BIL 133 HW4

Murat Sahin
191101010

1 FIRST RECURRENCE AND ITS SLIGHT GENERALIZATION

1.1 FIRST RECURRENCE

- $T(1) = \alpha$
- $T(n) = 2.T(\frac{n}{2}) + \beta.n$ for $n > 1$, and n is an exact power of 2.

  I will solve recurrence above by using the repertoire method.

  We can rewrite $T(n)$ as $T(n) = A(n).\alpha + B(n).\beta$. A and B are functions that solely depend on n.

  Now, we need to find what A(n) and B(n) are equal to.

  Think the case that $\alpha = 0$ and $\beta = 1$. Then, $T(n) = B(n)$ **(1)**

$$T(1) = 0$$
$$T(n) = 2.T(\tfrac{n}{2}) + n$$

  Lets look at the small cases.

  T(2) = 2

  T(4) = 8

  T(8) = 24

  It seems like solution for recurrence is $n.logn$. We need to prove it to be sure.

  Claim : Solution for recurrence above is $n.logn$.
  Proof : I will prove the claim by mathematical induction on n.
  Base Case : As base case I will consider case n=1. We know T(1) is equal to 0. It needs to be 0 from our formula too.

$$1.log1 = 1.0 = 0 = 0 \text{ Base case holds.}$$

Inductive Assumption : We assume that our claim is true for some integer $k \geq 1$, and k is an exact power of 2.

All we need to show is that claim is true for $2k$. So, $T(2k) = 2k.log2k$ must hold.

$T(2k) = 2T(k) + k$ We know that $T(k) = k.logk$ from our inductive assumption. From given recursion,

$$T(2k) = 2k.logk + 2k$$
$$T(2k) = 2k.(logk + 1)$$
$$T(2k) = 2k.(logk + log2)$$
$$T(2k) = 2k.log2k$$

Our claim is true for all integer $n \geq 1$, and n is an exact power of 2.
So, $B(n) = n.logn$ from equation (1) and above equation.

Think the case that $\alpha = 1$ and $\beta = 0$. Then, $T(n) = A(n)$

$$T(1) = 1$$
$$T(n) = 2.T(\tfrac{n}{2})$$

We know this recurrence. We can rewrite $T(n)$ as $T(n) = 2^{n-1}$.
Since $T(n) = A(n)$ for this case, $A(n) = 2^{n-1}$ from the equation above.
Now, we have two independent equations.

$$A(n) = 2^{n-1} \tag{1}$$

$$B(n) = n.logn \tag{2}$$

From these equations, T(n) will be:

$$T(n) = 2^{n-1}.\alpha + n.logn.\beta, \text{ and } n \text{ is an exact power of 2.}$$

1.2 SLIGHT GENERALIZATION

$T(n) = 2.T(\frac{n}{2}) + \beta.n + \gamma$ We can repeat previous steps to find A(n) and B(n) again. Only difference is we should set γ to 0 while doing them. Thus,

$$A(n) = 2^{n-1}$$
$$B(n) = n.logn$$

We need another independent equation. Think the case that $T(n) = 1$. Then, $\alpha = 1$, $\beta = 0$, $\gamma = -1$. So,

$$1 = A(n) - C(n)$$
Substitute A(n) from equality above,
$$C(n) = 2^{n-1} - 1$$

Thus,

$$T(n) = 2^{n-1}.\alpha + n.logn.\beta + (2^{n-1} - 1)\gamma, \text{ and } n \text{ is an exact power of 2.}$$

2 SECOND RECURRENCE AND ITS GENERALIZATION

2.1 SECOND RECURRENCE

- $T(1) = \alpha$
- $T(n) = 3.T(\frac{n}{3}) + \beta.n$ for $n > 1$, and n is an exact power of 3.

  I will solve recurrence above by using the repertoire method.

  We can rewrite $T(n)$ as $T(n) = A(n).\alpha + B(n).\beta$. A and B are functions that solely depend on n.

  Now, we need to find what A(n) and B(n) are equal to.

  Think the case that $\alpha = 0$ and $\beta = 1$. Then, $T(n) = B(n)$ **(1)**

$$T(1) = 0$$
$$T(n) = 3.T(\frac{n}{3}) + n$$

Lets look at the small cases.

T(3) = 3

T(9) = 18

T(27) = 81

It seems like solution for recurrence is $n.logn$. We need to prove it to be sure.

