

STA286 Lecture 15

Neil Montgomery

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Using full-blown Poisson process notation, we have that the expect value and the variance of the number of events in $[s, t]$ are both $\lambda(t - s)$.

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The time when the event happened is a continuous random variable, which we can call X . What is its distribution?

We can derive its cumulative distribution function from first principles. We know $P(X \leq x | N(t) = 1) = 0$ for $x \leq 0$ and $P(X \leq x | N(t) = 1) = 1$ for $x > t$.

cdf of “the time when that one thing happened” - cont'

Between 0 and t we have:

$$P(X \leq x | N(t) = 1) = 1 - P(X > x | N(t) = 1)$$

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density of “the time when that one thing happened”

Putting it all together, the cdf of the time X when the event occurred given $N(t) = 1$ is:

$$F_X(x) = \begin{cases} 0 & : x \leq 0 \\ \frac{x}{t} & : 0 < x \leq t \\ 1 & : x > t \end{cases}$$

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The density is flat between 0 and t . We call this a “uniform distribution” between 0 and t .

the uniform distributions

The random variable X , the result of “picking a real number at random between a and b ”, is modeled using a flat density:

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We say $X \sim U[a, b]$, with $U[0, 1]$ an important special case.

Poisson approximation to binomial

The limit $P(X(t) = k) \rightarrow P(N(t) = k)$ converges very fast, which means difficult Binomial calculations can be approximated very accurately using a Poisson probability calculations.

This was great in the 1960s, but not so important now. There is a discussion in the textbook and a handful of textbook exercises we'll call "optional" if you are intested.

waiting time to the first event of a Poisson process

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We can derive the cdf from first principles by observing the following:

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The density is therefore 0 when $t < 0$ and otherwise:

$$f_X(t) = \lambda e^{-\lambda t}$$

the exponential distributions

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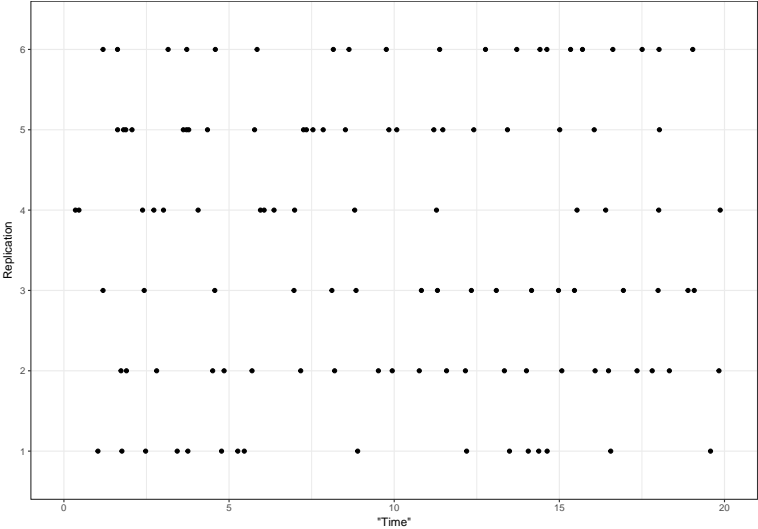
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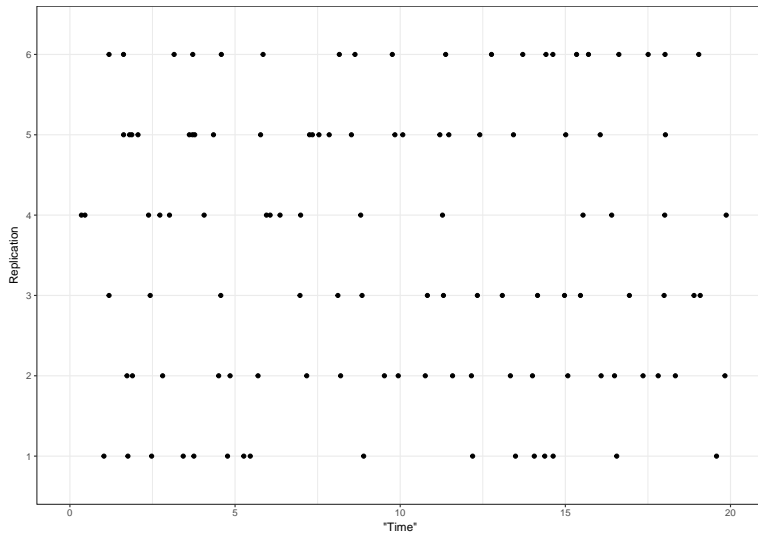
The Exponential distributions are the *only* (continuous) memoryless distributions.

DISASTROUS TREND SHOCKER PANIC HEADLINE

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Answer: 1, 4, 5

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So the density is (a long telescoping sum of work later...):

$$f(t) = F'(t) = \begin{cases} \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} & : t > 0 \\ 0 & : \text{otherwise.} \end{cases}$$

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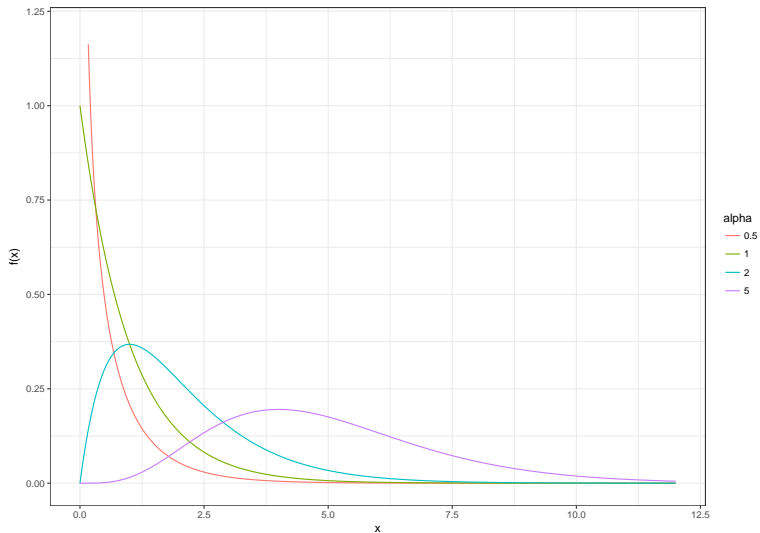
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$\alpha = 1$ is the special case of $\text{Exp}(\lambda)$.

pictures of some $\text{Gamma}(\alpha, 1)$ densities



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Using the m.g.f. argument it is clear that $X \sim \text{Gamma}(n, \lambda)$, which makes sense in the Poisson process context.