 1)}{P(N(t) = 1)}
\end{aligned}
$$

cdf of "the time when that one thing happened" - cont'

Between 0 and t we have:

$$
\begin{aligned}
P(X \leqslant x | N(t) = 1) &= 1 - P(X > x | N(t) = 1) \\
&= 1 - P(N(x) = 0 | N(t) = 1) \\
&= 1 - \frac{P(N(x) = 0, N(t) = 1)}{P(N(t) = 1)} \\
&= 1 - \frac{P(N(x) = 0, N(t) - N(x) = 1)}{P(N(t) = 1)} \\
&= 1 - \frac{P(N(x) = 0)P(N(t) - N(x) = 1)}{P(N(t) = 1)} \\
&= 1 - \frac{e^{-\lambda x}\lambda(t - x)e^{-\lambda(t-x)}}{\lambda t e^{-\lambda t}}
\end{aligned}
$$

cdf of "the time when that one thing happened" - cont'

Between 0 and t we have:

$$
\begin{aligned}
P(X \leqslant x | N(t) = 1) &= 1 - P(X > x | N(t) = 1) \\
&= 1 - P(N(x) = 0 | N(t) = 1) \\
&= 1 - \frac{P(N(x) = 0, N(t) = 1)}{P(N(t) = 1)} \\
&= 1 - \frac{P(N(x) = 0, N(t) - N(x) = 1)}{P(N(t) = 1)} \\
&= 1 - \frac{P(N(x) = 0)P(N(t) - N(x) = 1)}{P(N(t) = 1)} \\
&= 1 - \frac{e^{-\lambda x}\lambda(t - x)e^{-\lambda(t-x)}}{\lambda t e^{-\lambda t}} \\
&= 1 - \frac{t - x}{t} = \frac{x}{t}
\end{aligned}
$$

density of "the time when that one thing happened"

Putting it all together, the cdf of the time X when the event occurred given $N(t) = 1$ is:

$$F_X(x) = \begin{cases} 0 & : x \leqslant 0 \\ \frac{x}{t} & : 0 < x \leqslant t \\ 1 & : x > t \end{cases}$$

density of "the time when that one thing happened"

Putting it all together, the cdf of the time X when the event occurred given $N(t) = 1$ is:

$$F_X(x) = \begin{cases} 0 & : x \leqslant 0 \\ \frac{x}{t} & : 0 < x \leqslant t \\ 1 & : x > t \end{cases}$$

So the density is:

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} \frac{1}{t} & : 0 < x \leqslant t \\ 0 & : \text{ otherwise} \end{cases}$$

density of "the time when that one thing happened"

Putting it all together, the cdf of the time X when the event occurred given $N(t) = 1$ is:

$$F_X(x) = \begin{cases} 0 & : x \leqslant 0 \\ \frac{x}{t} & : 0 < x \leqslant t \\ 1 & : x > t \end{cases}$$

So the density is:

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} \frac{1}{t} & : 0 < x \leqslant t \\ 0 & : \text{ otherwise} \end{cases}$$

The density is flat between 0 and t. We call this a "uniform distribution" between 0 and t.

the uniform distributions

The random variable X, the result of "picking a real number at random between a and b", is modeled using a flat density:

$$f(x) = \begin{cases} \frac{1}{b-a} & : a < x < b \\ 0 & : \text{ otherwise} \end{cases}$$

the uniform distributions

The random variable X, the result of "picking a real number at random between a and b", is modeled using a flat density:

$$f(x) = \begin{cases} \frac{1}{b-a} & : a < x < b \\ 0 & : \text{ otherwise} \end{cases}$$

Easy to show $E(X) = (a+b)/2$ and $\text{Var}(X) = (b-a)^2/12$.

the uniform distributions

The random variable X, the result of "picking a real number at random between a and b", is modeled using a flat density:

$$f(x) = \begin{cases} \frac{1}{b-a} & : a < x < b \\ 0 & : \text{ otherwise} \end{cases}$$

Easy to show $E(X) = (a+b)/2$ and $\text{Var}(X) = (b-a)^2/12$.

mgf is easy to determine, but not really useful for anything.

the uniform distributions

The random variable X, the result of "picking a real number at random between a and b", is modeled using a flat density:

$$f(x) = \begin{cases} \frac{1}{b-a} & : a < x < b \\ 0 & : \text{ otherwise} \end{cases}$$

Easy to show $E(X) = (a+b)/2$ and $\mathrm{Var}(X) = (b-a)^2/12$.

mgf is easy to determine, but not really useful for anything.

We say $X \sim \mathrm{U}[a,b]$, with $\mathrm{U}[0,1]$ an important special case.

