Poisson Regression

Prof. Maria Tackett

01.26.22

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Announcements

- Week 03 & 04 reading:
  - BMLR: Chapter 4 - Poisson Regression
- Quiz 01 due Thu, Jan 27 at 3:30pm (start of lab)
- Mini-project 01: Analysis plan due Thu, Jan 27 at 11:59pm for feedback (optional)

Learning goals

- Describe properties of the Poisson random variable
- Write the Poisson regression model
- Describe how the Poisson regression differs from least-squares regression
- Interpret the coefficients for the Poisson regression model
- Compare two Poisson regression models

Scenarios to use Poisson regression

- Does the number of employers conducting on-campus interviews during a year differ for public and private colleges?
- Does the daily number of asthma-related visits to an Emergency Room differ depending on air pollution indices?
- Does the number of paint defects per square foot of wall differ based on the years of experience of the painter?

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Each response variable is a **count per a unit of time or space**.

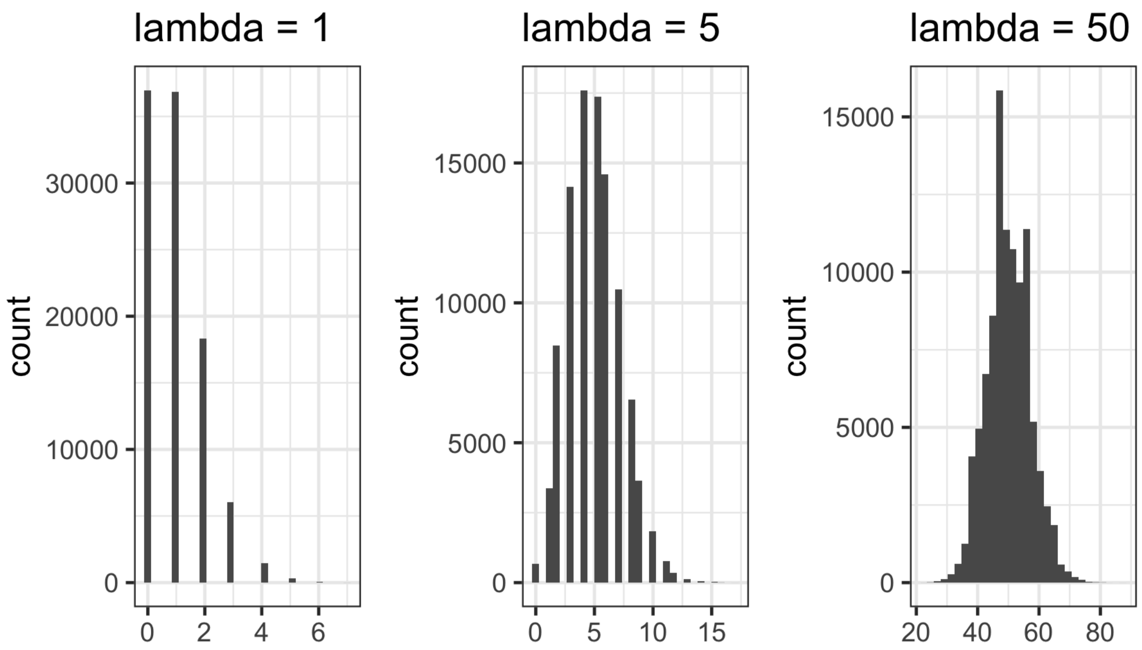
Poisson distribution

Let Y be the number of events in a given unit of time or space. Then Y can be modeled using a **Poisson distribution**

$$P(Y = y) = \frac{e^{-\lambda}\lambda^y}{y!} \qquad y = 0, 1, 2, \ldots, \infty$$

Features

- $E(Y) = Var(Y) = \lambda$
- The distribution is typically skewed right, particularly if λ is small
- The distribution becomes more symmetric as λ increases
  - If λ is sufficiently large, it can be approximated using a normal distribution (Click here for an example.)



	Mean	Variance
lambda = 1	0.99351	0.9902178
lambda = 5	4.99367	4.9865798
lambda = 50	49.99288	49.8962683

Example

The annual number of earthquakes registering at least 2.5 on the Richter Scale and having an epicenter within 40 miles of downtown Memphis follows a Poisson distribution with mean 6.5. **What is the probability there will be at 3 or fewer such earthquakes next year?**

$$P(Y \leq 3) = P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3)$$

$$= \frac{e^{-6.5}6.5^0}{0!} + \frac{e^{-6.5}6.5^1}{1!} + \frac{e^{-6.5}6.5^2}{2!} + \frac{e^{-6.5}6.5^3}{3!}$$

$$= 0.112$$

```
ppois(3, 6.5)
```

```
## [1] 0.1118496
```

Example adapted from [Introduction to Probability Theory Example 28-2]()

Poisson regression

The data: Household size in the Philippines

The data fHH1.csv come from the 2015 Family Income and Expenditure Survey conducted by the Philippine Statistics Authority.

