CMPE 133 HW 2

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1.

a)

p	q	$p*q$
0	0	0
0	1	1
1	0	1
1	1	0

b)

p	q	$p*p$	$q*q$	$(p*p)*(q*q)$
0	0	0	0	0
0	1	0	0	0
1	0	0	0	0
1	1	0	0	0

c) It is called XOR (Exclusive-OR) Gate.

2. Lets define a fresh variable f. Define i as smallest integer that holds $f \leq 2^i$.

I am going to proof $P(2^n)$ is true for all positive integers n.
I will proof it by mathematical induction on n.

Base cases: $P(2^0) = P(1)$, $P(2^1) = P(2)$ > both of them true according to the given informations

Inductive step: I am going to assume $P(2^n)$ and check if $P(2^{n+1})$ is true.
Since $P(n) \land P(2) \to P(2n)$, $P(2^n) \land P(2) \to P(2^{n+1})$, $P(2^{n+1})$ is true.
both of them true

Hence, $P(2^n)$ is true for all positive integers n. So, $P(2^i)$ is true.

If we use second information step by step $(2^i - f)$ times,
$P(2^i) \to P(2^i - 1)$
$P(2^i - 1) \to P(2^i - 2)$
$\vdots$
we can prove $P(f)$ is true.

Since f is a fresh variable, property P is true for all positive integers.

3.

a) I used a kind of invalid inductive reasoning.

b) It is not scientific.

c) It is not welcomed in our class, it is treated as garbage.

4. we use mathematical induction to prove $F(3n)$ is even for $n \geq 3$.

Base Case: For $n=1$ we should show that $F(3n)$ is even

$$F(3.1) = F(3) = \underbrace{F(2)}_{1} + \underbrace{F(1)}_{1} = 2 \rightarrow \text{it is even} \checkmark$$

Inductive Assumption: we are assuming $F(3n)$ is even.

Inductive Step: We need to show that $F(3(n+1))$ is even. we will use our assumption above.

$$F(3(n+1)) = F(3n+3) = F(3n+2) + F(3n+1)$$
$$= F(3n) + F(3n+1) + F(3n+1)$$
$$= \underbrace{F(3n)}_{\substack{\text{even} \\ \text{(assumption)}}} + \underbrace{2 \cdot F(3n+1)}_{\text{even}}$$

Since $F(3(n+1))$ is sum of two even numbers,

$F(3(n+1))$ is even.

Proof completed.

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a) $S(m,x)$ is a formula.

b) $B(m,f(m))$ is a formula.

c) $f(m)$ is not a formula because f is a function and it returns a term, which is not a truth value. Predicates can take terms as arguments.

d) $B(B(m,x),y)$ is not a valid formula because predicates can take terms as arguments, not truth values. First argument is $B(m,x)$ since B returns truth value, it is not valid.

e) $S(B(m),z)$ is not a valid formula because B must take two arguments. After that, S can't take a truth value as an argument.

f) $(B(x,y) \rightarrow (\exists z\ S(z,y)))$ is a formula.

g) $(S(x,y) \rightarrow S(y,f(f(x))))$ is a formula.

h) $(B(x) \rightarrow B(B(x)))$ is not a valid formula because B must take two arguments. After that, B can't take a truth value as an argument.

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a) $\forall x\ (P(x) \rightarrow A(m,x))$

