# AR, MA, and ARMA Models

## Lecture 09

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# AR models

# AR(1) models

From last time we derived the following properties for AR(1) models,

$$y_t = \delta + \phi\, y_{t-1} + w_t$$
$$w_t \overset{iid}{\sim} N(0, \sigma_w^2)$$

The process $y_t$ is stationary iff $|\phi| < 1$, and if stationary then

$$E(y_t) = \frac{\delta}{1-\phi}$$
$$Var(y_t) = \gamma(0) = \frac{\sigma_w^2}{1-\phi^2}$$
$$Cov(y_t, y_{t+h}) = \gamma(h) = \phi^h \frac{\sigma_w^2}{1-\phi^2} = \phi^h \gamma(0)$$
$$Corr(y_t, y_{t+h}) = \rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h$$

# AR(p) models

We can generalize from an AR(1) to an AR(p) model by simply adding additional autoregressive terms to the model.

$$
\begin{aligned}
AR(p): \quad y_t &= \delta + \phi_1 \, y_{t-1} + \phi_2 \, y_{t-2} + \cdots + \phi_p \, y_{t-p} + w_t \\
&= \delta + w_t + \sum_{i=1}^{p} \phi_i \, y_{t-i}
\end{aligned}
$$

What are the properities of $AR(p)$, specifically

1. Stationarity conditions?
2. Expected value?
3. Autocovariance / autocorrelation?

# Lag operator

The lag operator is convenience notation for writing out AR (and other) time series models.

We define the lag operator $L$ as follows,

$$L\, y_t = y_{t-1}$$

this can be generalized where,

$$\begin{aligned} L^2 y_t &= L\,(L\, y_t) \\ &= L\, y_{t-1} \\ &= y_{t-2} \end{aligned}$$

therefore,

$$L^k\, y_t = y_{t-k}$$

# Lag polynomial

We can rewrite the $\text{AR}(p)$ model using the lag operator,

$$\begin{aligned} y_t &= \delta + \phi_1\, y_{t-1} + \phi_2\, y_{t-2} + \ldots + \phi_p\, y_{t-p} + w_t \\ &= \delta + \phi_1\, L\, y_t + \phi_2\, L^2\, y_t + \ldots + \phi_p\, L^p\, y_t + w_t \end{aligned}$$

If we group all of the $y_t$ terms, we get the following

$$\begin{aligned} \delta + w_t &= y_t - \phi_1\, L\, y_t - \phi_2\, L^2\, y_t - \ldots - \phi_p\, L^p\, y_t \\ &= (1 - \phi_1\, L - \phi_2\, L^2 - \ldots - \phi_p\, L^p)\, y_t \end{aligned}$$

This polynomial of lags

$$\phi_p(L) = (1 - \phi_1\, L - \phi_2\, L^2 - \cdots - \phi_p\, L^p)$$

is called the characteristic polynomial of the AR process.

# Stationarity of $\text{AR}(p)$ processes

**Claim**: An $\text{AR}(p)$ process is stationary if the roots of the characteristic polynomial lay *outside* the complex unit circle

If we define $\lambda = 1/L$ then we can rewrite the characteristic polynomial as

$$(\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_{p-1}\lambda - \phi_p)$$

then as a corollary of the preceeding claim is that an $\text{AR}(p)$ process is stationary if the roots of this new polynomial are *inside* the complex unit circle, i.e. $|\lambda| < 1$