Goal: Understand the association between household size and various characteristics of the household

Response:

- `total`: Number of people in the household other than the head

Predictors:

- `location`: Where the house is located
- `age`: Age of the head of household
- `roof`: Type of roof on the residence (proxy for wealth)

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The data

```
hh_data <- read_csv("data/fHH1.csv")
hh_data %>% slice(1:5) %>% kable()
```

location	age	total	numLT5	roof
CentralLuzon	65	0	0	Predominantly Strong Material
MetroManila	75	3	0	Predominantly Strong Material
DavaoRegion	54	4	0	Predominantly Strong Material
Visayas	49	3	0	Predominantly Strong Material
MetroManila	74	3	0	Predominantly Strong Material

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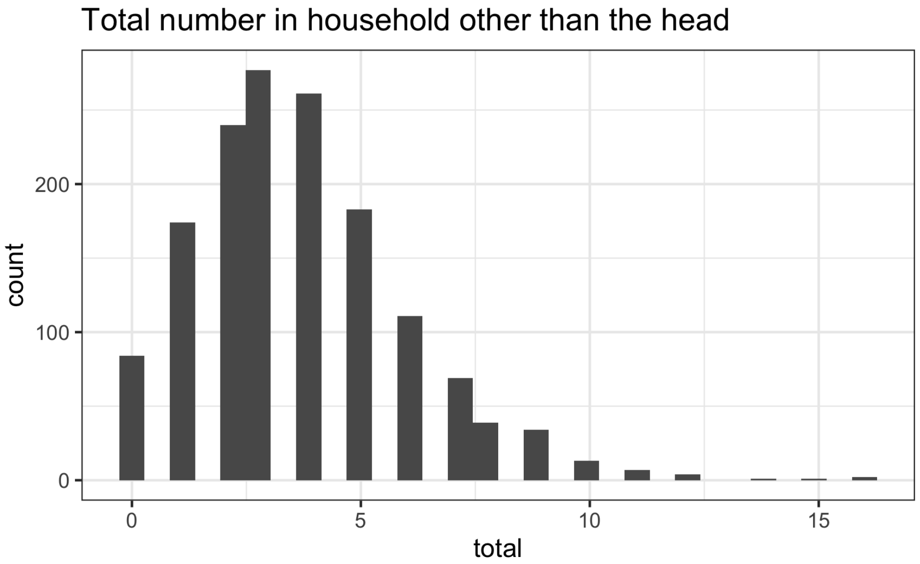
Response variable



mean	var
3.685	5.534

Why the least-squares model doesn't work

The goal is to model λ, the expected number of people in the household (other than the head), as a function of the predictors (covariates)

We might be tempted to try a linear model

$$\lambda_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}$$

This model won't work because...

- It could produce negative values of λ for certain values of the predictors
- The equal variance assumption required to conduct inference for linear regression is violated.

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Poisson regression model

If $Y_i \sim Poisson$ with $\lambda = \lambda_i$ for the given values $x_{i1}, \dots, x_{ip}$, then

$$\log(\lambda_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}$$

- Each observation can have a different value of λ based on its value of the predictors $x_1, \dots, x_p$
- λ determines the mean and variance, so we don't need to estimate a separate error term