b) $\exists_x\ (P(x) \land A(x,m))$

c) $A(m,m)$

d) $\neg \exists x\ \forall y\ (S(x) \land L(y) \rightarrow B(x,y))$

e) $\neg \exists y\ \forall x\ (S(x) \land L(y) \rightarrow B(x,y))$

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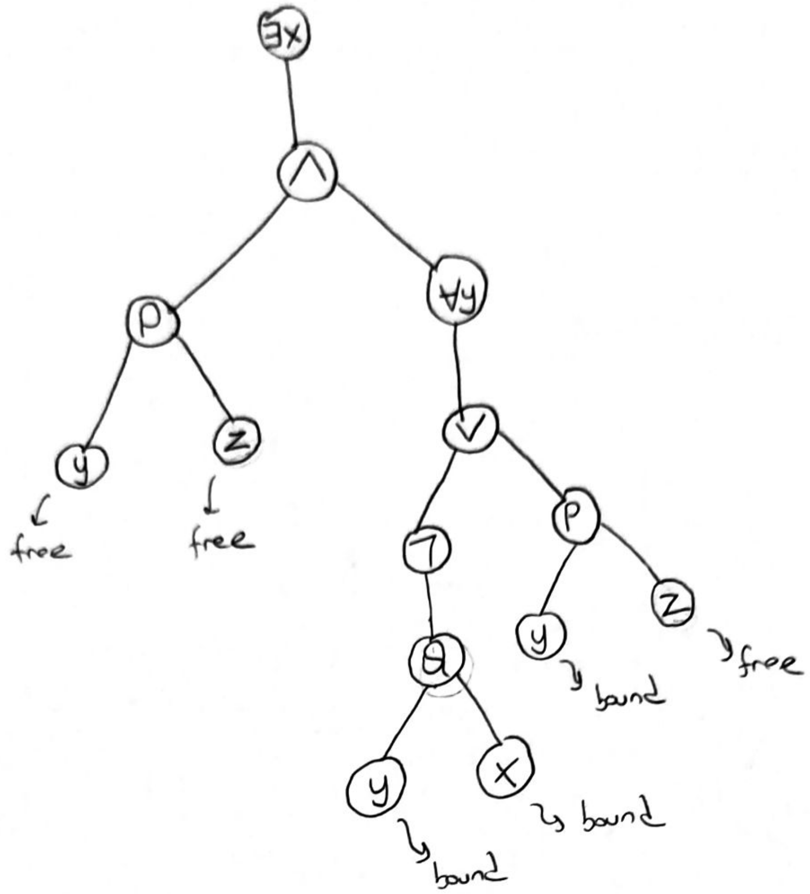
a,b)



c) Yes, variable y has 1 free and 2 bound occurences.

d)

d.1)

$\varphi[w/x] = \varphi = \exists x(P(y,z) \wedge (\forall y(\neg Q(y,x) \vee P(y,z))))$ → No free x

$\varphi[w/y] = \exists x(P(w,z) \wedge (\forall y(\neg Q(y,x) \vee P(y,z))))$

$\varphi[f(x)/y] =$ } substition can't be done   x is not free for y in φ

$\varphi[g(y,z)/z] =$ y is not free for z in φ

d.2) All of them are free for x in φ.

d.3) w and g(y,z) are free for y in φ. f(x) gets captured by $\exists x$.

e) It is whole subtree under $\exists x$. $P(y,z) \wedge (\forall y(\neg Q(y,x) \vee P(y,z)))$

f) It is $P(y,z)$. $\forall x$ limits its scope.

8

a) $\varphi \to (q_1 \wedge q_2) \vdash (\varphi \to q_1) \wedge (\varphi \to q_2)$

1.	$\varphi \to (q_1 \wedge q_2)$	premise
2.	φ	assumption
3.	$q_1 \wedge q_2$	$\to_e$ 2,1
4.	q_1	$\wedge_{e_1}$ 3
5.	$\varphi \to q_1$	$\to_i$ 2-4
6.	φ	assumption
7.	$q_1 \wedge q_2$	$\to_e$ 6,1
8.	q_2	$\wedge_{e_2}$ 7
9.	$\varphi \to q_2$	$\to_i$ 6-8
10.	$(\varphi \to q_1) \wedge (\varphi \to q_2)$	$\wedge_i$ 5,9

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b) $\varphi \to \forall x\, Q(x) \vdash \forall x\, (\varphi \to Q(x))$

1.	$\varphi \to \forall x\, Q(x)$	premise
2.	x_0	
3.	φ	assumption
4.	$\forall x\, Q(x)$	$\to_e$ 3,1
5.	$Q(x_0)$	$\forall x_e$ 4
6.	$\varphi \to Q(x_0)$	$\to_i$ 3-5
7.	$\forall x\, (\varphi \to Q(x))$	$\forall x_i$ 2-6

c) $\forall x\,(P(x) \to Q(x)) \vdash \forall x\, P(x) \to \forall x\, Q(x)$

1.	$\forall x\,(P(x) \to Q(x))$	premise
2.	$\forall x\, P(x)$	assumption
3.	x_0 $P(x_0) \to Q(x_0)$	$\forall x e$ 1
4.	$P(x_0)$	$\forall x e$ 2
5.	$Q(x_0)$	$\to e$ 4,3
6.	$\forall x\, Q(x)$	$\forall x i$ 3-5
7.	$\forall x\, P(x) \to \forall x\, Q(x)$	$\to i$ 2-6

