STA286 Lecture 17

Neil Montgomery

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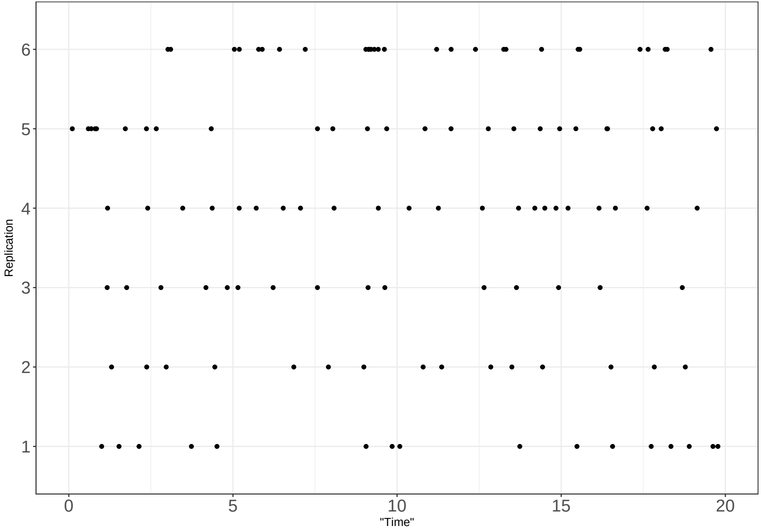
DISASTROUS TREND SHOCKER PANIC HEADLINE

complete randomness is hard for humans



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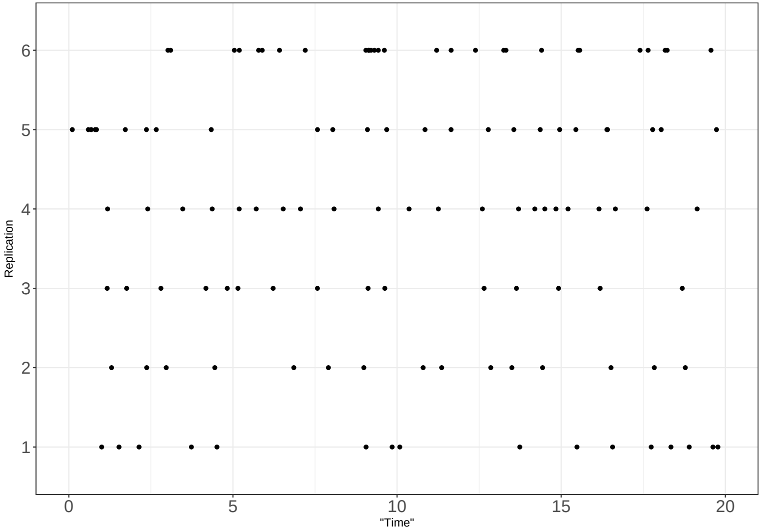

Answer: 1, 5, 6

waiting time to the n^{th} event of a Poisson process

Let's say we have a Poisson process $N(t)$ with rate λ. The time of the ~~first~~ n^{th} event is random. Call this time X.

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So the density is (a long telescoping sum of work later...):

$$f(t) = F'(t) = \begin{cases} \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} & : t > 0 \\ 0 & : \text{otherwise.} \end{cases}$$

the gamma distributions

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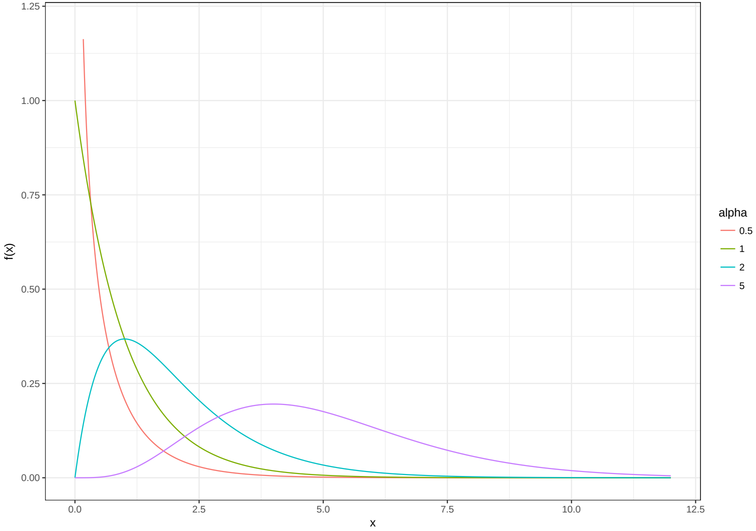
$\alpha = 1$ is the special case of $\text{Exp}(\lambda)$.

pictures of some Gamma(α, 1) densities



properties of gamma distributions

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Using the m.g.f. argument it is clear that $X \sim \text{Gamma}(n, \lambda)$, which makes sense in the Poisson process context.

summary of the Bernoulli-Poisson-o-sphere

Starting with a Bernoulli(p) process, we have the following:

What?	Discrete Version	Comments	Continuous Version
Count	Binomial(n, p)	Sum of n Bernoulli(p). Fix $E(X(t)) = np = \lambda t$ fixed $n \to \infty \ldots$	$\ldots$ Poisson(λt)
Inter-arrival	Geometric(p)	"Memoryless"	Exponential(λ)
Wait for r^{th} event	NegBin(r, p)		Gamma(n, λ)
Look back after 1	"Discrete Uniform"	($\leftarrow$ not done)	Uniform$(0, t)$

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Although it makes me scarlet with rage.

the “normal” distributions

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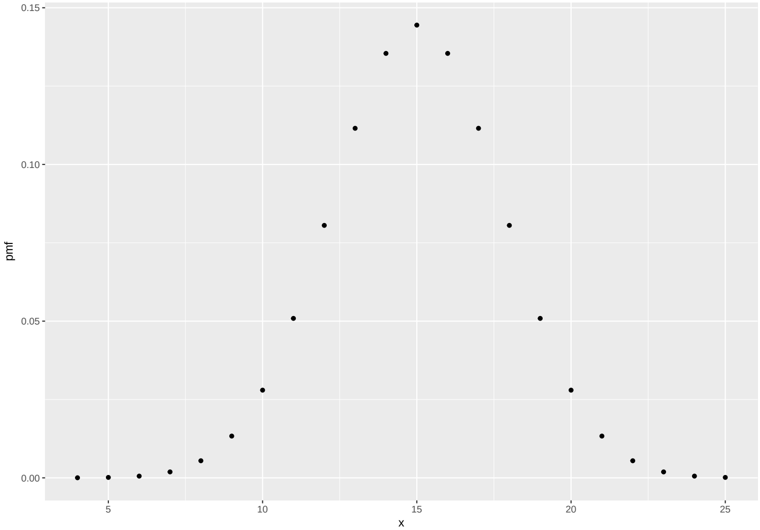
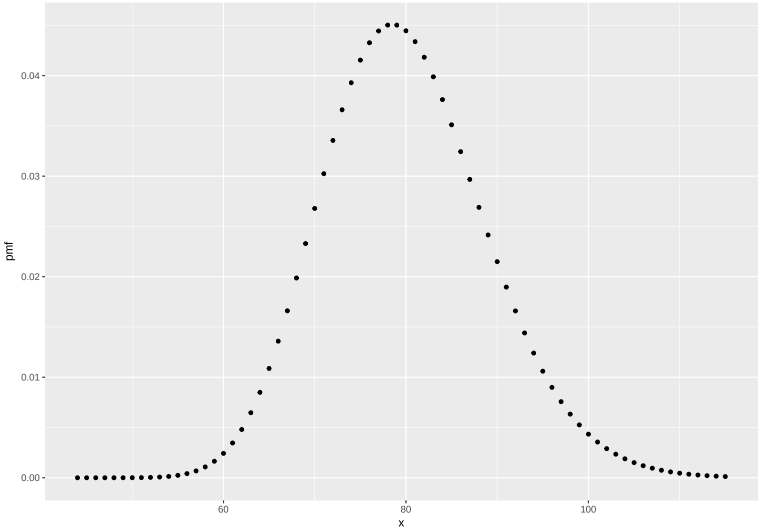
Let's see what happens with the sum is of not a small number of terms...

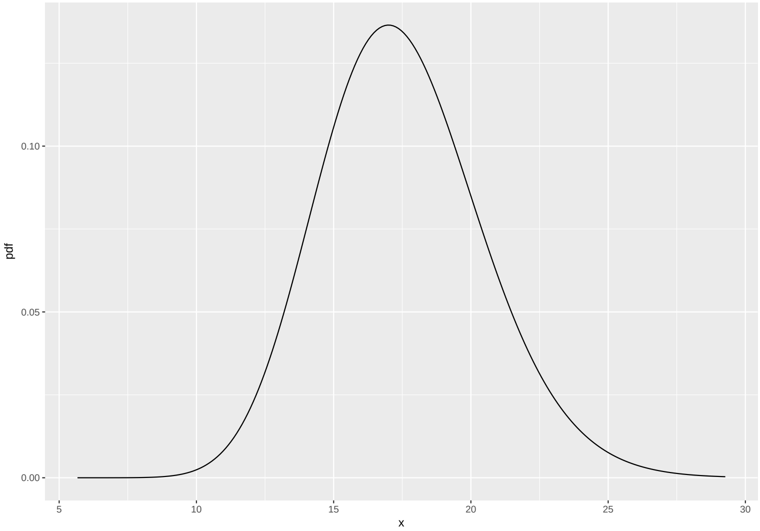
Binomial(30, 0.5)



NegBin(40, 0.5)



Gamma(35, 2)



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Height, test score, lab measurement, etc.

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But this fact undersells the critical importance of the normal distributions.

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We say X has a normal distribution with parameters μ and σ, or $X \sim N(\mu, \sigma)$ if its density is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

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A special case is the "standard normal" which is $Z \sim N(0, 1)$