STA286 Lecture 17

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waiting time to the n^{th} event of a Poisson process

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So the density is (a long telescoping sum of work later...):

$$f(t) = F'(t) = \begin{cases} \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} & : t > 0 \\ 0 & : \text{otherwise.} \end{cases}$$

the gamma distributions

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Definition: the *gamma function* is defined as:

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Many interesting properties, including $\Gamma(n) = (n-1)!$ for integer $n \geqslant 1$.

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The following function is a valid density for $\alpha > 0$ and $\lambda > 0$:

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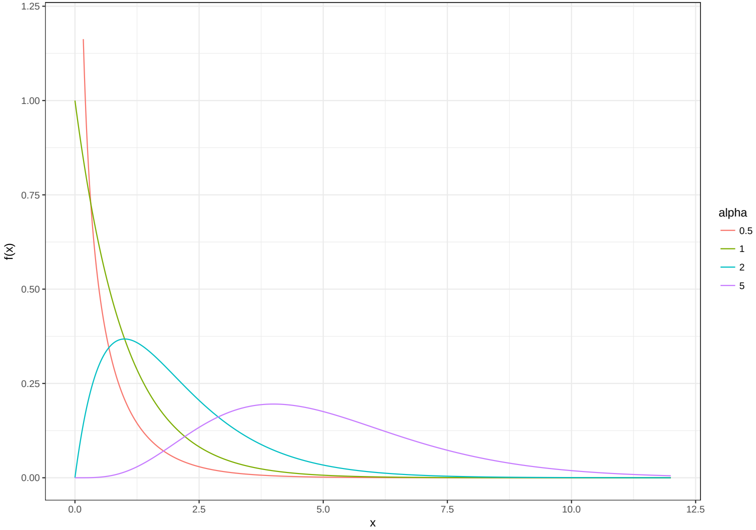
$\alpha = 1$ is the special case of $\text{Exp}(\lambda)$.

pictures of some Gamma(α, 1) densities



properties of gamma distributions

If $X \sim \text{Gamma}(\alpha, \lambda)$, its moment generating function can be found to be:

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Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. $\text{Exp}(\lambda)$. What is the distribution of $X = X_1 + X_2 + \cdots + X_n$?

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Using the m.g.f. argument it is clear that $X \sim \text{Gamma}(n, \lambda)$, which makes sense in the Poisson process context.

summary of the Bernoulli-Poisson-o-sphere

Starting with a Bernoulli(p) process, we have the following:

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Count			
Inter-arrival			
Wait for r^{th} event			
Look back after 1			

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Look back after 1	"Discrete Uniform"	$(\leftarrow$ not done$)$	Uniform$(0, t)$

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But it doesn't matter.

Although it makes me scarlet with rage.

the “normal” distributions

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We've already seen many examples of things that can be considered as *sums of random variables*.

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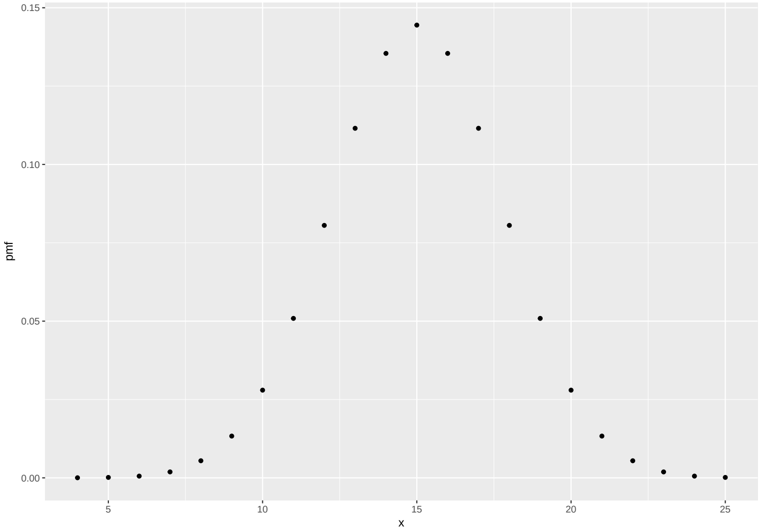
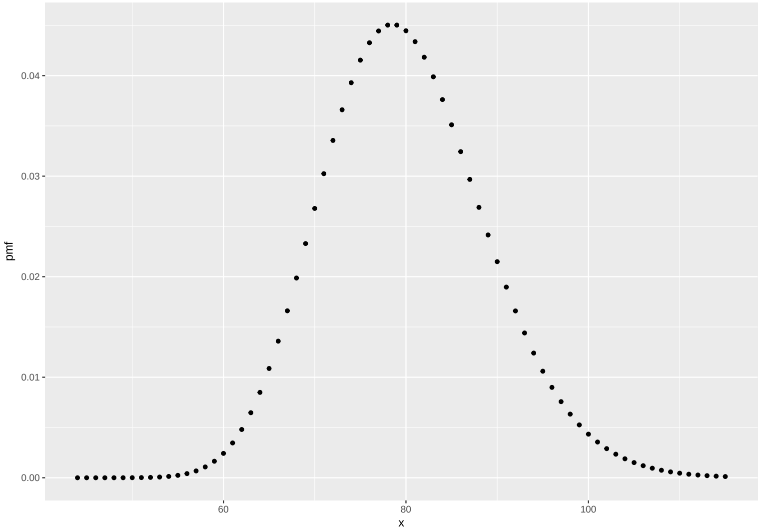
Let's see what happens with the sum is of not a small number of terms...