Claim : Solution for recurrence above is $n.logn$.
Proof : I will prove the claim by mathematical induction on n.
Base Case : As base case I will consider case n=1. We know T(1) is equal to 0. It needs to be 0 from our formula too.

$$1.log1 = 1.0 = 0 = 0 \text{ Base case holds.}$$

Inductive Assumption : We assume that our claim is true for some integer $k \geq 1$, and k is an exact power of 3.

All we need to show is that claim is true for $3k$. So, $T(3k) = 3k.log3k$ must hold.

$T(3k) = 3T(k) + k$ We know that $T(k) = k.logk$ from our inductive assumption. From given recursion,

$$T(3k) = 3k.logk + 3k$$
$$T(3k) = 3k.(logk + 1)$$
$$T(3k) = 3k.(logk + log3)$$
$$T(3k) = 3k.log3k$$

Our claim is true for all integer $n \geq 1$, and n is an exact power of 3.
So, $B(n) = n.logn$ from equation (1) and above equation.

Think the case that $\alpha = 1$ and $\beta = 0$. Then, $T(n) = A(n)$

$$T(1) = 1$$
$$T(n) = 3.T(\tfrac{n}{3})$$

We know this recurrence. We can rewrite $T(n)$ as $T(n) = 3^{n-1}$.

Since $T(n) = A(n)$ for this case, $A(n) = 3^{n-1}$ from the equation above.
Now, we have two independent equations.

$$A(n) = 3^{n-1} \tag{3}$$

$$B(n) = n.logn \tag{4}$$

From these equations, T(n) will be:

$$T(n) = 3^{n-1}.\alpha + n.logn.\beta, \text{ and } n \text{ is an exact power of 3.}$$

2.2 SLIGHT GENERALIZATION

$T(n) = 3.T(\frac{n}{3}) + \beta.n + \gamma$ We can repeat previous steps to find A(n) and B(n) again. Only difference is we should set γ to 0 while doing them. Thus,

$$A(n) = 3^{n-1}$$
$$B(n) = n.logn$$

We need another independent equation. Think the case that $T(n) = 1$. Then, $\alpha = 1$, $\beta = 0$, $\gamma = -1$. So,

$$1 = A(n) - C(n)$$
Substitute A(n) from equality above,
$$C(n) = 3^{n-1} - 1$$

Thus,

$$T(n) = 3^{n-1}.\alpha + n.logn.\beta + (3^{n-1} - 1)\gamma, \text{ and } n \text{ is an exact power of 3.}$$

3 AN UGLY LOOKING RECURRENCE TEACHING THE IMPORTANCE OF PLAYING WITH SMALL INSTANCES

Lets look at the small cases.

$$Q_0 = \alpha$$
$$Q_1 = \beta$$
$$Q_2 = \frac{1+Q_1}{Q_0} = \frac{1+\beta}{\alpha} \ (1)$$
$$Q_3 = \frac{1+Q_2}{Q_1} = \frac{1+\alpha+\beta}{\alpha.\beta} \ (2)$$
$$Q_4 = \frac{1+Q_3}{Q_2} = \frac{1+\alpha}{\beta} \ (3)$$
$$Q_5 = \alpha(4)$$
$$Q_6 = \beta(5)$$
It will repeat itself after n=4.

Claim : Define m as : $m = n\%5$.

If $m = 0$, $Q_n = \alpha$,
If $m = 1$, $Q_n = \beta$,
If $m = 2$, $Q_n = \frac{1+\beta}{\alpha}$,
If $m = 3$, $Q_n = \frac{1+\alpha+\beta}{\alpha.\beta}$,
If $m = 4$, $Q_n = \frac{1+\alpha}{\beta}$,for all $n \geq 2$.

Proof : I will prove this claim by mathematical induction on n.

Base Case : As base cases, we will consider cases n=2,3,4,5,6.

If n=2, m=2, Q_n must be equal to $\frac{1+\beta}{\alpha}$. From eq.1, it is true.
If n=3, m=3, Q_n must be equal to $\frac{1+\alpha+\beta}{\alpha.\beta}$. From eq.2, it is true.
If n=4, m=4, Q_n must be equal to $\frac{1+\alpha}{\beta}$. From eq.1, it is true.
If n=5, m=0, Q_n must be equal to α. From eq.4, it is true.
If n=6, m=1, Q_n must be equal to β. From eq.5, it is true.

