BİL 031/133 Combinatorics and Graph Theory

HOMEWORK 5 (45 Points)

Due Date: July 26, 2020

1 [10 POINTS] THINK RECURSIVELY

Give a recurrence for the function $\mathscr{F} : \mathbb{Z}_+ \to \mathbb{Z}_+$, where $\mathscr{F}(n)$ is defined as the number of binary strings of length n with no three consecutive 1's in it.

Notice that $\mathscr{F}(1) = 2$, since both the strings "0" and "1" does not have 3 consecutive 1's in it. Similarly, $\mathscr{F}(2) = 4$, since the strings "00", "01", "10", "11" do not have 3 consecutive 1's in them. $\mathscr{F}(3) = 7$, however, since the string "111" has 3 consecutive 1's in it.

Recall that you need to prove that your recurrence is correct by using mathematical induction.

2 [5 POINTS] SIZES OF SETS

Let A and B be two countably-infinite sets. Prove that there exists sets A', B' and functions f and g satisfying the following conditions:

- $A' \subsetneq A$,
- $B' \subsetneq B$,
- $f : A' \to B$ is one-to-one and onto,
- $g : B' \to A$ is one-to-one and onto.

3 [10 Points] Shooting the Damned Rabbit

In this question, we investigate a game played by a hunter and a rabbit on the integer line where the rabbit hops along the integer line and the hunter tries to shoot the rabbit. The hunter and the rabbit alternate turns, where the hunter shoots, then the rabbit jumps, (then the hunter shoots, then the rabbits jumps, and so on...) until the hunter hits the rabbit. Rules are as following:

- The rabbit starts at a position $s \in \mathbb{Z}$ on the number line. Then, the hunter takes its first shot.
- Every time the hunter takes its shot, then the rabbit leaps $z \in \mathbb{Z}$ spaces, i.e., z may be positive or negative. If z is positive, then the rabbit jumps to the right. If z is negative, then the rabbit jumps to the left. Note that it always jumps the same amount in the same direction.
- The tricky part is that the values of s and z are unknown to the hunter!

  Example: The rabbit may start at 6 and always jump +4 spaces so that its location changes as $6, 10, 14\ldots$ However, the hunter cannot observe this locations. The hunter only knows that the rabbit moves according to above rules.

Imagine that you are the shooter in this game. It may seem as if you are helpless at first. After all you never know where the rabbit is during the game. *So, how can you shoot it without knowing where it is?*
Indeed, this is what we ask in the question: Devise an algorithm to determine where you shoot at each turn so that you are guaranteed to hit the rabbit eventually. It might take an extraordinary long time for your algorithm to hit the rabbit, but it has to guarantee that it will in a finite amount of time.

Hint: Recall the proof we give in the lectures to show that the set of rational numbers are countable.

4 [5 Points] Languages

Let L be a language such that both L and $\bar{L}$ are recursively enumerable. Argue as formally as you can that both L and $\bar{L}$ are decidable.

5 [5 Points] Number of Square Integer Matrices

We say that a matrix is a **square matrix** if it has equal number of rows and columns. We say that a matrix is an **integer matrix** if all its entries are integers.
What can you say about the cardinality of the set of square integer matrices? Is it countable or uncountable?

6 [5 POINTS] AN EQUIVALENCE RELATION

We define the $\sim$ relation over the collection of sets as follows. For any two sets S_1 and S_2, we say that $S_1 \sim S_2$ if and only if there is a bijection $f : S_1 \to S_2$. Prove that $\sim$ is an equivalence relation.
(Recall that a relation is an equivalence relation if it is reflexive, symmetric and transitive.)

7 [5 POINTS] SPACE CONSIDERATIONS FOR MERGE SORT

We have seen several sorting algorithms in class. Merge Sort algorithm, which uses the **divide and conquer** approach, is not an **in place** algorithm, i.e., the amount of extra space it needs is not bounded by a constant. In this problem, you are asked to calculate the amount of extra space used by Merge Sort.