Poisson approximation to binomial

The limit $P(X(t) = k) \to P(N(t) = k)$ converges very fast, which means difficult Binomial calculations can be approximated very accurately using a Poisson probability calculations.

This was great in the 1960s, but not so important now. There is a discussion in the textbook and a handful of textbook exercises we'll call "optional" if you are intested.

waiting time to the first event of a Poisson process

In a Poisson process with rate λ, the first event will happen at some random time X. What is the distribution of X?

We can derive the cdf from first principles by observing the following:

$$P(X > t) = P(N(t) = 0)$$

waiting time to the first event of a Poisson process

In a Poisson process with rate λ, the first event will happen at some random time X. What is the distribution of X?

We can derive the cdf from first principles by observing the following:

$$P(X > t) = P(N(t) = 0)$$

The cdf is 0 for $t < 0$. Otherwise:

$$F_X(t) = P(X \leqslant t) = 1 - P(X > t) = 1 - P(N(t) = 0)$$

waiting time to the first event of a Poisson process

In a Poisson process with rate λ, the first event will happen at some random time X. What is the distribution of X?

We can derive the cdf from first principles by observing the following:

$$P(X > t) = P(N(t) = 0)$$

The cdf is 0 for $t < 0$. Otherwise:

$$F_X(t) = P(X \leqslant t) = 1 - P(X > t) = 1 - P(N(t) = 0) = 1 - \frac{(\lambda x)^0 e^{-\lambda t}}{0!} = 1 - e^{-\lambda t}$$

waiting time to the first event of a Poisson process

In a Poisson process with rate λ, the first event will happen at some random time X. What is the distribution of X?

We can derive the cdf from first principles by observing the following:

$$P(X > t) = P(N(t) = 0)$$

The cdf is 0 for $t < 0$. Otherwise:

$$F_X(t) = P(X \leqslant t) = 1 - P(X > t) = 1 - P(N(t) = 0) = 1 - \frac{(\lambda x)^0 e^{-\lambda t}}{0!} = 1 - e^{-\lambda t}$$

The density is therefore 0 when $t < 0$ and otherwise:

$$f_X(t) = \lambda e^{-\lambda t}$$

the exponential distributions

We say X has an exponential distribution with rate $\lambda > 0$ when it has density $f_X(t) = \lambda e^{-\lambda t}$ for $t > 0$.

the exponential distributions

We say X has an exponential distribution with rate $\lambda > 0$ when it has density $f_X(t) = \lambda e^{-\lambda t}$ for $t > 0$.

The m.g.f. is:

$$M_X(s) = \int_0^\infty e^{st}\lambda e^{-\lambda t}\,dt = \frac{\lambda}{\lambda - s}$$

the exponential distributions

We say X has an exponential distribution with rate $\lambda > 0$ when it has density $f_X(t) = \lambda e^{-\lambda t}$ for $t > 0$.

The m.g.f. is:

$$M_X(s) = \int_0^\infty e^{st} \lambda e^{-\lambda t}\, dt = \frac{\lambda}{\lambda - s}$$

The mean and variance are $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$

the exponential distributions

We say X has an exponential distribution with rate $\lambda > 0$ when it has density $f_X(t) = \lambda e^{-\lambda t}$ for $t > 0$.

The m.g.f. is:

$$M_X(s) = \int_0^\infty e^{st} \lambda e^{-\lambda t} \, dt = \frac{\lambda}{\lambda - s}$$

The mean and variance are $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$

Exponential distributions have the *memoryless* property, which is a crucial aspect of Poisson process as a model for “complete randomness”.

the exponential distributions

We say X has an exponential distribution with rate $\lambda > 0$ when it has density $f_X(t) = \lambda e^{-\lambda t}$ for $t > 0$.

The m.g.f. is:

$$M_X(s) = \int_0^\infty e^{st} \lambda e^{-\lambda t} \, dt = \frac{\lambda}{\lambda - s}$$

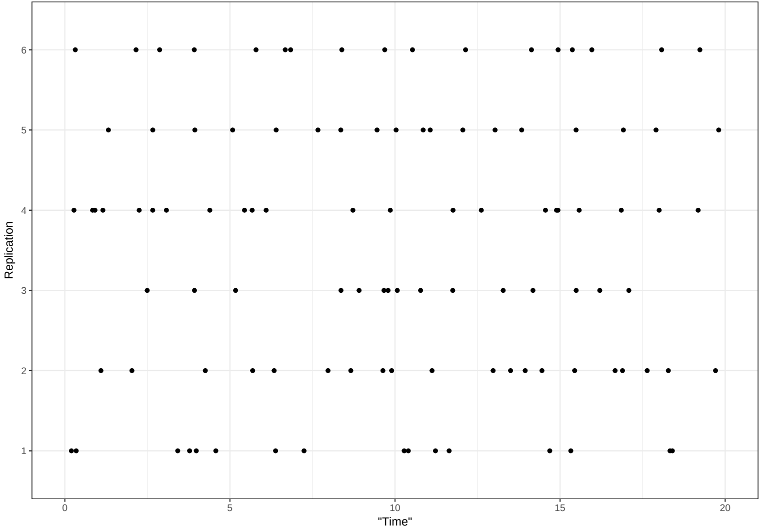
The mean and variance are $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$