d) $\forall x\,(P(x) \land Q(x)) \vdash \forall x P(x) \land \forall x\, Q(x)$

1.	$\forall x\,(P(x) \land Q(x))$	premise
2.	x_0 $P(x_0) \land Q(x_0)$	$\forall x e$ 1
3.	$P(x_0)$	$\land e_1$ 2
4.	$\forall x\, P(x)$	$\forall x i$ 2-4
5.	x_0 $P(x_0) \land Q(x_0)$	$\forall x e$ 1
6.	$Q(x_0)$	$\land e_2$ 5
7.	$\forall x\, Q(x)$	$\forall x i$ 5-6
8.	$\forall x P(x) \land \forall x\, Q(x)$	$\land i$ 4,7

e) $\forall x P(x) \vee \forall x Q(x) \vdash \forall x (P(x) \vee Q(x))$

1.	$\forall x P(x) \vee \forall x Q(x)$	premise
2.	x_0	
3.	$\forall x P(x)$	assumption
4.	$P(x_0)$	$\forall x e$ 3
5.	$P(x_0) \vee Q(x_0)$	$\vee i_1$ 4
6.	$\forall x Q(x)$	assumption
7.	$Q(x_0)$	$\forall x e$ 6
8.	$P(x_0) \vee Q(x_0)$	$\vee i_2$ 7
9.	$P(x_0) \vee Q(x_0)$	$\vee e$ 1,3-5,6-8
10.	$\forall x (P(x) \vee Q(x))$	$\forall x i$ 2-9

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f) $\exists x (P(x) \wedge Q(x)) \vdash \exists x P(x) \wedge \exists x Q(x)$

1.	$\exists x (P(x) \wedge Q(x))$	premise
2.	x_0 $P(x_0) \wedge Q(x_0)$	assumption
3.	$P(x_0)$	$\wedge e_1$ 2
4.	$\exists x P(x)$	$\exists x i$ 3
5.	$\exists x P(x)$	$\exists x e$ 1,2-4
6.	x_0 $P(x_0) \wedge Q(x_0)$	assumption
7.	$Q(x_0)$	$\wedge e_2$ 6
8.	$\exists x Q(x)$	$\exists x i$ 7
9.	$\exists x Q(x)$	$\exists x e$ 1,6-8
10.	$\exists x P(x) \wedge \exists x Q(x)$	$\wedge i$ 5,9

g) $\exists x F(x) \vee \exists x G(x) \vdash \exists x (F(x) \vee G(x))$

1.	$\exists x F(x) \vee \exists x G(x)$	premise
2.	$\exists x F(x)$	assumption
3.	x_0 $F(x_0)$	assumption
4.	$F(x_0) \vee G(x_0)$	$\vee i_1$ 3
5.	$\exists x (F(x) \vee G(x))$	$\exists x i$ 4
6.	$\exists x (F(x) \vee G(x))$	$\exists x e$ 2,3-5
7.	$\exists x G(x)$	assumption
8.	x_0 $G(x_0)$	assumption
9.	$F(x_0) \vee G(x_0)$	$\vee i_2$ 8
10.	$\exists x (F(x) \vee G(x))$	$\exists x i$ 9
11.	$\exists x (F(x) \vee G(x))$	$\exists x e$ 7,8-10
12.	$\exists x (F(x) \vee G(x))$	$\vee e$ 1,2-6,7-11

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h) $\forall x \forall y (S(y) \to F(x)) \vdash \exists y S(y) \to \forall x F(x)$

1.	$\forall x \forall y (S(y) \to F(x))$	premise
2.	$\exists y S(y)$	assumption
3.	y_0 $S(y_0)$	assumption
4.	x_0 $\forall x (S(y_0) \to F(x))$	$\forall y e$ 1
5.	$S(y_0) \to F(x_0)$	$\forall x e$ 4
6.	$F(x_0)$	$\to e$ 3,5
7.	$\forall x F(x)$	$\forall x i$ 4-6
8.	$\forall x F(x)$	$\exists y e$ 2,3-7
9.	$\exists y S(y) \to \forall x F(x)$	$\to i$ 2-8