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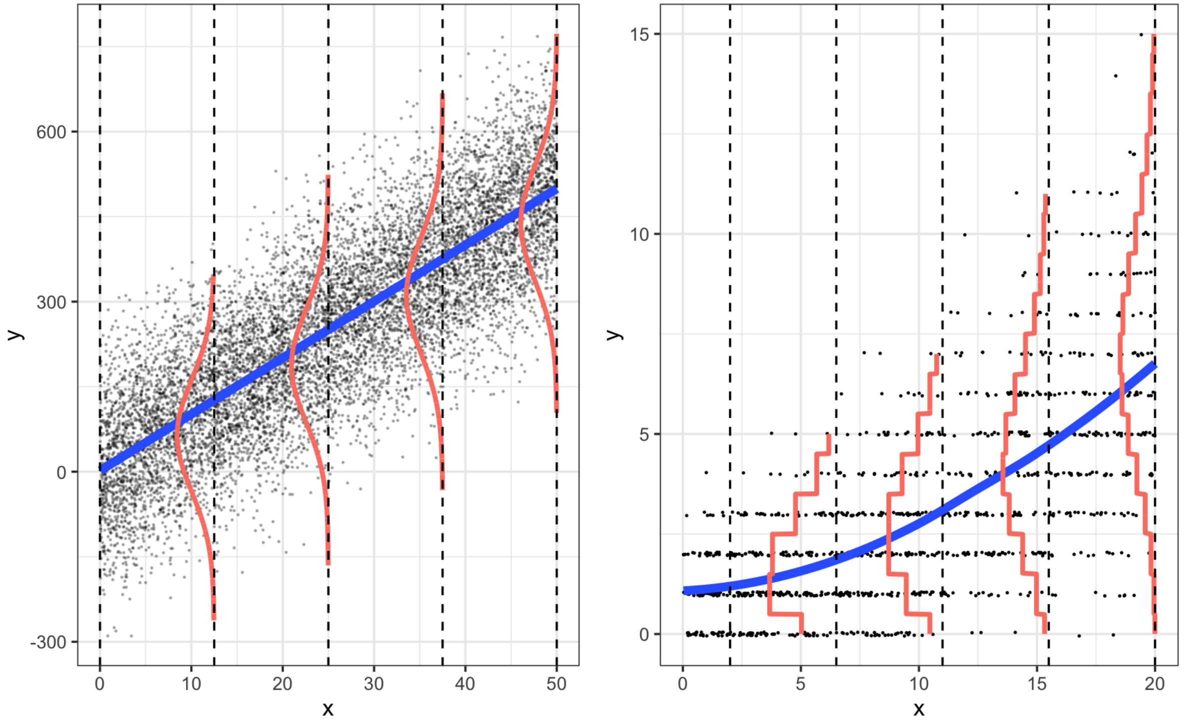
Poisson vs. multiple linear regression



Regression models: Linear regression (left) and Poisson regression (right).

From [BMLR Figure 4.1]

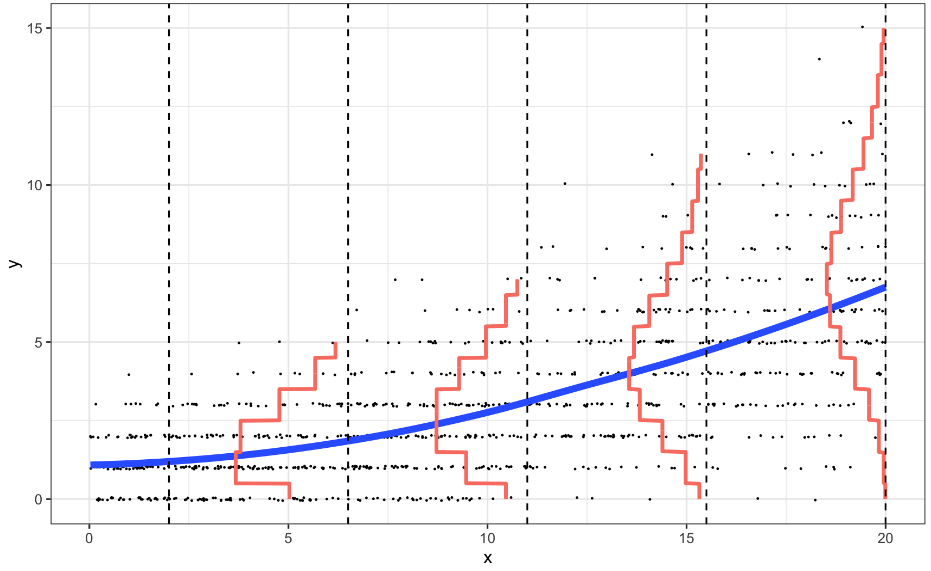
Assumptions for Poisson regression

Poisson response: The response variable is a count per unit of time or space, described by a Poisson distribution, at each level of the predictor(s)

Independence: The observations must be independent of one another

Mean = Variance: The mean must equal the variance

Linearity: The log of the mean rate, $\log(\lambda)$, must be a linear function of the predictor(s)



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Model 1: Number in household vs. age