# Example AR(1)

# Example AR(2)

# AR(2) Stationarity Conditions

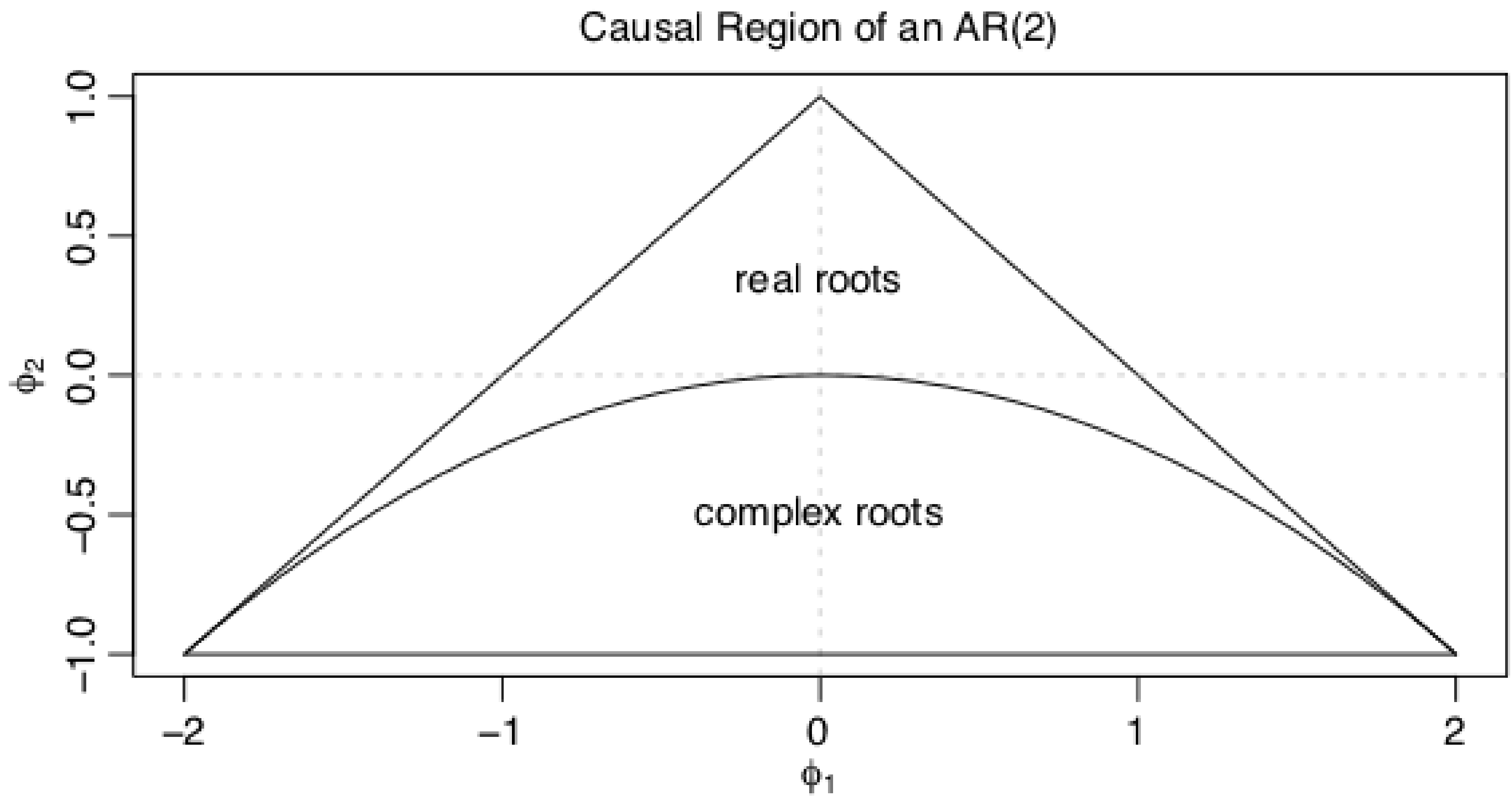



*Fig. 3.3. Causal region for an AR(2) in terms of the parameters.*

From Shumway & Stofer 4th ed.

# Proof Sketch

We can rewrite the $\mathrm{AR}(\mathrm{p})$ model into an $\mathrm{AR}(1)$ form using matrix notation

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t$$
$$\boldsymbol{Y}_t = \boldsymbol{\delta} + \boldsymbol{F} \boldsymbol{Y}_{t-1} + \boldsymbol{w}_t$$

where

$$\underset{p\times 1}{\boldsymbol{Y}_t} = [y_t, y_{t-1}, y_{t-2}, \ldots, y_{t-p+1}]'$$
$$\underset{p\times 1}{\boldsymbol{\delta}} = [\delta, 0, 0, \ldots, 0]'$$
$$\underset{p\times 1}{\boldsymbol{w}_t} = [w_t, 0, 0, \ldots, 0]'$$

$$\underset{p\times p}{\boldsymbol{F}} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

This construction is an example of a state space model (also called a dynamic linear model), which are

# Putting it together

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \delta \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \delta + \sum_{i=1}^{p} \phi_i \, y_{t-i} + w_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix}$$

