Modeling two-level longitudinal data

Inference cont'd

03.23.22

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Announcements

- Mini-project 02
  - draft due **Thu, Mar 24 at noon** (peer review in lab)
  - presentations **Mon, Mar 28** in class
  - report due **Mon, Mar 28** at 11:59pm
- Looking ahead
  - HW 04 due **Fri, Apr 1** (assigned later this week)
  - Final project - optional draft due **Fri, Apr 15**, final report due **Wed, Apr 27**
- DataFest: April 1 - 3 in Penn Pavilion
  - Click here to register

Learning goals

- Compare multilevel models
- Conduct inference for random effects
- Conduct inference for fixed effects

Comparing multilevel models

Data: Charter schools in MN

The data set `charter-long.csv` contains standardized test scores and demographic information for schools in Minneapolis, MN from 2008 to 2010. The data were collected by the Minnesota Department of Education. Understanding the effectiveness of charter schools is of particular interest, since they often incorporate unique methods of instruction and learning that differ from public schools.

- **`MathAvgScore`**: Average MCA-II score for all 6th grade students in a school (response variable)
- **`urban`**: urban (1) or rural (0) location school location
- **`charter`**: charter school (1) or a non-charter public school (0)
- **`schPctfree`**: proportion of students who receive free or reduced lunches in a school (based on 2010 figures).
- **`year08`**: Years since 2008

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Data

```r
charter <- read_csv("data/charter-long.csv")
```

schoolName	year08	urban	charter	schPctfree	MathAvgScore
RIPPLESIDE ELEMENTARY	0	0	0	0.363	652.8
RIPPLESIDE ELEMENTARY	1	0	0	0.363	656.6
RIPPLESIDE ELEMENTARY	2	0	0	0.363	652.6
RICHARD ALLEN MATH&SCIENCE ACADEMY	0	1	1	0.545	NA
RICHARD ALLEN MATH&SCIENCE ACADEMY	1	1	1	0.545	NA
RICHARD ALLEN MATH&SCIENCE ACADEMY	2	1	1	0.545	631.2

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Compare two models

Full model

$$
\begin{aligned}
Y_{ij} = &[\alpha_0 + \alpha_1 Charter_i + \alpha_2 Urban_i + \alpha_3 schpctfree_i \\
&+ \beta_0 Year08_{ij} + \beta_1 Charter_i : Year08_{ij} + \beta_2 Urban_i : Year_{ij}] \\
&+ [u_i + v_i Year08_{ij} + \epsilon_{ij}]
\end{aligned}
$$

where

$$
\begin{bmatrix} u_i \\ v_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{bmatrix} \right) \text{ and } \epsilon_{ij} \sim N(0, \sigma^2)
$$

Estimated fixed effects: $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$

Variance components to estimate: $\sigma, \sigma_u^2, \sigma_v^2, \rho_{uv}$ (Note: $\sigma_{uv} = \rho_{uv}\sigma_u\sigma_v$

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Compare two models

Null Model (simplified variance structure)

$$
\begin{aligned}
Y_{ij} = & [\alpha_0 + \alpha_1 Charter_i + \alpha_2 Urban_i + \alpha_3 schpctfree_i \\
& + \beta_0 Year08_{ij} + \beta_1 Charter_i : Year08_{ij} + \beta_2 Urban_i : Year_{ij}] \\
& + [u_i + \epsilon_{ij}]
\end{aligned}
$$

where

$$u_i \sim N(0, \sigma_u^2) \text{ and } \epsilon_{ij} \sim N(0, \sigma^2)$$

Estimated fixed effects: $\alpha_0, \alpha_1, \alpha_3, \alpha_3, \beta_0, \beta_1, \beta_2$

Variance components to estimate: σ, σ_u^2

Full and reduced models

```
full_model <- lmer(MathAvgScore ~ charter + urban + schPctfree +
                     charter:year08  + urban:year08 + year08 +
                     (year08|schoolid), REML = T, data = charter)
```

```
reduced_model <-  lmer(MathAvgScore ~ charter + urban + schPctfree +
                     charter:year08  + urban:year08 + year08 +
                       (1 | schoolid), REML = T, data = charter)
```

Hypotheses

$$
\begin{aligned}
&H_0 : \sigma_v = \rho_{uv} = 0 \\
&H_a : \text{at least one of the parameters is not equal to 0}
\end{aligned}
$$

Note: $\rho_{uv} \neq 0 \Rightarrow \sigma_v \neq 0$

Bootstrapping

- **Bootstrapping** is from the phrase "pulling oneself up by one’s bootstraps"
- Accomplishing a difficult task without any outside help
- **Task**: conduct inference for model parameters (fixed and random effects) using only the sample data
- Two types of bootstrapping
    - Parametric bootstrapping (use for LRT to test variance components)
    - Nonparametric bootstrapping (use to test fixed effects)

Parametric bootstrapping for likelihood ratio test

1️⃣ Fit the null (reduced) model to obtain the fixed effects and variance components (*parametric* part).

