$$\underset{\sim}{Y} \sim N(\underline{X}\,\underline{\beta}, \Sigma)$$

$$\underline{Y} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{n \times 1} \qquad \underline{\beta} = \underbrace{\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{k+1} \end{pmatrix}}_{k+1 \times 1}$$

$$\underset{n \times k+1}{\underline{X}} = \begin{pmatrix} 1 & x_{11} & x_{21} & \cdots & x_{k1} \\ 1 & x_{12} & x_{22} & \cdots & x_{k2} \\ 1 & x_{13} & x_{23} & \cdots & x_{k3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{1n} & x_{2n} & \cdots & x_{kn} \end{pmatrix}$$

$$\underset{n \times n}{\underline{\Sigma}} = \begin{pmatrix} \sigma^2 & & & 0 \\ & \sigma^2 & & \\ & & \ddots & \\ 0 & & & \sigma^2 \end{pmatrix}$$

$$= \sigma^2 \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \; 1 \end{pmatrix}$$

$$\ell(y, x, \beta, \Sigma) = -\frac{n}{2} \log 2\pi - \frac{1}{2} \log\left(\det(\Sigma)\right) - \frac{1}{2}(y - X\beta)' \Sigma^{-1} (y - X\beta)$$

$$\propto -\frac{1}{2} \log\left(\det\left(\sigma^2 I\right)\right) - \frac{1}{2}(y - X\beta)' 1/\sigma^2 I^{-1} (y - X\beta)$$

$$\propto -\frac{1}{2} \log(\sigma^2)^n - \frac{1}{2\sigma^2}(\underline{y} - \underline{X}\underline{\beta})'(\underline{y} - \underline{X}\underline{\beta})$$

$\boxed{\text{MLE} - \beta}$

$$\frac{\partial \ell(\cdot)}{\partial \underline{\beta}} = -\frac{1}{2\sigma^2} \frac{\partial}{\partial \underline{\beta}}\left[(y - X\beta)'(y - X\beta)\right]$$

$$= \frac{1}{2\sigma^2} \frac{\partial}{\partial \beta}\left[y'y - \underbrace{y'X\beta}_{1\times 1} - \underbrace{\beta X' y}_{1\times 1} + \beta' X' X \beta\right]$$

$$= \frac{1}{2\sigma^2}\left[-\cancel{2} X'y + \cancel{2} X'X\beta\right] = 0$$

$$-X'y + X'X\beta = 0$$

$$X'X\beta = X'y$$

$$(X'X)^{-1}(X'X)\beta = (X'X)^{-1}(X'y)$$

$$\hat{\beta}_{MLE} = (X'X)^{-1}X'y$$

$$\hat{y} = X\hat{\beta}_{MLE} = X(X'X)^{-1}X'y$$

$$\ell(\cdot) \propto -\frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(\underline{Y}-\underline{X}\underline{\beta})'(\underline{Y}-\underline{X}\underline{\beta})$$

$$\frac{\partial \ell(\cdot)}{\partial \sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2(\sigma^2)^2}(Y-X\beta)'(Y-X\beta)$$

$$= \frac{1}{2\sigma^2}\left(-n + \frac{1}{\sigma^2}(\underline{Y}-\underline{X}\underline{\beta})'(\underline{Y}-\underline{X}\underline{\beta})\right) = 0$$

$$\boxed{\text{Assume } \sigma^2 > 0}$$

$$-n + \frac{1}{\sigma^2}(\underline{Y}-\underline{X}\underline{\beta})'(\underline{Y}-\underline{X}\underline{\beta}) = 0$$

$$\sigma^2 = \frac{1}{n}(Y-X\beta)'(Y-X\beta)$$ plug in $\hat{\beta} \Rightarrow X\hat{\beta} = \hat{Y}$

$$= \frac{1}{n}(Y-\hat{Y})'(Y-\hat{Y}) = \frac{1}{n}\sum_{i=1}^{n}(Y_i-\hat{Y}_i)^2$$