Model 1: Household vs. Age

```
model1 <- glm(total ~ age, data = hh_data, family = poisson)

tidy(model1) %>%
  kable(digits = 4)
```

term	estimate	std.error	statistic	p.value
(Intercept)	1.5499	0.0503	30.8290	0
age	-0.0047	0.0009	-5.0258	0

$$\log(\hat{\lambda}) = 1.5499 - 0.0047\ age$$

The coefficient for **age** is -0.0047. Interpret this coefficient in context. Select all that apply. Click here to submit your response.

a. The mean household size is predicted to decrease by 0.0047 for each year older the head of the household is.

b. The mean household size is predicted to multiply by a factor of 0.9953 for each year older the head of the household is.

c. The mean household size is predicted to decrease by 0.9953 for each year older the head of the household is.

d. The mean household size is predicted to multiply by a factor of 0.47% for each year older the head of the household is.

e. The mean household size is predicted to decrease by 0.47% for each year older the head of the household is.

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the household is.

03:00

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Understanding the interpretation

Let's derive the change in predicted mean when we go from x to $x + 1$

(see boardwork)

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Is the coefficient of age statistically significant?

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	1.5499	0.0503	30.8290	0	1.4512	1.6482
age	-0.0047	0.0009	-5.0258	0	-0.0065	-0.0029

$$H_0 : \beta_1 = 0 \text{ vs. } H_a : \beta_1 \neq 0$$

Test statistic

$$Z = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)} = \frac{-0.0047 - 0}{0.0009} = -5.026 \text{ (using exact values)}$$

P-value

$$P(|Z| > |-5.026|) = 5.01 \times 10^{-7} \approx 0$$

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What are plausible values for the coefficient of age?

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	1.5499	0.0503	30.8290	0	1.4512	1.6482
age	-0.0047	0.0009	-5.0258	0	-0.0065	-0.0029

95% confidence interval for the coefficient of **age**

$$\hat{\beta}_1 \pm Z^* \times SE(\hat{\beta}_1)$$

$$-0.0047 \pm 1.96 \times 0.0009 = \mathbf{(-.0065, -0.0029)}$$

Interpret the interval in terms of the change in mean household size.

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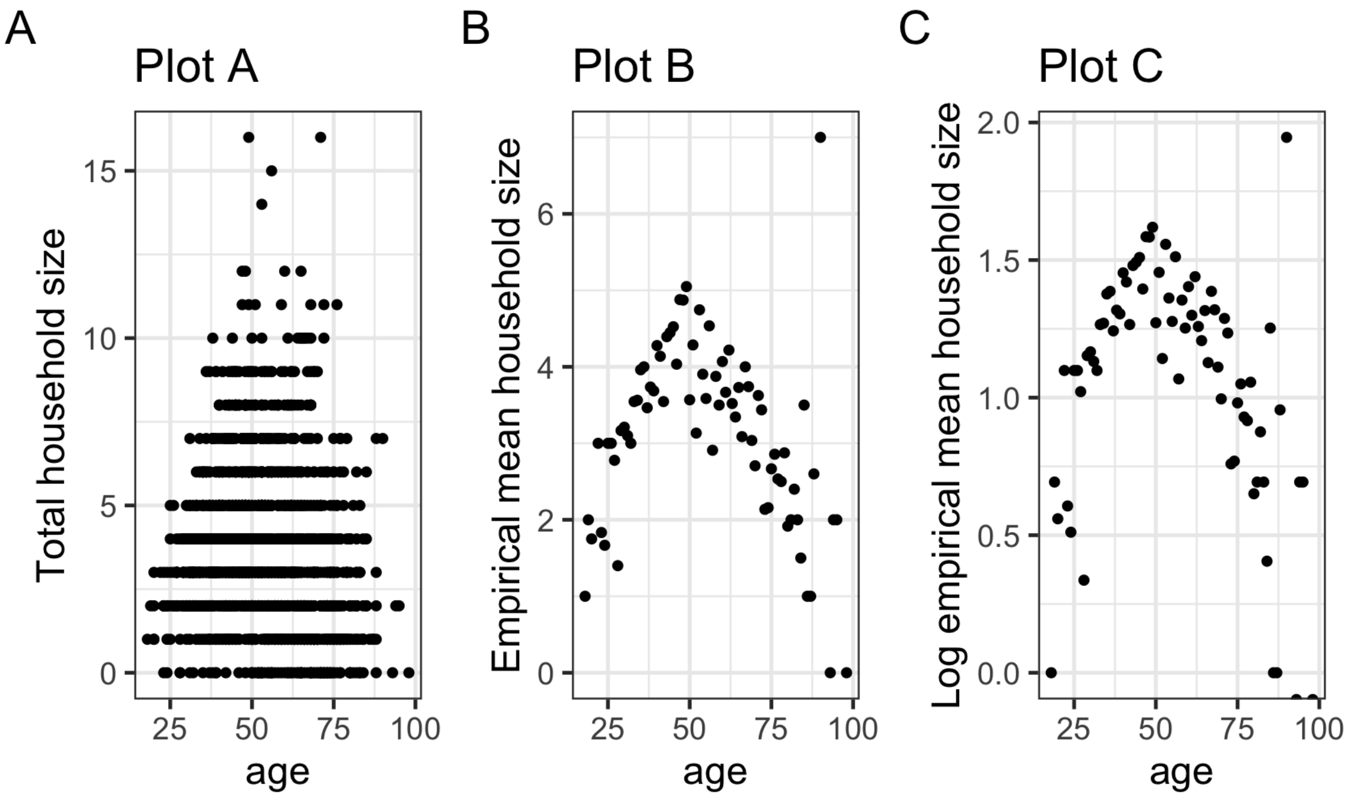
Which plot can best help us determine whether Model 1 is a good fit?



