Lec 8

$$y_t = \delta + \beta t + v_t$$

$$d_t^1 = y_t - y_{t-1}$$

$$= (\delta + \beta t + v_t) - (\delta + \beta(t-1) + v_{t-1})$$

$$= \beta + v_t - v_{t-1}$$

$$y_t = \delta + \beta t + \gamma t^2 + w_t$$

$$d_t^2 = d_t^1 - d_{t-1}^1 = (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})$$

$$= (\cancel{\delta} + \cancel{\beta t} + \gamma t^2 + v_t) - (\cancel{\delta} + \beta(\cancel{t} - \cancel{1}) + \gamma(t-1)^2 + v_{t-1})$$
$$- (\cancel{\delta} + \beta(\cancel{t} - \cancel{1}) + \gamma(t-1)^2 + v_{t-1}) + (\cancel{\delta} + \beta(\cancel{t} - \cancel{2}) + \gamma(t-2)^2 + v_{t-2})$$

$$= \gamma t^2 - 2\gamma(t-1)^2 + \gamma(t-2)^2 + w_t - 2v_{t-1} + v_{t-2}$$

$$= \gamma \cancel{t^2} - 2\gamma(\cancel{t^2} - \cancel{2t} + 1) + \gamma(\cancel{t^2} - \cancel{4t} + 4) + v_t - 2v_{t-1} + v_{t-2}$$

$$= 2\gamma + v_t - 2v_{t-1} + v_{t-2}$$

AR(1) w/ ∞ terms

$$
\begin{aligned}
y_t &= \delta + \phi y_{t-1} + w_t \\
&= \delta + \phi(\delta + \phi y_{t-2} + w_{t-1}) + w_t \\
&= \delta + \phi\delta + \phi y_{t-2} + w_t + \phi w_{t-1} \\
&= \delta + \phi\delta + \phi^2\delta + \phi y_{t-3} + w_t + \phi w_{t-1} + \phi^2 w_{t-2} \\
&= \left(\sum_{i=0}^{\infty} \phi^i \delta\right) + \left(\sum_{i=0}^{\infty} \phi^i w_{t-i}\right)
\end{aligned}
$$

$$
\begin{aligned}
E(y_t) &= E\left(\sum_{i=0}^{\infty} \phi^i \delta\right) + E\left(\sum_{i=0}^{\infty} \phi^i w_{t-i}\right) \\
&= \sum_{i=0}^{\infty} E(\phi^i \delta) = \sum_{i=0}^{\infty} \phi^i \delta < \infty \quad \text{iff } |\phi| < 1 \\
&= \begin{cases} \dfrac{\delta}{1-\phi} & \text{if } |\phi| < 1 \\ \infty & \text{otherwise} \end{cases}
\end{aligned}
$$

$$Var(y_t) = Var\left(\sum_{i=0}^{\infty} \phi^i \delta\right) + Var\left(\sum_{i=0}^{\infty} \phi^i w_{t-i}\right)$$

$$= Var\left(w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 v_{t-3} + \dots\right)$$

$$= \sigma^2 + \phi^2 \sigma_v^2 + \phi^4 \sigma_v^2 + \phi^6 \sigma_v^2 + \dots$$

$$= \begin{cases} \dfrac{\sigma_v^2}{1-\phi^2} & \phi^2 < 1 \\ \infty & \text{otherwise} \end{cases}$$

$$\gamma(h) = Cov(y_t, y_{t+h})$$

$$\gamma(0) = Cov(y_t, y_t) = Var(y_t) = \frac{\sigma_v^3}{1-\phi^2}$$

$$\gamma(1) = E\left(\left(y_t - E(y_t)\right)\left(y_{t+1} - E(y_{t+1})\right)\right)$$

$$= E\Big(\left(\cancel{v_t} + \phi v_{t-1} + \phi^2 v_{t-2} + \phi^3 v_{t-3} + \dots\right)$$
$$\left(v_{t-1} + \phi v_{t-2} + \phi^2 w_{t-3} + \phi^3 v_{t-4} + \dots\right)\Big)$$

$$= E\left(\phi v_{t-1}^2 + \phi^3 v_{t-2}^2 + \phi^5 v_{t-3} + \dots\right)$$

$$= \phi \sigma_v^2 + \phi^3 \sigma_v^2 + \phi^5 \sigma_v^2 + \dots$$

$$= \phi \sigma_v^2 \left(1 + \phi^2 + \phi^4 + \phi^6 + \dots\right) = \phi \frac{\sigma_v^2}{1-\phi^2} = \phi \gamma(0)$$

$$\gamma(2) = E\Big(\big(y_t - E(y_t)\big)\big(y_{t+2} - E(y_{t+2})\big)\Big)$$

$$= E\Big(\big(\cancel{v_t} + \cancel{\phi v_{t-1}} + \phi^2 v_{t+2} + \phi^3 v_{t+3} + \dots\big)$$

$$\big(v_{t+2} + \phi v_{t+3} + \phi^2 v_{t+4} + \dots$$

$$= \phi^2 \sigma_v^2 + \phi^4 \sigma_v^2 + \phi^6 \sigma_v^2 + \dots$$

$$= \phi^2 \frac{\sigma_v^2}{1-\phi^2} = \phi^2 \gamma(0)$$

$$\gamma(h) = \phi^h \gamma(0)$$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h$$