Inductive Hypothesis : Our claim is true for all integer $2 \leq n \leq k$.

Inductive Step: All we need to do is show that claim is true for k+1 for cases $k\%5 = 0, 1, 2, 3, 4$.

If $k\%5 = 0$, $(k-1)\%5 = 4$. Then, $Q_k = \alpha$ and $Q_{k-1} = \frac{1+\alpha}{\beta}$ from inductive hypothesis. Since $(k+1)\%5 = 1$, we want Q_{k+1} to be β.

$$Q_{k+1} = \frac{1+Q_k}{Q_{k-1}} \text{ (from given recursion.)}$$
$$Q_{k+1} = \frac{1+\alpha}{\frac{1+\alpha}{\beta}} = \beta \text{ as desired.}$$

If $k\%5 = 1$, $(k-1)\%5 = 0$. Then, $Q_k = \beta$ and $Q_{k-1} = \alpha$ from inductive hypothesis. Since $(k+1)\%5 = 2$, we want Q_{k+1} to be $\frac{1+\beta}{\alpha}$.

$$Q_{k+1} = \frac{1+Q_k}{Q_{k-1}} \text{ (from given recursion.)}$$
$$Q_{k+1} = \frac{1+\beta}{\alpha} = \frac{1+\beta}{\alpha} \text{ as desired.}$$

If $k\%5 = 2$, $(k-1)\%5 = 1$. Then, $Q_k = \frac{1+\beta}{\alpha}$ and $Q_{k-1} = \beta$ from inductive hypothesis. Since $(k+1)\%5 = 3$, we want Q_{k+1} to be $\frac{1+\alpha+\beta}{\alpha.\beta}$.

$$Q_{k+1} = \frac{1+Q_k}{Q_{k-1}} \text{ (from given recursion.)}$$
$$Q_{k+1} = \frac{1+\frac{1+\beta}{\alpha}}{\beta} = \frac{1+\alpha+\beta}{\alpha.\beta} \text{ as desired.}$$

If $k\%5 = 3$, $(k-1)\%5 = 2$. Then, $Q_k = \frac{1+\alpha+\beta}{\alpha.\beta}$ and $Q_{k-1} = \frac{1+\beta}{\alpha}$ from inductive hypothesis. Since $(k+1)\%5 = 4$, we want Q_{k+1} to be $\frac{1+\alpha}{\beta}$.

$$Q_{k+1} = \frac{1+Q_k}{Q_{k-1}} \text{ (from given recursion.)}$$
$$Q_{k+1} = \frac{1+\frac{1+\alpha+\beta}{\alpha.\beta}}{\frac{1+\beta}{\alpha}} = \frac{1+\alpha}{\beta} \text{ as desired.}$$

If $k\%5 = 4$, $(k-1)\%5 = 3$. Then, $Q_k = \frac{1+\alpha}{\beta}$ and $Q_{k-1} = \frac{1+\alpha+\beta}{\alpha.\beta}$ from inductive hypothesis. Since $(k+1)\%5 = 0$, we want Q_{k+1} to be α.

$$Q_{k+1} = \frac{1+Q_k}{Q_{k-1}} \text{ (from given recursion.)}$$
$$Q_{k+1} = \frac{1+\frac{1+\alpha}{\beta}}{\frac{1+\alpha+\beta}{\alpha.\beta}} = \alpha \text{ as desired.}$$

Therefore, claim is true.

4 PROOF BY MATHEMATICAL INDUCTION

Claim : $n! > 2^n$ for $n \geq 4$.

Proof : I will prove this claim by mathematical induction on n.
Base Case : As base case, we consider case n = 4.

$$4! > 2^4$$
$$24 > 16 \text{ our base case holds.}$$

Inductive Assumption : We assume claim holds for some integer $k > 4$.
Inductive Step : All we need to show is that claim holds for k+1. So $(k+1)! > 2^{k+1}$ must be true.

$$k! > 2^k \text{ holds from our inductive assumption.}$$

We know that $k+1 > 5$, so if we multiply left side with k+1 and right side with 2, inequality still holds.

$$(k+1).k! > 2.2^k$$
$$(k+1)! > 2^{k+1}$$

Thus, equality holds for k+1 too. Therefore, $n! > 2^n$ for $n \geq 4$.

