STA286 Lecture 25

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watching even more paint dry

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So just to bother you, I'll use $n = 130$.

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From the t_{129} distribution we get $t_{129,0.025} = 1.979$. So the 95% confidence interval is:

$$\overline{x} \pm t_{129,0.025} \frac{s}{\sqrt{n}} = 4.143 \pm 1.979 \frac{1.037}{\sqrt{130}}$$

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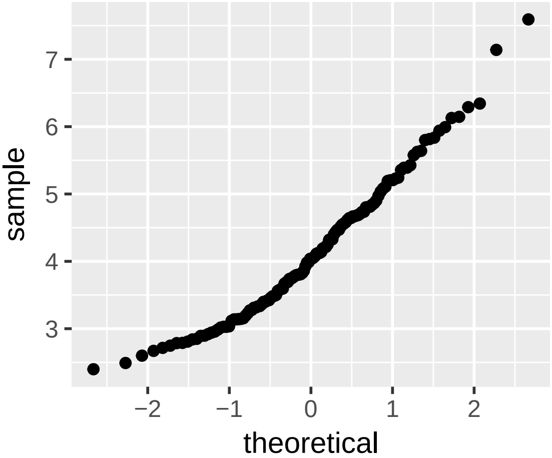
or

$$[3.963, 4.323]$$

verifying the model assumption(s)

In this case there is only one assumption (that can be verified)—that the underlying distribution is normal. Here is a normal quantile plot of the data:



model assumption conclusion | robustness of "t procedure"

The normal distribution assumption has been violated. It seems the underlying distribution is skewed right.

However, the sample size $n = 130$ is large, so by the speed of convergence of the CLT and its buddy Mr. Slutsky, we're still OK.

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Dirty secret: I simulated the data to get the example sample of size 130.

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As an example of what is called the "robustness" of this confidence interval against violations of the normality assumption, I did a quick simulation (code embedded in notes).

The proportion of the 10^4 simulated confidence intervals that captured the true mean is (for this simulation—changes every time I render the lecture notes):

$$0.9444$$

prediction, as opposed to estimation

To get the interval estimate of μ we used the fact that $\overline{X} - \mu$ is normal with variance $\mathrm{Var}\left(\overline{X} - \mu\right) = \sigma^2/n$ to obtain $(\overline{X} - \mu)/(\sigma/\sqrt{n}) \sim N(0,1)$, etc.

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If the population is normal, so will be $\overline{X} - X$, and its mean will be $E\left(\overline{X} - X\right) = \mu - \mu = 0$

prediction "interval"

Put it all together to get:

$$\frac{\overline{X} - X}{\sigma\sqrt{1 + \frac{1}{n}}} \sim N(0, 1)$$

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Deal with unknown σ right now—replace it with S from the sample, to get:

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A $100 \cdot (1 - \alpha)\%$ *prediction interval* can be obtained by solving for X in:

$$P\left(-t_{n-1,\alpha/2} < \frac{\overline{X} - X}{S\sqrt{1 + \frac{1}{n}}} < t_{n-1,\alpha/2}\right)$$

prediction interval example

The formula is:

$$\overline{X} \pm t_{n-1,\alpha/2} S \sqrt{1 + \frac{1}{n}}$$

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Just that little '1' under the square root—but it makes all the difference. It guarantees that the prediction can never be better than the variance in the population itself, no matter what the sample size.

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Using the paint example, with $t_{129,0.025} = 1.97852$ and

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in the formula gives:

$$4.143 \pm 1.979 \cdot 1.037 \sqrt{1 + \frac{1}{130}} \qquad \text{or} \qquad [2.084, 6.203]$$

prediction interval model assumptions

Normal population is the only assumption.

Suppose the population is not normal. What might happen to the following as n gets large?

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So the population really has to be normal, or the P.I. formula doesn't work.

The paint drying P.I. we calculated is therefore not that useful.

the two-sample problem (normal populations)

We've solved the case of one numerical variable in a dataset with a normal population.

Often you'll have a numerical variable in one column, and a "grouping" variable in another column that categorizes the observations into two groups.

Variable	Group
3.85	2
6.06	2
3.28	1
4.85	2
5.34	1
6.03	2
$\vdots$	$\vdots$

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Variable	Group
X_{21}	2
X_{22}	2
X_{11}	1
X_{23}	2
X_{12}	1
X_{24}	2
$\vdots$	$\vdots$

the two-sample problem (normal populations) with equal variances

We have two populations $N(\mu_1, \sigma)$ and $N(\mu_2, \sigma)$, and the goal is to estimate $\theta = \mu_1 - \mu_2$.

Gather independent samples: $X_{11}, \ldots, X_{1n_1}$ i.i.d. $N(\mu_1, \sigma)$ and $X_{21}, \ldots, X_{2n_2}$ i.i.d. $N(\mu_2, \sigma)$.

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\end{aligned}
$$

We need to figure out what to do about σ^2.