# Proof sketch (cont.)

So just like the original $\text{AR}(1)$ we can expand out the autoregressive equation

$$
\begin{aligned}
\boldsymbol{Y}_t &= \boldsymbol{\delta} + \boldsymbol{w}_t + \boldsymbol{F}\,\boldsymbol{\xi}_{t-1} \\
&= \boldsymbol{\delta} + \boldsymbol{w}_t + \boldsymbol{F}\,(\boldsymbol{\delta} + \boldsymbol{w}_{t-1}) + \boldsymbol{F}^2\,(\boldsymbol{\delta} + \boldsymbol{w}_{t-2}) + \cdots \\
&\qquad\qquad + \boldsymbol{F}^{t-1}\,(\boldsymbol{\delta} + \boldsymbol{w}_1) + \boldsymbol{F}^t\,(\boldsymbol{\delta} + \boldsymbol{w}_0) \\
&= \left(\sum_{i=0}^{t} F^i\right)\boldsymbol{\delta} + \sum_{i=0}^{t} F^i\, w_{t-i}
\end{aligned}
$$

and therefore we need $\lim\limits_{t\to\infty} F^t \to 0$ so that $\lim\limits_{t\to\infty} \sum_{i=0}^{t} F^i < \infty$.

# Proof sketch (cont.)

We can find the eigen decomposition such that $\boldsymbol{F} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^{-1}$ where the columns of $\boldsymbol{Q}$ are the eigenvectors of $\boldsymbol{F}$ and $\boldsymbol{\Lambda}$ is a diagonal matrix of the corresponding eigenvalues.

A useful property of the eigen decomposition is that

$$\boldsymbol{F}^{i} = \boldsymbol{Q}\boldsymbol{\Lambda}^{i}\boldsymbol{Q}^{-1}$$

Using this property we can rewrite our equation from the previous slide as

$$\begin{aligned}\boldsymbol{Y}_t &= (\sum_{i=0}^{t} F^{i})\boldsymbol{\delta} + \sum_{i=0}^{t} F^{i}\, w_{t-i} \\ &= \left(\sum_{i=0}^{t} \boldsymbol{Q}\boldsymbol{\Lambda}^{i}\boldsymbol{Q}^{-1}\right)\boldsymbol{\delta} + \sum_{i=0}^{t} \boldsymbol{Q}\boldsymbol{\Lambda}^{i}\boldsymbol{Q}^{-1}\, w_{t-i}\end{aligned}$$

# Proof sketch (cont.)

$$\mathbf{\Lambda}^i = \begin{bmatrix} \lambda_1^i & 0 & \cdots & 0 \\ 0 & \lambda_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p^i \end{bmatrix}$$

Therefore, $\lim_{t\to\infty} F^t \to 0$ when $\lim_{t\to\infty} \Lambda^t \to 0$ which requires that

$$|\lambda_i| < 1 \quad \text{for all } i$$

# Proof sketch (cont.)

Eigenvalues are defined such that for $\boldsymbol{\lambda}$,

$$\det(\boldsymbol{F} - \boldsymbol{\lambda}\,\boldsymbol{I}) = 0$$

based on our definition of $\boldsymbol{F}$ our eigenvalues will therefore be the roots of

$$\lambda^p - \phi_1\,\lambda^{p-1} - \phi_2\,\lambda^{p-2} - \cdots - \phi_{p_1}\,\lambda^1 - \phi_p = 0$$

which if we multiply by $1/\lambda^p$ where $L = 1/\lambda$ gives

$$1 - \phi_1\,L - \phi_2\,L^2 - \cdots - \phi_{p_1}\,L^{p-1} - \phi_p\,L^p = 0$$

# Properties of $\text{AR}(2)$

For a *stationary* $\text{AR}(2)$ process,

# Properties of $\mathrm{AR}(2)$ (cont.)

# Properties of $\text{AR}(2)$

For a *stationary* $\text{AR}(2)$ process,

$$E(y_t) = \frac{\delta}{1 - \phi_1 - \phi_2}$$

$$Var(y_t) = \gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma_w^2$$

$$Cov(y_t, y_{t+h}) = \gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2)$$

$$Corr(y_t, y_{t+h}) = \rho(h) = \phi_1\, \rho(h-1) + \phi_2\, \rho(h-2)$$

# Properties of $AR(p)$

For a *stationary* $AR(p)$ process,

$$E(y_t) = \frac{\delta}{1 - \phi_1 - \phi_2 - \ldots - \phi_p}$$

$$Var(y_t) = \gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \ldots + \phi_p\gamma(p) + \sigma_w^2$$

$$Cov(y_t, y_{t+h}) = \gamma(h) = \phi_1\gamma(h-1) + \phi_2\gamma(h-2) + \ldots + \phi_p\gamma(h-p)$$

$$Corr(y_t, y_{t+h}) = \rho(h) = \phi_1\,\rho(h-1) + \phi_2\,\rho(h-2) + \ldots + \phi_p\,\rho(h-p)$$

