STA286 Lecture 16

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Using full-blown Poisson process notation, we have that the expect value and the variance of the number of events in $[s, t]$ are both $\lambda(t - s)$.

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The time when the event happened is a continuous random variable, which we can call X. What is it's distribution?

We can derive its cumulative disribution function from first principles. We know $P(X \leqslant x | N(t) = 1) = 0$ for $x \leqslant 0$ and $P(X \leqslant x | N(t) = 1) = 1$ for $x > t$.

cdf of “the time when that one thing happened” - cont’

Between 0 and t we have:

$$P(X \leqslant x | N(t) = 1) = 1 - P(X > x | N(t) = 1)$$

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density of "the time when that one thing happened"

Putting it all together, the cdf of the time X when the event occurred given $N(t) = 1$ is:

$$F_X(x) = \begin{cases} 0 & : x \leqslant 0 \\ \frac{x}{t} & : 0 < x \leqslant t \\ 1 & : x > t \end{cases}$$

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The density is flat between 0 and t. We call this a “uniform distribution” between 0 and t.

the uniform distributions

The random variable X, the result of "picking a real number at random between a and b", is modeled using a flat density:

$$f(x) = \begin{cases} \frac{1}{b-a} & : a < x < b \\ 0 & : \text{ otherwise} \end{cases}$$

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We say $X \sim \mathrm{U}[a,b]$, with $\mathrm{U}[0,1]$ an important special case.

Poisson approximation to binomial

The limit $P(X(t) = k) \to P(N(t) = k)$ converges very fast, which means difficult Binomial calculations can be approximated very accurately using a Poisson probability calculations.

This was great in the 1960s, but not so important now. There is a discussion in the textbook and a handful of textbook exercises we'll call "optional" if you are intested.

waiting time to the first event of a Poisson process

In a Poisson process with rate λ, the first event will happen at some random time X. What is the distribution of X?

We can derive the cdf from first principles by observing the following:

$$P(X > t) = P(N(t) = 0)$$

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The density is therefore 0 when $t < 0$ and otherwise:

$$f_X(t) = \lambda e^{-\lambda t}$$

the exponential distributions

We say X has an exponential distribution with rate $\lambda > 0$ when it has density $f_X(t) = \lambda e^{-\lambda t}$ for $t > 0$.

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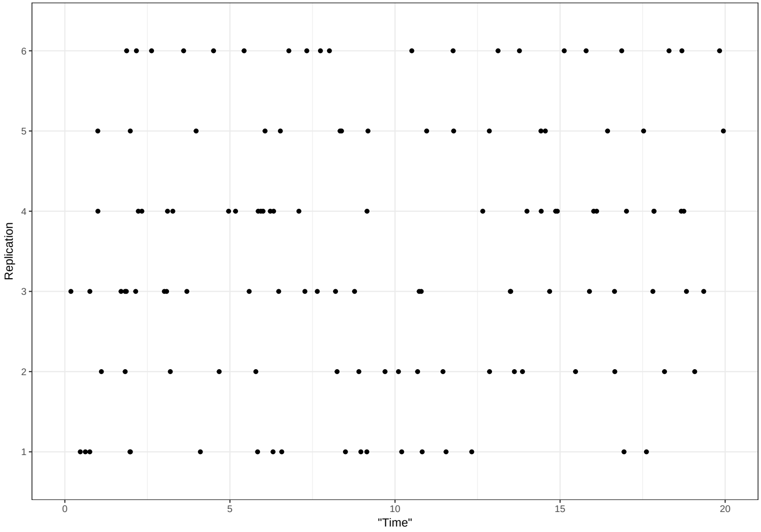
The Exponential distributions are the *only* (continuous) memoryless distributions.

DISASTROUS TREND SHOCKER PANIC HEADLINE

complete randomness is hard for humans



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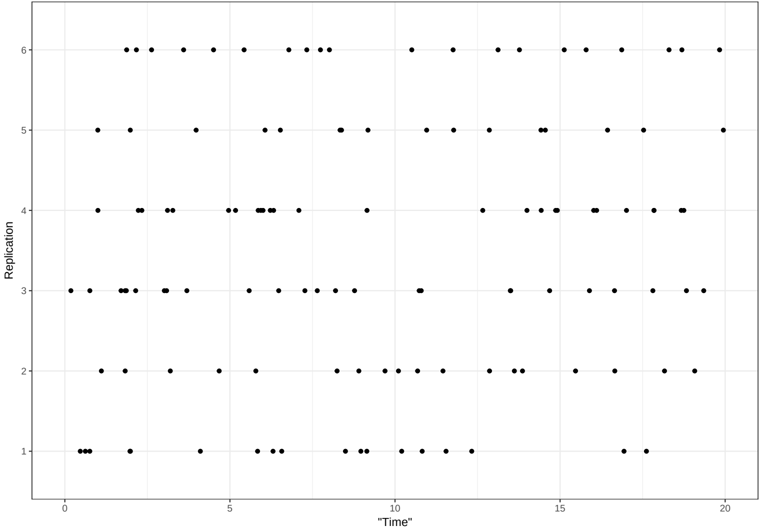

Answer: 1, 3, 4

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What can we say about X? Can we completely describe its distribution?

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So the density is (a long telescoping sum of work later...):

$$f(t) = F'(t) = \begin{cases} \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} & : t > 0 \\ 0 & : \text{otherwise.} \end{cases}$$

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Definition: the *gamma function* is defined as:

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Many interesting properties, including $\Gamma(n) = (n-1)!$ for integer $n \geqslant 1$.

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The following function is a valid density for $\alpha > 0$ and $\lambda > 0$:

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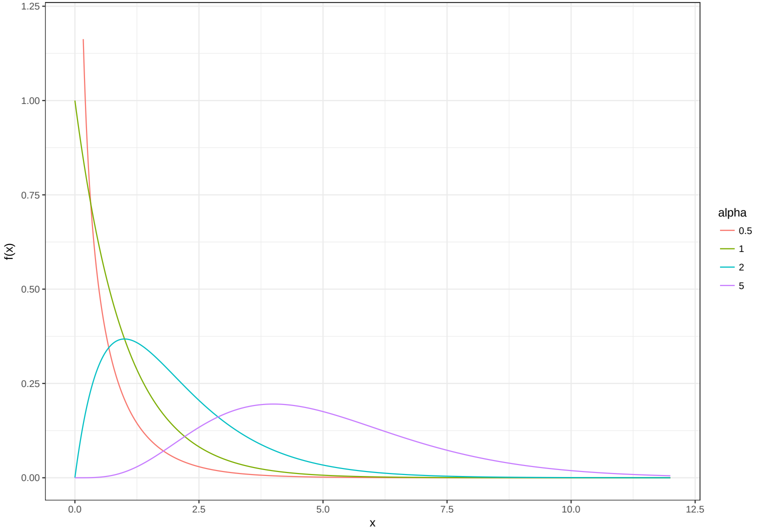
The parameters α and λ are called the *shape* and scale parameters. We say $X \sim \text{Gamma}(\alpha, \lambda)$

$\alpha = 1$ is the special case of $\text{Exp}(\lambda)$.

pictures of some Gamma(α, 1) densities



properties of gamma distributions

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Using the m.g.f. argument it is clear that $X \sim \text{Gamma}(n, \lambda)$, which makes sense in the Poisson process context.