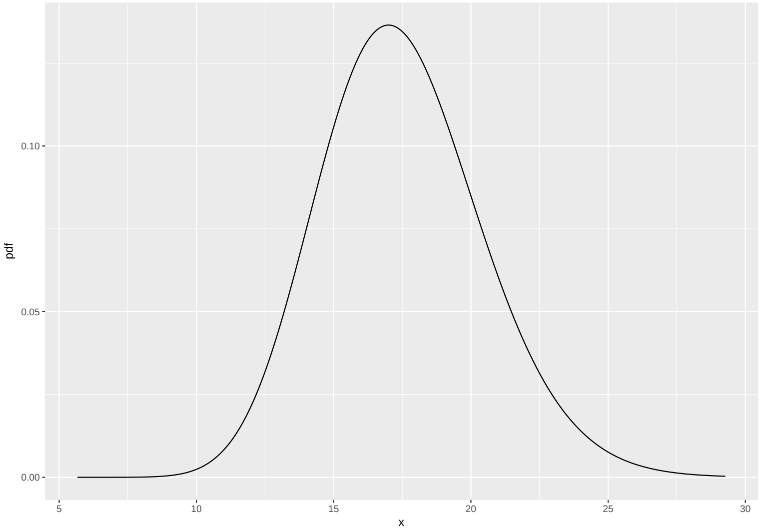
Binomial(30, 0.5)



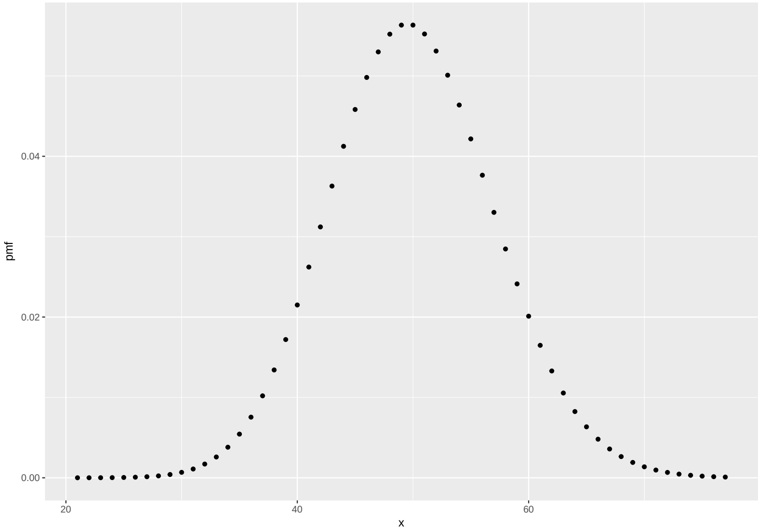
NegBin(40, 0.5)



Gamma(35, 2)



Poisson(50)



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Height, test score, lab measurement, etc.

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But this fact undersells the critical importance of the normal distributions.

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We say X has a normal distribution with parameters μ and σ, or $X \sim N(\mu, \sigma)$, when it has this density.