Exponential distributions have the *memoryless* property, which is a crucial aspect of Poisson process as a model for "complete randomness".

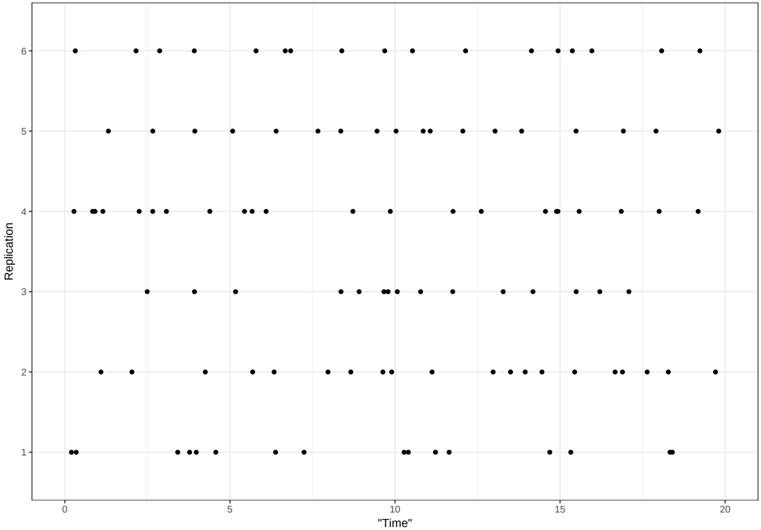
The Exponential distributions are the *only* (continuous) memoryless distributions.

DISASTROUS TREND SHOCKER PANIC HEADLINE

complete randomness is hard for humans



complete randomness is hard for humans



Answer: 1, 3, 4

waiting time to the n^{th} event of a Poisson process

Let's say we have a Poisson process $N(t)$ with rate λ. The time of the ~~first~~ n^{th} event is random. Call this time X.

What can we say about X? Can we completely describe its distribution?

waiting time to the n^{th} event of a Poisson process

Let's say we have a Poisson process $N(t)$ with rate λ. The time of the ~~first~~ n^{th} event is random. Call this time X.

What can we say about X? Can we completely describe its distribution?

Yes, because $F(t) = 1 - P(X > t)$, and $\{X > t\}$ *is exactly equivalent to* $\{N(t) \leqslant n - 1\}$, so we can derive the cdf for X.

waiting time to the n^{th} event of a Poisson process

Let's say we have a Poisson process $N(t)$ with rate λ. The time of the ~~first~~ n^{th} event is random. Call this time X.

What can we say about X? Can we completely describe its distribution?

Yes, because $F(t) = 1 - P(X > t)$, and $\{X > t\}$ *is exactly equivalent to* $\{N(t) \leqslant n-1\}$, so we can derive the cdf for X.

$$F(t) = P(X \leqslant t) = \begin{cases} 0 & : t \leqslant 0 \\ 1 - \sum_{i=0}^{n-1} \frac{[\lambda t]^i}{i!} e^{-\lambda t} & : t > 0 \end{cases}.$$

waiting time to the n^{th} event of a Poisson process

Let's say we have a Poisson process $N(t)$ with rate λ. The time of the ~~first~~ n^{th} event is random. Call this time X.

What can we say about X? Can we completely describe its distribution?

Yes, because $F(t) = 1 - P(X > t)$, and $\{X > t\}$ *is exactly equivalent to* $\{N(t) \leqslant n - 1\}$, so we can derive the cdf for X.

$$F(t) = P(X \leqslant t) = \begin{cases} 0 & : t \leqslant 0 \\ 1 - \sum_{i=0}^{n-1} \frac{[\lambda t]^i}{i!} e^{-\lambda t} & : t > 0 \end{cases}.$$

So the density is (a long telescoping sum of work later...):

$$f(t) = F'(t) = \begin{cases} \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} & : t > 0 \\ 0 & : \text{otherwise.} \end{cases}$$

the gamma distributions

The density is a special class of a larger family of distributions.

the gamma distributions

The density is a special class of a larger family of distributions.

