STA286 Lecture 32

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the "two-sample t-test"

A more realistic hypothesis testing scenario.

Two populations: $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$. The obvious hypotheses will always be:

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

The "parameter" is $\theta = \mu_1 - \mu_2$, estimated (as usual) by $\overline{X_1} - \overline{X_2}$ from samples of sizes n_1 and n_2.

Two possibilities:

$$\frac{\overline{X_1} - \overline{X_2}}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2} \qquad \text{or} \qquad \frac{\overline{X_1} - \overline{X_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \sim t_\nu$$

two-sample t-test example

Modified from 10.106. Can nutritional counselling change blood cholesterol level? A group of 15 people received counseling for 8 weeks. A group of 18 people did not.

The readings are made available by the textbook in the following terrible manner:

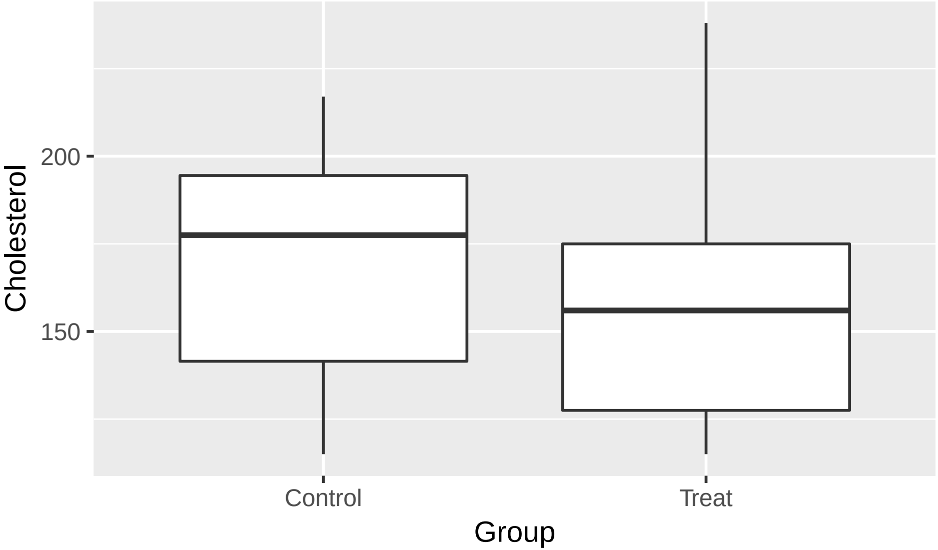
Treat	Control
129	151
131	132
154	196
172	195
115	188
126	198
175	187
191	168
122	115
238	165
159	137
156	208
176	133
175	217
126	191
	193
	140
	146

two-sample t-test example

A Real Dataset:

ID	Group	Cholesterol
2	Treat	131
4	Treat	172
26	Control	168
19	Control	151
23	Control	188
20	Control	132
9	Treat	122
3	Treat	154
13	Treat	176
32	Control	217
8	Treat	191
10	Treat	238
24	Control	198
27	Control	115
36	Control	146
12	Treat	156
33	Control	191
35	Control	140
7	Treat	175
34	Control	193
31	Control	133
6	Treat	126
15	Treat	126
1	Treat	129
22	Control	195
14	Treat	175
28	Control	165
30	Control	208
21	Control	196

two-sample t-test example - plot



two-sample t-test example - equal variance version

Group	n	X_bar	S
Control	18	170.00	30.79
Treat	15	156.33	33.09

```
##
##  Two Sample t-test
##
## data:  Cholesterol by Group
## t = 1.23, df = 31, p-value = 0.23
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  -9.0417 36.3750
## sample estimates:
## mean in group Control   mean in group Treat
##                170.00                156.33
```

two-sample t-test example - no variance assumption version

Group	n	X_bar	S
Control	18	170.00	30.79
Treat	15	156.33	33.09

```
##
##  Welch Two Sample t-test
##
## data:  Cholesterol by Group
## t = 1.219, df = 29.04, p-value = 0.233
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  -9.25835 36.59169
## sample estimates:
## mean in group Control   mean in group Treat
##               170.000               156.333
```

two-sample t-test example - or is it?

Question 10.54. Nine people had breathing rates measured with and without elevated CO levels.

Subject	WithCO	WithoutCO
1	30	30
2	45	40
3	26	25
4	25	23
5	34	30
6	51	49
7	46	41
8	32	35
9	30	28

Does CO impact breathing frequency?

two-sample t-test example - or is it?

Two populations are $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$.

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

The two samples are $X_{11}, \ldots, X_{19}$ and $X_{21}, \ldots, X_{29}$.

two-sample t-test example - or is it?