Model 2: Add a quadratic effect for `age`

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Model 2: Add a quadratic effect for age

```
hh_data <- hh_data %>%
  mutate(age2 = age*age)

model2 <- glm(total ~ age + age2, data = hh_data, family = poisson)
tidy(model2, conf.int = T) %>%
  kable(digits = 4)
```

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	-0.3325	0.1788	-1.8594	0.063	-0.6863	0.0148
age	0.0709	0.0069	10.2877	0.000	0.0575	0.0845
age2	-0.0007	0.0001	-11.0578	0.000	-0.0008	-0.0006

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Model 2: Add a quadratic effect for age

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We can determine whether to keep age^2 in the model in two ways:

1️⃣ Use the p-value (or confidence interval) for the coefficient (since we are adding a single term to the model)

2️⃣ Conduct a drop-in-deviance test

Deviance

A deviance is a way to measure how the observed data deviates from the model predictions.

- It's a measure unexplained variability in the response variable (similar to SSE in linear regression)
- Lower deviance means the model is a better fit to the data

We can calculate the "deviance residual" for each observation in the data (more on the formula later). Let $(\text{deviance residual}_i$ be the deviance residual for the i^{th} observation, then

$$\text{deviance} = \sum (\text{deviance residual})_i^2$$

Note: Deviance is also known as the "residual deviance"

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Drop-in-Deviance Test

We can use a drop-in-deviance test to compare two models. To conduct the test

1️⃣ Compute the deviance for each model

2️⃣ Calculate the drop in deviance

$$\text{drop-in-deviance} = \text{Deviance(reduced model) - Deviance(larger model)}$$

3️⃣ Given the reduced model is the true model $(H_0 \text{ true})$, then

$$\text{drop-in-deviance} \sim \chi_d^2$$

where d is the difference in degrees of freedom between the two models (i.e., the difference in number of terms)

Drop-in-deviance to compare Model1 and Model2

```
anova(model1, model2, test = "Chisq") %>%
  kable(digits = 3)
```

Resid. Df	Resid. Dev	Df	Deviance	Pr(>Chi)
1498	2337.089	NA	NA	NA
1497	2200.944	1	136.145	0

- Write the null and alternative hypotheses.
- What does the value 2337.089 tell you?
- What does the value 1 tell you?
- What is your conclusion?

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Add `location` to the model?

Suppose we want to add **`location`** to the model, so we compare the following models:

Model A: $\lambda_i = \beta_0 + \beta_1\ age_i + \beta_2\ age_i^2$

Model B:
$\lambda_i = \beta_0 + \beta_1\ age_i + \beta_2\ age_i^2 + \beta_3\ Loc1_i + \beta_4\ Loc2_i + \beta_5\ Loc3_i + \beta_6\ Loc4_i$

Which of the following are reliable ways to determine if **`location`** should be added to the model? Select all that apply. Click here to submit your response.

Drop-in-deviance test, Use the p-value for each coefficient, Likelihood ratio test, Nested F Test, BIC

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Looking ahead

- For next time - Chapter 4 - Poisson Regression
  - Sections 4.4 - 4.9

Acknowledgements

These slides are based on content in BMLR Chapter 2 - Beyond Least Squares: Using Likelihoods