TOBB ETU COMPUTER ENGINEERING DEPARTMENT

BİL 133 Combinatorics and Graph Theory

HOMEWORK 2 (34 Points)

Due Date: June 9, 2020

1 **[3 POINTS]** INTRODUCING NEW CONNECTIVES TO PROPOSITIONAL LOGIC

Let $*$ be a new logical connective such that $p * q$ does *not* hold iff(if and only if) p and q are either both false or both true.

- $[1 Point]$ Write down the truth table for $p * q$
- $[1 Point]$ Write down the truth table for $(p * p) * (q * q)$
- $[1 Point]$ You should know $*$ already as a logic gate in circuit design. What is it called?

2 **[5 POINTS]** MATHEMATICAL INDUCTION WITH WEIRD IMPLICATIONS

We spent a great deal of time on mathematical induction. Everybody shall be comfortable with the following idea:
If $P(1)$ is true, and $P(i) \rightarrow P(i+1)$ is true for all positive integers i, then $P(i)$ is true for all positive integers i.
Sometimes, we may not have such implications handy but have only a weird set of implications at hand. Assume we have the following information about the property P on positive integers:

- $P(1)$ and $P(2)$ are true,
- $P(n) \rightarrow P(n-1)$ is true for all $n > 1$, and

- $P(n) \wedge P(2) \rightarrow P(2n)$ for all positive integers n.

Is it possible for us to conclude that $P(n)$ is true for all integers n, from that much of information about property P? Justify your claim.

3 [3 POINTS] ON REASONING

Assume that you are asked you to prove that $\mathscr{F}(3n)$ is even for all $n \geq 1$, where $\mathscr{F}(n)$ is the n^{th} Fibonacci number.

Further assume that you performed the following set of actions. You computed the value of $\mathscr{F}(n)$ for thousands of different values of n. In all these instances, you noticed that $\mathscr{F}(n)$ is even if and only if n is divisible by 3. You then claimed that the thousands of observations you made constitutes as a proof of correctness of the given assertion.

- $[1 Points]$ What kind of reasoning you used in the above process?
- $[1 Points]$ Is your methodology scientific?
- $[1 Points]$ Is the scientific methodology welcomed in our class, or is it treated as garbage?

4 [4 POINTS] MATHEMATICAL INDUCTION IN ACTION

Prove that $\mathscr{F}(3n)$ is even for all $n \geq 1$, where $\mathscr{F}(n)$ is the n^{th} Fibonacci number.

5 [4 POINTS] ON SYNTAX OF PREDICATE LOGIC

Let m be a constant, f a function symbol with one argument and S and B two predicate symbols, each with two arguments. Which of the following strings are formulas in predicate logic? Specify a reason for failure for strings which aren't.

- $[0,5 Points]$ $S(m,x)$
- $[0,5 Points]$ $B(m,f(m))$
- $[0,5 Points]$ $f(m)$
- $[0,5 Points]$ $B(B(m,x),y)$
- $[0,5 Points]$ $S(B(m),z)$
- $[0,5 Points]$ $(B(x,y) \rightarrow (\exists z S(z,y)))$
- $[0,5 Points]$ $(S(x,y) \rightarrow S(y,f(f(x))))$
- $[0,5 Points]$ $(B(x) \rightarrow B(B(x)))$

6 [3 POINTS] TRANSLATION FORM ENGLISH TO THE LANGUAGE OF PREDICATE LOGIC

Use the predicates

$$A(x,y): x \text{ admires } y$$
$$B(x,y): x \text{ attended } y$$
$$P(x): x \text{ is a professor}$$
$$S(x): x \text{ is a student}$$
$$L(x): x \text{ is a lecture}$$

and the constant

$$m: \text{ Mary}$$

to translate the following declarative sentences into the language of predicate logic:

- $[0,5 Point]$ Mary admires every professor. (Hint: The answer is *not* $\forall x A(m, P(x))$ since it is not even a formula!)
- $[0,5 Point]$ Some professor admires Mary.
- $[0,5 Point]$ Mary admires herself.
- $[0,5 Point]$ No student attended every lecture.
- $[1 Point]$ No lecture was attended by every student.

7 [4 POINTS] FREE/BOUND VARIABLES AND SUBSTITUTION OF TERMS FOR VARIABLES

Let ϕ be $\exists x(P(y,z) \wedge (\forall y(\neg Q(y,x) \vee P(y,z))))$, where P and Q are predicates with two arguments.

- $[0.5 Points]$ Draw the parse tree of ϕ.
- $[0.5 Points]$ Identify those variable leaves which occur free and those which occur bound in ϕ.
- $[0.5 Points]$ Is there a variable in ϕ which has free and bound occurrences? Explain.
- $[1.5 Points]$ Consider the terms w (w is a variable), $f(x)$ and $g(y,z)$, where f and g are function symbols with one, respectively two, arguments.

  $[0.5 Points]$ Compute $\phi[w/x]$, $\phi[w/y]$, $\phi[f(x)/y]$, and $\phi[g(y,z)/z]$.

  $[0.5 Points]$ Which of w, $f(x)$ and $g(y,z)$ are free for x in ϕ?

  $[0.5 Points]$ Which of w, $f(x)$ and $g(y,z)$ are free for y in ϕ?

- $[0.5 Points]$ What is the scope of $\exists x$ in ϕ?
- $[0.5 Points]$ Suppose that we change ϕ to $\exists x(P(y,z) \land (\forall x(\neg Q(x,x) \lor P(x,z))))$. What is the scope of $\exists x$ now?

8 [8 POINTS] LOGIC PROOFS

Notice that the rules for $\forall$ are very similar to those for $\land$ and those for $\exists$ are just like those for $\lor$. Let's elaborate this statement in this problem.

- $[1 Point]$ Find a (propositional) proof for $\phi \to (q_1 \land q_2) \vdash (\phi \to q_1) \land (\phi \to q_2)$.
- $[1 Point]$ Find a (predicate) proof for $\phi \to \forall x Q(x) \vdash \forall x(\phi \to Q(x))$, provided that x is not free in ϕ. (Hint: Whenever you used $\land$ rules in the propositional proof of the first item, use $\forall$ rules for this item).
- $[1 Point]$ Find a proof for $\forall x(P(x) \to Q(x)) \vdash \forall x P(x) \to \forall x Q(x)$.
- $[1 Point]$ Prove $\forall x(P(x) \land Q(x)) \vdash \forall x P(x) \land \forall x Q(x)$.
- $[1 Point]$ Prove $\forall x P(x) \lor \forall x Q(x) \vdash \forall x(P(x) \lor Q(x))$.
- $[1 Point]$ Prove $\exists x(P(x) \land Q(x)) \vdash \exists x P(x) \land \exists x Q(x)$.
- $[1 Point]$ Prove $\exists x F(x) \lor \exists x G(x) \vdash \exists x(F(x) \lor G(x))$
- $[1 Point]$ Prove $\forall x \forall y(S(y) \to F(x)) \vdash \exists y S(y) \to \forall x F(x)$.