5 IVERSON'S CONVENTION

Lets consider cases $x = 0$, $x > 0$ and $x < 0$ one by one. This cases contains every possible cases.

Case $x = 0 : x.(0 - 0) = 0$
Case $x > 0 : x.(1 - 0) = x$
Case $x < 0 : x.(0 - 1) = -x$

It reminds something familiar. It is actually absolute value of x.

We can rewrite our statement as :

$$|x|$$

6 EVALUATING A SIMPLE SUM

$$\sum_k [i \leq j \leq k \leq n] = \sum_{j \leq k \leq n} 1$$

$$= (\sum_{1 \leq k \leq n} 1) - (\sum_{1 \leq k \leq j-1} 1) \text{ (from commutative law)}$$

We know these sums.

$$= \frac{n.(n+1)}{2} - \frac{(j-1).j}{2}$$

7 SUMMATION FACTOR METHOD

To find a closed form, we can manipulate recurrence in order to change it to a familiar form. To do that, we need to find what summation factor is. We should multiply both sides with s_n. $s_n.2.T_n = s_n.n.T_{n-1} + s_n.3.n!$. Think the familiar form $s_n.a_n.T_n = s_n.b_n.T_{n-1} + s_n.c_n$.In this case, $a_n = 2$, $b_n = n$ and $c_n = 3.n!$. Following equality should hold $s_{n-1}.a_{n-1} = s_n.b_n$.

$$s_n = \frac{a_{n-1}.a_{n-2}.a_{n-3}...a}{b_n.b_{n-1}.b_{n-2}...b_2} = \frac{2.2.2...2}{n.(n-1).(n-2)...2} = \frac{2^{n-1}}{b!}$$

Now, multiply both sides with our summation factor.

$$\frac{2^{n-1}}{b!}.2.T_n = \frac{2^{n-1}}{b!}.n.T_{n-1} + \frac{2^{n-1}}{n!}.3.n!$$
$$\frac{2^n}{n!}.T_n = \frac{2^{n-1}}{(n-1)!}.T_{n-1} + 3.2^{n-1}$$

Let $P_n = \frac{2^n}{n!}.T_n$. Then, $P_n = P_{n-1} + 3.2^{n-1}$. Then, $P_0 = 5$. Lets rewrite our last situation.

$$P_0 = 5$$
$$P_n = P_{n-1} + 3.2^{n-1}$$

This recursion can be rewritten as:

$$P_n = 5 + \sum_{k=1}^{n} 3.2^{k-1}$$
$$P_n = 5 + 3.\sum_{k=1}^{n} 2^{k-1} \text{ (from distributive law)}$$
$$P_n = 5 + 3.\sum_{k=0}^{n-1} 2^{k} \text{ We know this sum.}$$
$$P_n = 5 + 3.\frac{(1-2^n)}{(1-2)}$$

$$P_n = 3.2^n + 2$$

Now, lets find T_n.

$$T_n = (3.2^n + 2).\frac{n!}{2^n}$$
$$T_n = (3.n! + \frac{2.n!}{2^n})$$
$$T_n = 3.n! + \frac{n!}{2^{n-1}}$$

This is the closed form for our recurrence.

8 EVALUATING A SUM

Lets evaluate $A_n = \sum_{k=0}^{n} kx^k$ and substitute n by ∞. So, we should find A_∞.

$A_{n+1} = A_n + (n+1).x^{n+1}$ (1) by taking the last term out.
$A_{n+1} = \sum_{k=1}^{n+1} kx^k = \sum_{k=0}^{n}(k+1)x^{k+1} = x.\sum_{k=0}^{n}(k+1)x^k$ from commutative rule followed by distributive rule.
$= x.(\sum_{k=0}^{n} kx^k + \sum_{k=0}^{n} x^k) = x.(A_n + \frac{1-x^{n+1}}{1-x})$ (2) (From geometric series)

From equation 1 and equation 2,

$$A_n + (n+1).x^{n+1} = x.A_n + x.\frac{1-x^{n+1}}{1-x}$$

Replace n by ∞. Since $|x| < 1$ from the definition, $x^\infty = 0$.

$$A_\infty = x.A_\infty + x.\frac{1}{1-x}$$
$$A_\infty = \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

We evaluated the sum.