STA286 Lecture 21

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a quick note—n "large enough" in the Bernoulli(p) special case

The probability p is in some sense a "shape" parameter for Binomial distributions, in the sense that the close p is to 0 (or 1), the more right (left) skewed the distribution is.

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We'll revisit this issue when the time comes to discuss ways to evaluate the empirical accuracy of a normal approximation.

more normal-based "sampling distributions"

The main focus will be on $\overline{X}$ and friends, because the most common statistical problem is to make statements about an unknown population mean μ and $\overline{X}$ will be used as the guess.

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but typically σ is also unknown, and $\sqrt{S^2}$ will be used as *its* guess.

the distribution of the sample variance S^2 - I

First we'll consider the distribution of Z^2 when $Z \sim N(0,1)$.

$$M_{Z^2}(t) = E\left(e^{tZ^2}\right) = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

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\end{aligned}
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The integrand is a $N\left(0, \frac{0.5}{0.5-t}\right)$ density, so the integral equals 1, leaving

$M_{Z^2}(t) = \left(\frac{0.5}{0.5-t}\right)^{0.5}$

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Therefore: If $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma)$, then:

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Theorem: If X and Y are independent with $X \sim \chi^2_n$ and $X + Y \sim \chi^2_{n+m}$, then $Y \sim \chi^2_m$.

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Magical truth: If $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma)$, then $\overline{X}$ and S^2 are independent.

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Finally: If $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma)$:

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$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} \left(X_i - \overline{X} + \overline{X} - \mu \right)^2$$

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Conclusion: $\frac{n-1}{\sigma^2} S^2 \sim \chi^2_{n-1}$

the t distributions - I

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However, when the constant σ is replaced with the random variable S, the result is no longer $N(0,1)$.

It turns out it is possible to derive the density of:

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

the t distributions - II

The density $f(t)$ of T is is primarily nice to look at:

$$f(t) = \frac{\Gamma[(\nu+1)/2]}{\Gamma[\nu/2]\sqrt{\pi}\nu}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2}$$

where ν is called the "degrees of freedom" and in this case is $n-1$. You can think of $n-1$ as having been "inherited" from the denominator of T.

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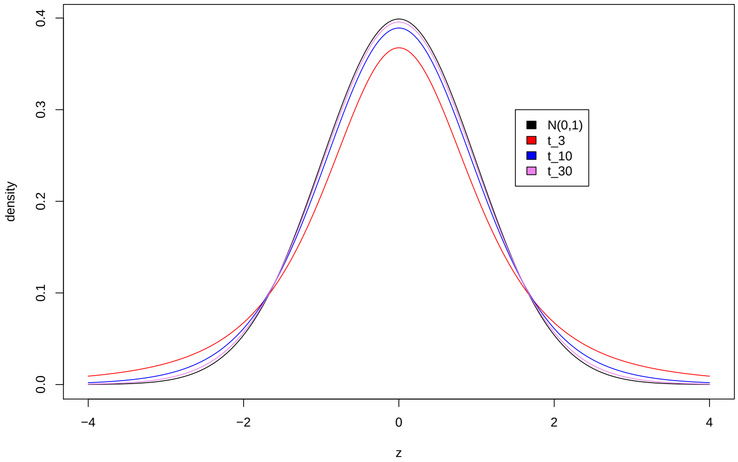
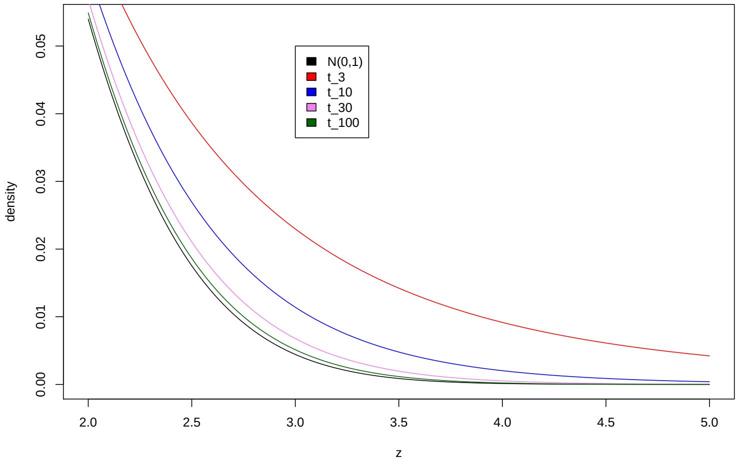
- ...in other words *as the sample size gets large, T starts to look like $Z \sim N(0,1)$*
- no anti-derivative, so a table of t probabilities needed on tests.

overall pictures of t_ν



pictures of t_ν in the "tail"



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$$F = \frac{U/m}{V/n} \sim F_{m,n}$$

or "F has an F distribution with m and n degrees of freedom."

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The density is nasty, etc.

pictures of some F distributions