Two populations are $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$.

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

The two samples are $X_{11}, \ldots, X_{19}$ and $X_{21}, \ldots, X_{29}$.

But they are surely not independent. We should examine the differences $D_1, \ldots, D_9$, which will be $N(\mu_D, \sigma_D)$ where $\mu_D = \mu_1 - \mu_2$.

two-sample t-test example - or is it?

Two populations are $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$.

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

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But they are surely not independent. We should examine the differences $D_1, \ldots, D_9$, which will be $N(\mu_D, \sigma_D)$ where $\mu_D = \mu_1 - \mu_2$.

Here's a one-sample case where the null and alternatives are actually self-evident:

$$H_0 : \mu_D = 0$$
$$H_1 : \mu_D \neq 0$$

two-sample t-test example - or is it?

The analysis:

n	X_Bar_1	X_Bar_2	S_1	S_2	X_bar_D	S_D
9	35.444	33.444	9.462	8.502	2	2.55

```
##
##  One Sample t-test
##
## data:  co$WithCO - co$WithoutCO
## t = 2.353, df = 8, p-value = 0.0464
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
##  0.0402733 3.9597267
## sample estimates:
## mean of x
##         2
```

single proportion example (something funny happens)

From the second test, that gas company "knew" the proportion of defective meters was 0.01. Let's change that to "assumes" (perhaps based on some industry knowledge). As usual, the single sample scenarios tend to be a bit contrived.

Work Team Beta inspects 2000 meters and finds 24 defective ones. Is there evidence that the company's assumption is inaccurate?

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Work Team Beta inspects 2000 meters and finds 24 defective ones. Is there evidence that the company's assumption is inaccurate?

$$H_0 : p = 0.01$$
$$H_1 : p \neq 0.01$$

single proportion example (something funny happens)

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$$H_0 : p = 0.01$$
$$H_1 : p \neq 0.01$$

We'll use the MLE $\hat{p}$, for which we know:

$$\hat{p} \sim^{approx} N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$$

To calculate the p-value (or to get a critical region) we plug the H_0 value to obtain the null distribution. This happens to eliminate the unknown variance problem!

single proportion example

single proportion example

We observe $\hat{p}_{obs} = 0.012$. What is the p-value?

$$
\begin{aligned}
P(\hat{p} < 0.008) + P(\hat{p} > 0.012) &= P\left(Z < \frac{0.008 - 0.01}{\sqrt{\frac{0.01(1-0.01)}{2000}}}\right) + P\left(Z > \frac{0.012 - 0.01}{\sqrt{\frac{0.01(1-0.01)}{2000}}}\right) \\
&= P(Z < -0.899) + P(Z > 0.899) \\
&= 0.369
\end{aligned}
$$

two proportion example - a little trick

Much more natural.

Let's say Work Team Beta found $x_1 = 24$ defective meters in $n_1 = 2000$ inspections, and Work Team Delta found $x_2 = 14$ in $n_2 = 1500$ inspections. Do the teams find defectives at the same rate?

We are comparing a Bernoulli(p_1) with a Bernoulli(p_2). The null and alternative are self-evident:

$$H_0 : p_1 = p_2 \qquad H_1 : p_1 \neq p_2$$

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We are comparing a Bernoulli(p_1) with a Bernoulli(p_2). The null and alternative are self-evident:

$$H_0 : p_1 = p_2 \qquad H_1 : p_1 \neq p_2$$

We will use $\hat{p}_1 - \hat{p}_2$, which we know satisfies:

$$\hat{p}_1 - \hat{p}_2 \sim^{approx} N\left(p_1 - p_2, \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}\right)$$

two proportion example - null distribution little trick

When computing the p-value, we plug in the H_0 fact that $p_1 = p_2$, which we will denote by just p. The variance of the "null distribution" reduces to:

$$p(1-p)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

two proportion example - null distribution little trick

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We don't know p. Use the data, which, under the null hypothesis, are just 0's and 1's from the same Bernoulli(p) distribution. We pool them together to get:

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

two proportion example - null distribution little trick

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$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

So the null distribtion is:

$$\hat{p}_1 - \hat{p}_2 \sim^{approx} N\left(0, \sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}\right)$$

two proportion example

In our example we had $x_1 = 24$, $n_1 = 2000$, $x_2 = 14$, $n_2 = 1500$. So:

$$\hat{p} = 0.010857$$

and the standard deviation of the null distribution is 0.00354.

Also, $\hat{p}_1 - \hat{p}_2 = 0.002667$

The p-value is 0.451228 based on $2P(Z < -0.75337)$