TOBB ETU COMPUTER ENGINEERING DEPARTMENT

BİL 133 Combinatorics and Graph Theory

HOMEWORK 3 (25 Points)

Due Date: June 22, 2020

1 [5 POINTS] PROVING HUMMINGBIRDS ARE SMALL

Given the following premises:

(P1) "All hummingbirds are richly colored."
(P2) "No large birds live on honey."
(P3) "Birds that do not live on honey are dull in color."

Prove the following conclusion:

(C1) "Hummingbirds are small."

(*a*) First, explain your reasoning informally.

(*b*) Then, given the following predicates:

$P(x) =$ "x is a hummingbird"
$Q(x) =$ "x is large"
$R(x) =$ "x lives on honey"
$S(x) =$ "x is richly colored"

Formally state the premises (P1), (P2), (P3) and the conclusion (C1) in predicate logic.

(*c*) Finally, assuming that the premises (P1), (P2) and (P3) are given, prove the conclusion (C1).

2 [4 POINTS] UPPERBOUND FOR A VARIATION OF TOWER OF HANOI

Consider the variation of the Tower of Hanoi Problem, where we have 4 pegs instead of 3. Let $T(n)$ denote the minimum number of moves sufficient to carry n disks from one peg to another while obeying Lucas' rules when there are only 3 pegs as we defined in class. Now, let $S(n)$ denote the minimum number of moves sufficient to carry n disks from one peg to another while obeying Lucas' rules when there are 4 pegs.

First, determine the values of $S(0), S(1)$ and $S(2)$.

Then, show that $S(n) \leq 2S(k) + 2T(n-k-1) + 1$ for any $k \in \{1, 2, \ldots, n-1\}$.

3 **[4 POINTS]** INDUCTION PROOF FOR GEOMETRY

You learned in high school that the sum of the internal angles in an n-sided polygon (where $n \geq 3$) is $(n-2)\pi$. In that problem, you are asked to prove that fact by using mathematical induction on the number of sides of polygons.
You can take the fact that the sum of the internal angles of any triangle is π as a premise.

4 **[5 POINTS]** ANOTHER VARIATION OF TOWER OF HANOI

Consider the variation of the Tower of Hanoi problem, where we have $2n$ disks of n different sizes, two of each size. As in the classical version, there are 3 pegs, and all the $2n$ disks are initially stacked in one of the pegs. The goal is to transfer the pile of the disks to another peg. As usual, only one disk can be moved at a time, and a disk cannot be placed onto a smaller one. Notice that a disk can be placed on the other identical disk (since otherwise the initial configuration would be problematic.)
How many moves are necessary and sufficient to transfer the entire tower from one peg to another?

- Look at small cases first and try to see a pattern.
- $[1 Point]$ Define a function $T(n)$ that corresponds to the quantity of interest. (Make sure that your definition is unambiguous.)
- $[1 Point]$ Find an algorithm to solve the problem. Use the performance of your algorithm to obtain a recurrence **inequality** for $T(n)$.
- $[1 Point]$ Prove that the upperbound you proved in the previous step is tight, by proving a matching lowerbound for $T(n)$. In order to do that the algorithm you found in the previous step has to be the optimal one!
- $[1 Point]$ By combining your matching upper and lower bounds, you can obtain a recurrence equality that defines $T(n)$. Please state your recurrence equality.
- $[1 Point]$ Solve your recurrence.

5 [7 Points] Number of Binary Strings in the Special Form

Give a recurrence for the function $\mathscr{F} : \mathbb{Z}_+ \to \mathbb{Z}_+$, where $\mathscr{F}(n)$ is defined as the number of binary strings of length n with no two consecutive 1's in it.

Notice that $\mathscr{F}(1) = 2$, since both the strings "0" and "1" does not have 2 consecutive 1's in it. Similarly, $\mathscr{F}(2) = 3$, since the strings "00", "01", and "10" does not have 2 consecutive 1's in it, whereas the string "11" has two consecutive 1's in it.

Recall that you need to prove that your recurrence is correct by using mathematical induction.