STA286 Lecture 20

Neil Montgomery

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basic result for $\overline{X}$ in general

For a sample $X_1, \ldots, X_n$ i.i.d. from *any* distribution with mean μ and variance σ^2, the following are always true:

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\begin{aligned}
E\left(\overline{X}\right) &= E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
\mathrm{Var}\left(\overline{X}\right) &= \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)
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$$SD(\overline{X}) = \frac{\sigma}{\sqrt{n}}$$

full distribution of $\overline{X}$ when sample is normal

For a sample $X_1, \ldots, X_n$ i.i.d. $N(\mu, \sigma)$ we have:

$$M_{\sum_{i=1}^{n} X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

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The seemingly impossible task is to determine the distribution of $\overline{X}$ when the underlying distribution is not normal (i.e., almost always.)

the actual central limit theorem

For any random variable X with mean μ and variance σ^2, consider a sample $X_1, \ldots, X_n$ from the same distribution.

Now consider the sample average of this sample: $\overline{X}_n$ (because n will be changing below. . .).

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The actual Central Limit Theorem (CLT) says:

$$P\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \leqslant u\right)$$

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where $Z \sim N(0, 1)$.

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$$\lim_{n\to\infty} F_n(u) = F_Z(u)$$

where $Z \sim N(0,1)$.

the value of the CLT is in the speed of convergence

Useful limit theorems are ones where the convergence is fast—the CLT is such an example.

$$P\left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \leqslant u\right) \approx P(Z \leqslant u)$$

for n "large enough".

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- $n = 30$ for mildly skewed X.

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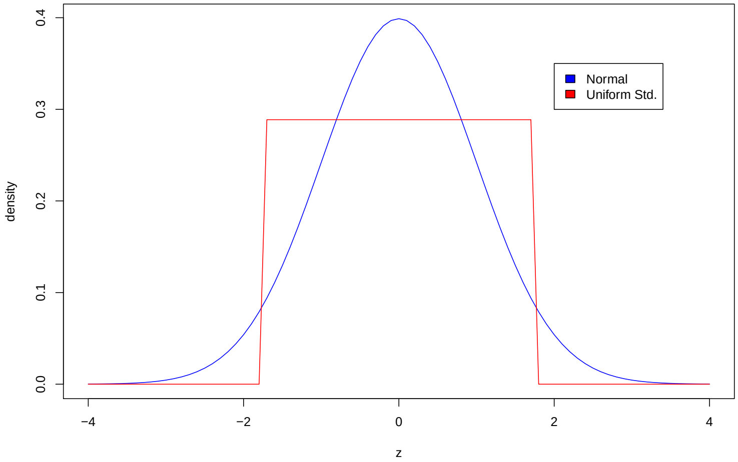
- $n = 10$ good enough for symmetric distributions without outliers.
- $n = 30$ for mildly skewed X.
- $n > 60$ for more skewed.

how large is large enough case I - symmetric/no outliers

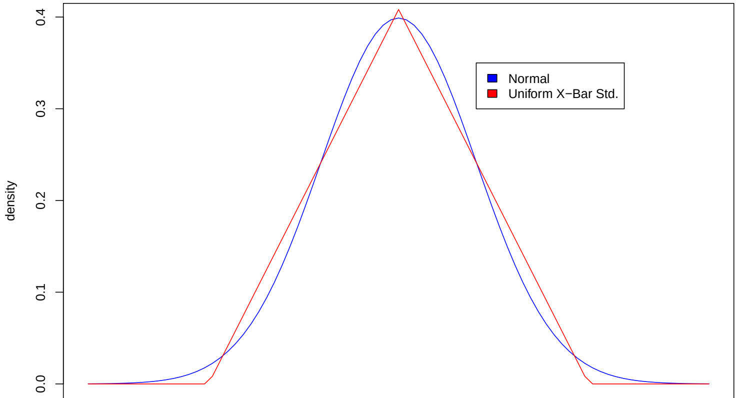
Consider a Uniform[0,1] distribution. The mean is 0.5 and the standard deviation is $1/\sqrt{12}$. Here is a plot of a *standardized* uniform density versus a $N(0,1)$ density:



how large is large enough case I - symmetric/no outliers

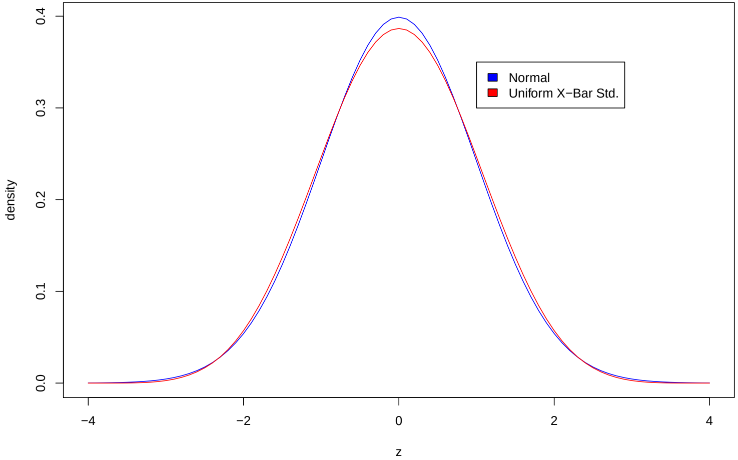
Now consider X_1, X_2 i.i.d. Uniform[0,1], and its sample average $\overline{X}$, which will have mean 0.5 and standard deviation $1/\sqrt{12 \cdot 2}$.