2️⃣ Use the estimated fixed effects and variance components to generate a new set of response values with the same sample size and associated covariates for each observation as the original data (*bootstrap* part).

3️⃣ Fit the full and null models to the newly generated data.

4️⃣ Compute the likelihood test statistic comparing the models from the previous step.

5️⃣ Repeat steps 2 - 4 many times (~ 1000).

Parametric bootstrapping for likelihood ratio test

6️⃣ Create a histogram of the likelihood ratio statistics to get the distribution of likelihood ratio statistic under the null hypothesis.

7️⃣ Get the p-value by calculating the proportion of bootstrapped test statistics greater than the observed statistic.

LRT using χ^2 and parametric bootstrap

Likelihood ratio test using χ^2 distribution

	npar	AIC	BIC	logLik	deviance	Chisq	Df	Pr(>Chisq)
reduced_model	9	9952.992	10002.11	-4967.496	9934.992	NA	NA	NA
full_model	11	9953.793	10013.83	-4965.897	9931.793	3.199	2	0.202

Likelihood ratio test using parametric bootstrap

term	npar	AIC	BIC	logLik	deviance	statistic	df	Pr_boot..Chisq.
m0	9	9952.992	10002.11	-4967.496	9934.992	NA	NA	NA
mA	11	9953.793	10013.83	-4965.897	9931.793	3.199347	2	0.144

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Inference for fixed effects

- The output for multilevel models do not contain p-values for the coefficients of fixed effects
- The exact distribution of the test statistic under the null hypothesis (no fixed effect) is unknown, because the exact degrees of freedom are unknown
  - Finding suitable approximations is an area of ongoing research
- We can use likelihood ratio test with an approximate χ^2 distribution to test these effects, since we're not testing on the boundary and fixed effects do not have limited ranges
  - Some research suggests the p-values are too low but approximations are generally pretty good
  - Can also calculate the p-values using parametric bootstrap approach

Go to **`lecture-20.Rmd`** to test whether **`schPctFree`** should be included in the current model using likelihood ratio test with the χ^2 distribution and the parametric bootstrap.

Methods to conduct inference for individual coefficients

- Use the t-value $\left(\frac{estimate}{std.error}\right)$ in the model output
  - General rule: Coefficients with |t-value| > 2 considered to be statistically significant, i.e., different from 0
- Calculate confidence intervals using nonparametric bootstrapping

Nonparametric bootstrapping

1️⃣ Take a sample, with replacement, of size n (the size of the original data) (called *case resampling*).

2️⃣ Fit the model to obtain estimates of the coefficients.

3️⃣ Repeat steps 1 - 2 many times (~ 1000) to obtain the bootstrap distribution.

4️⃣ Get the coefficients for the 95% confidence interval by taking the middle 95% of the bootstrap distribution.

Go to **`lecture-20.Rmd`** for an example of nonparametric bootstrapping to estimate the 95% confidence interval for the model coefficients.

Bootstrapped CI for coefficients

```
confint(reduced_model, method = "boot", level = 0.95, oldNames = F) %>%
  kable(digits = 3)
```

	2.5 %	97.5 %
sd_(Intercept)|schoolid	4.274	4.878
sigma	2.855	3.102
(Intercept)	659.438	661.410
charter	-4.355	-1.606
urban	-1.974	-0.266
schPctfree	-19.542	-16.288
year08	1.229	1.806
charter:year08	0.446	1.676

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Summary: Model comparisons

Methods to compare models with different fixed effects

- Likelihood ratio tests based on χ^2 distribution
- Likelihood ratio test based on parametric bootstrapped p-values
- AIC or BIC

Methods to compare models with different variance components

- Likelihood ratio test based on parametric bootstrapped p-values
- AIC or BIC

Summary: Understanding the model

Methods to understand individual coefficients

- Bootstrap confidence intervals
- Pseudo R^2 for variance components (if meaning is unchanged between models)

Methods to understand data structure

- Calculate intraclass correlation coefficient using unconditional means model

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Acknowledgements

The content in the slides from

- BMLR: Chapter 9 - Two-level Longitudinal Data
  - Sections 9.1 - 9.6
- Hierarchical Linear Modeling with Maximum Likelihood, Restricted Maximum Likelihood, and Fully Bayesian Estimation by Peter Boedeker
- *Applied longitudinal data analysis: Modeling change and event occurrence*. by J.D. Singer and J.B. Willett
  - Online copy available through Duke library

Acknowledgements

- *Extending the linear model with R.* by Julian Faraway
    - Online copy available through Duke library
- More mathematical details in *Computational methods for mixed models* by Douglas Bates (a vignette for the lme4 R package)

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