Definition: the *gamma function* is defined as:

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u}\, du, \qquad \alpha > 0.$$

Many interesting properties, including $\Gamma(n) = (n-1)!$ for integer $n \geqslant 1$.

the gamma distributions

The density is a special class of a larger family of distributions.

Definition: the *gamma function* is defined as:

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u}\, du, \qquad \alpha > 0.$$

Many interesting properties, including $\Gamma(n) = (n-1)!$ for integer $n \geqslant 1$.

The following function is a valid density for $\alpha > 0$ and $\lambda > 0$:

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & : x > 0 \\ 0 & : \text{ otherwise.} \end{cases}$$

the gamma distributions

The density is a special class of a larger family of distributions.

Definition: the *gamma function* is defined as:

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u}\, du, \qquad \alpha > 0.$$

Many interesting properties, including $\Gamma(n) = (n-1)!$ for integer $n \geqslant 1$.

The following function is a valid density for $\alpha > 0$ and $\lambda > 0$:

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & : x > 0 \\ 0 & : \text{ otherwise.} \end{cases}$$

The parameters α and λ are called the *shape* and scale parameters. We say $X \sim \text{Gamma}(\alpha, \lambda)$

the gamma distributions

The density is a special class of a larger family of distributions.

Definition: the *gamma function* is defined as:

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u}\, du, \qquad \alpha > 0.$$

Many interesting properties, including $\Gamma(n) = (n-1)!$ for integer $n \geqslant 1$.

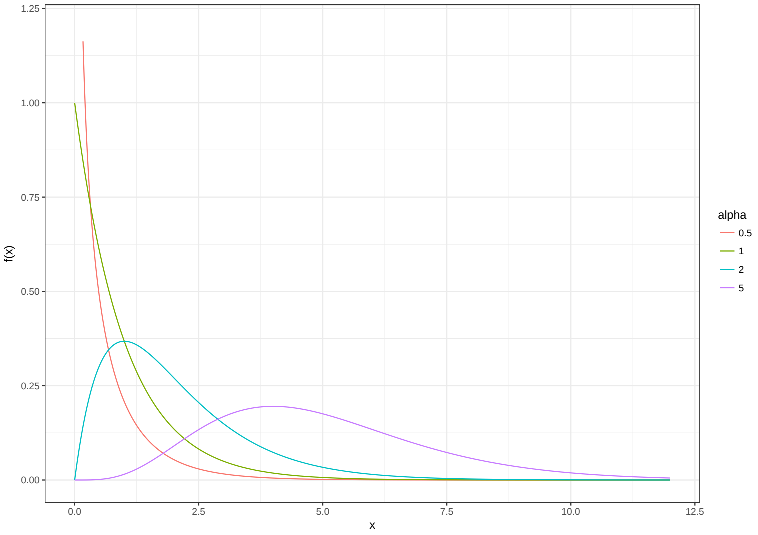
The following function is a valid density for $\alpha > 0$ and $\lambda > 0$:

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & : x > 0 \\ 0 & : \text{ otherwise.} \end{cases}$$

The parameters α and λ are called the *shape* and scale parameters. We say $X \sim \text{Gamma}(\alpha, \lambda)$

$\alpha = 1$ is the special case of $\text{Exp}(\lambda)$.

pictures of some Gamma(α, 1) densities



properties of gamma distributions

If $X \sim \text{Gamma}(\alpha, \lambda)$, its moment generating function can be found to be:

$$M_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^{\alpha}$$

properties of gamma distributions

If $X \sim \text{Gamma}(\alpha, \lambda)$, its moment generating function can be found to be:

$$M_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^{\alpha}$$

(so the mean and variance are $\frac{\alpha}{\lambda}$ and $\frac{\alpha}{\lambda^2}$)

properties of gamma distributions

If $X \sim \text{Gamma}(\alpha, \lambda)$, its moment generating function can be found to be:

$$M_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^{\alpha}$$

(so the mean and variance are $\frac{\alpha}{\lambda}$ and $\frac{\alpha}{\lambda^2}$)

Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. $\text{Exp}(\lambda)$. What is the distribution of $X = X_1 + X_2 + \cdots + X_n$?

properties of gamma distributions

If $X \sim \text{Gamma}(\alpha, \lambda)$, its moment generating function can be found to be:

$$M_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^\alpha$$

(so the mean and variance are $\frac{\alpha}{\lambda}$ and $\frac{\alpha}{\lambda^2}$)

Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. $\text{Exp}(\lambda)$. What is the distribution of $X = X_1 + X_2 + \cdots + X_n$?

Using the m.g.f. argument it is clear that $X \sim \text{Gamma}(n, \lambda)$, which makes sense in the Poisson process context.