Here is a picture of the density for $\frac{\overline{X} - 0.5}{(1/\sqrt{12 \cdot 2})/\sqrt{2}}$, along with the density for $Z \sim N(0,1)$.



how large is large enough case I - symmetric/no outliers

Same as before but now with X_1, X_2, X_3, X_4, X_5 (i.e. $n = 5$):



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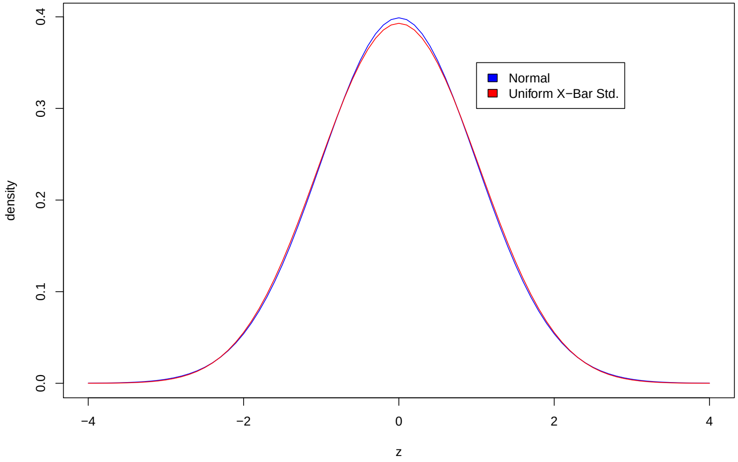
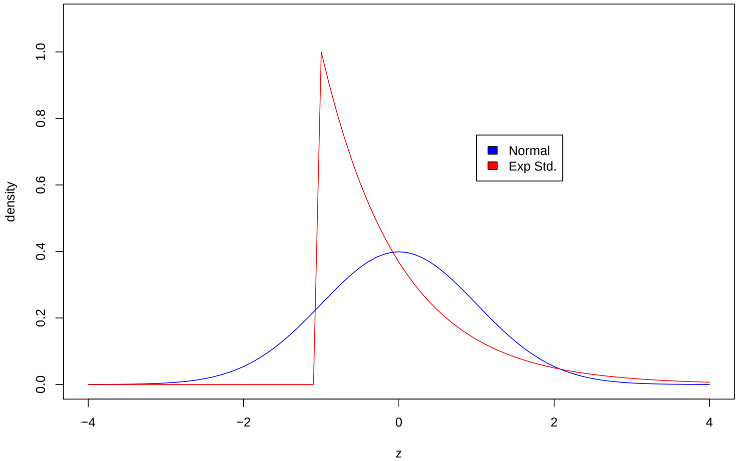
Now with $n = 10$.



how large is large enough case II - skewed

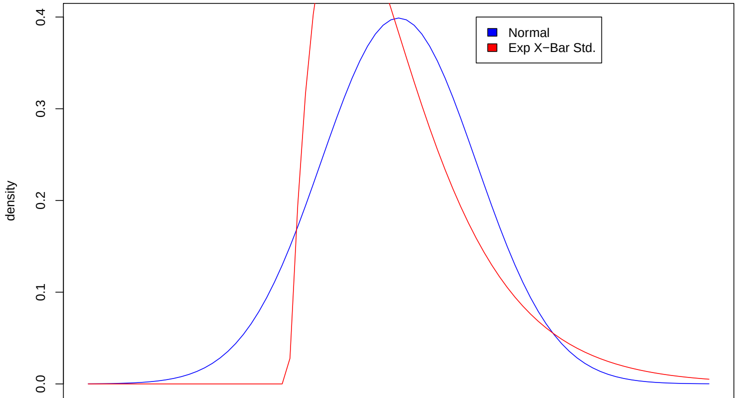
Now let the "underlying" distribution be Exp(1), which has mean and standard deviation both equal to $1/1 = 1$. Here's the standardized density along with a $N(0, 1)$ density:



how large is large enough case II - skewed

Now consider X_1, X_2 i.i.d. Exp(1), and its sample average $\overline{X}$, which will have mean 1 and standard deviation $1/\sqrt{2}$.

Here is a picture of the density for $\frac{\overline{X}-1}{1/\sqrt{2}}$, along with the density for $Z \sim N(0,1)$.



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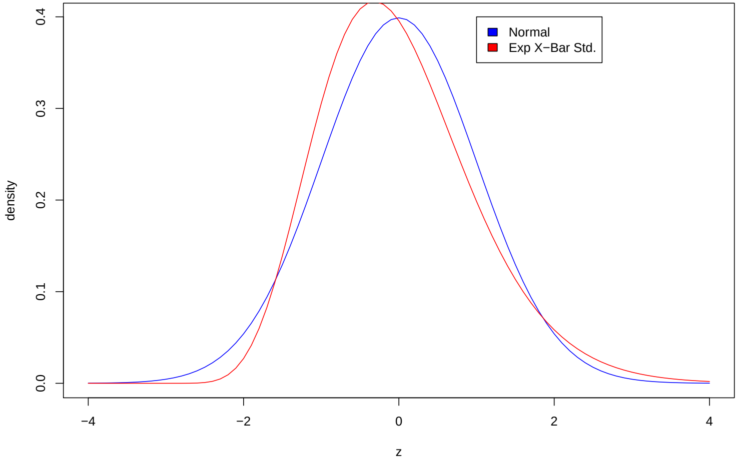
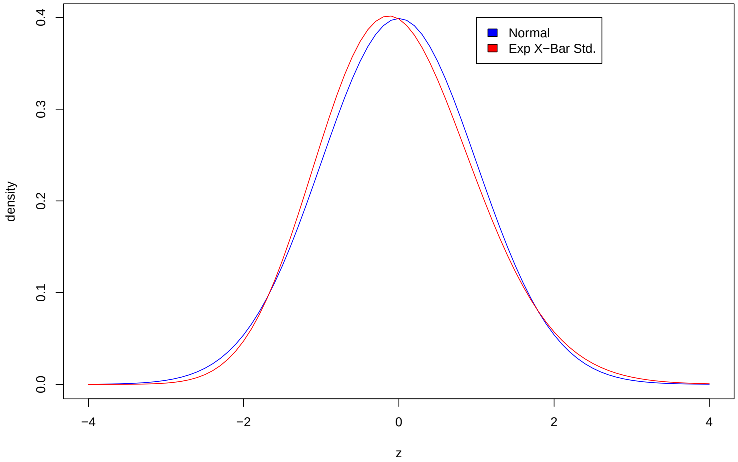
Now with $n = 10$.



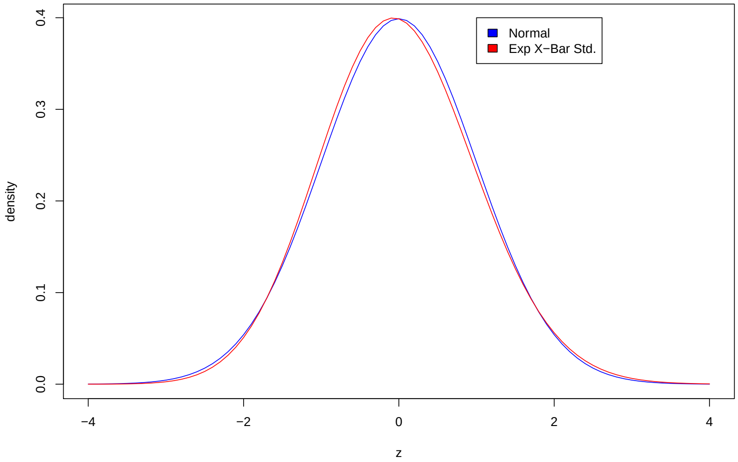
how large is large enough case II - skewed

Try $n = 60$



how large is large enough case II - skewed

Try $n = 200$



normal approximation applications

Given $X_1, \ldots, X_n$ i.i.d with mean μ and variance σ^2, as long as n is large enough, any of the following approximations hold. Pick the most convenient:

$$\sum_{i=1}^{n} X_i \sim^{approx} N(n\mu, \sqrt{n}\sigma)$$

$$\overline{X} \sim^{approx} N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim^{approx} N(0, 1)$$

normal approximation example

Piston lifetimes in a Diesel engine follow a roughly symmetric distribution with mean 3.4 years and standard deviation 2.1 years.

What is the chance that the average life of 25 engines exceeds 4 years?

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another normal approximation example

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If X_i is 0 or 1 as item i is not defective, or defective, respectively, then $X_i \sim \text{Bernoulli}(0.01)$. We want $P\left(\sum_{i=1}^{n} X_i < 80\right)$.

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In this case $n = 10000$, $\mu = p$, and $\sigma^2 = p(1-p)$. So:

$$\sum_{i=1}^{n} X_i \sim^{approx} N(100, \sqrt{99})$$

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