# Moving Average (MA) Processes

# MA(1)

A moving average process is similar to an AR process, except that the autoregression is on the error term.

$$\text{MA}(1): \qquad y_t = \delta + w_t + \theta\, w_{t-1}$$

Properties:

# MA(1) - properties

For a *stationary* AR(p) process,

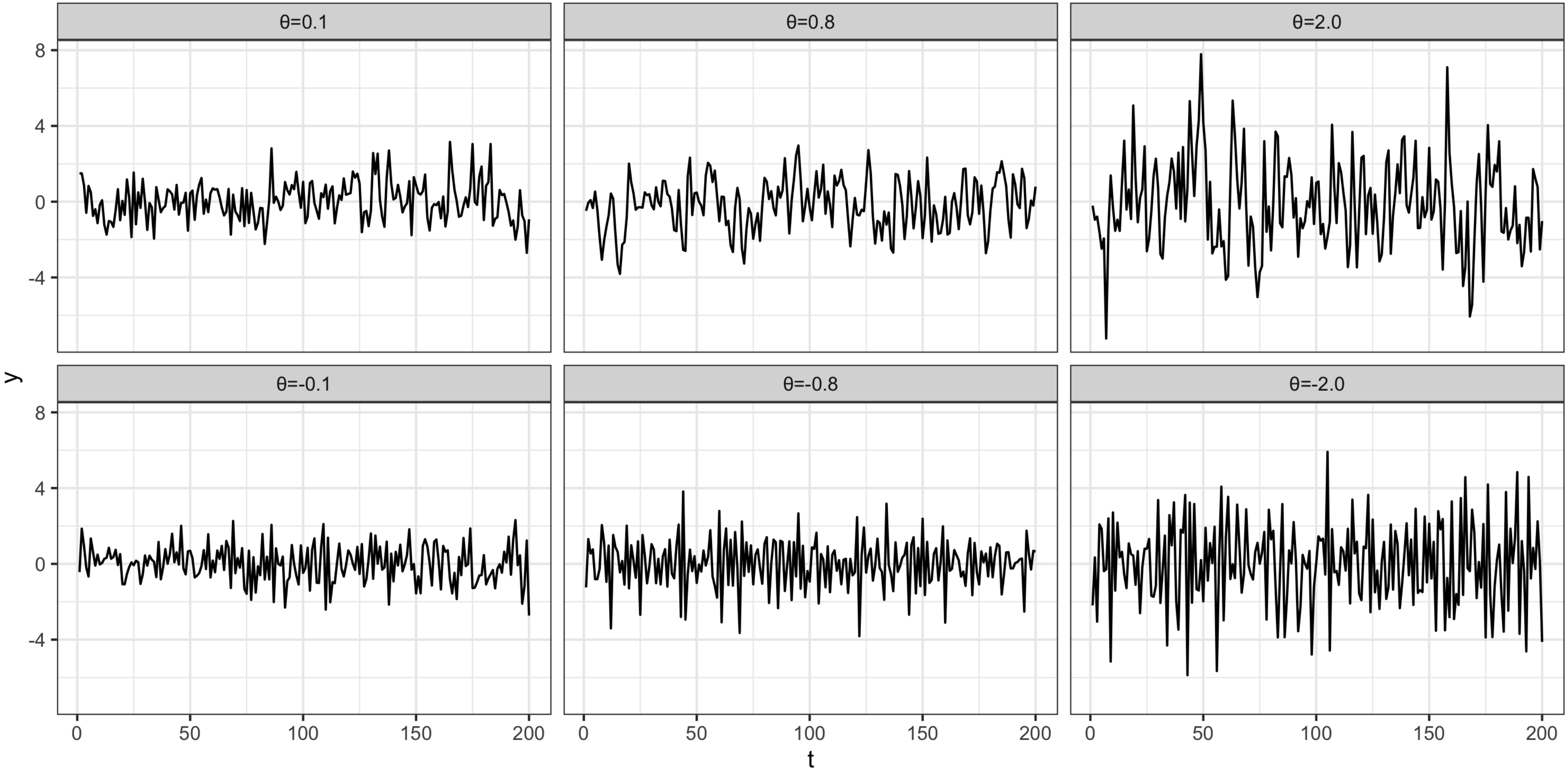
$$E(y_t) = \delta$$

$$Var(y_t) = \gamma(0) = \sigma^2(1 + \theta^2)$$

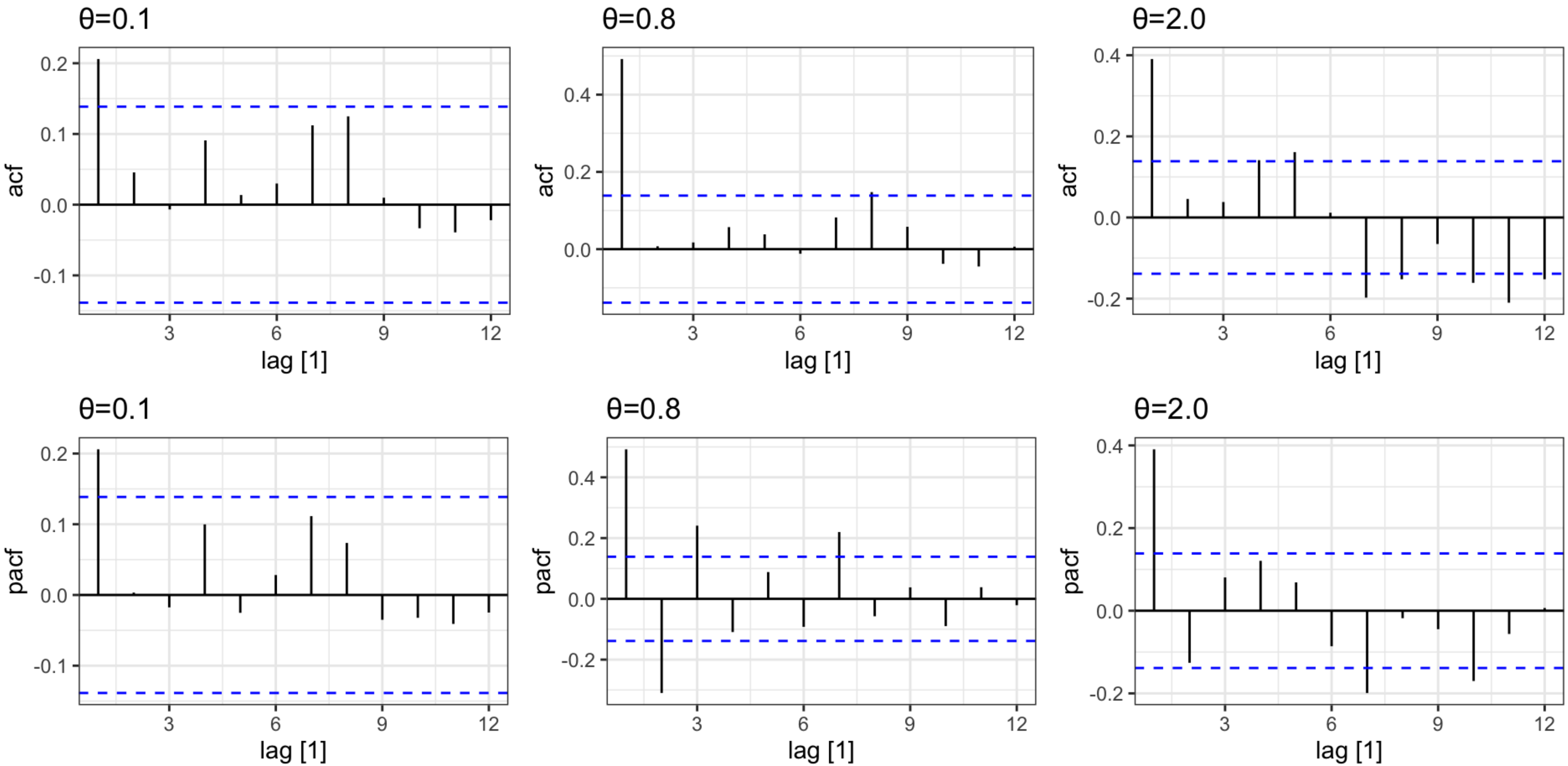
$$Cov(y_t, y_{t+h}) = \gamma(h) = \begin{cases} \sigma^2(1 + \theta^2) & \text{if } h = 0 \\ \theta\sigma^2 & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Corr(y_t, y_{t+h}) = \rho(h) = \begin{cases} 1 & \text{if } h = 0 \\ \theta/(1 + \theta^2) & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

# Example time series



# MA(1) ACF & pACF



# MA(q)

$$y_t = \delta + w_t + \theta_1 \, w_{t-1} + \theta_2 \, w_{t-2} + \cdots + \theta_q \, w_{t-q}$$

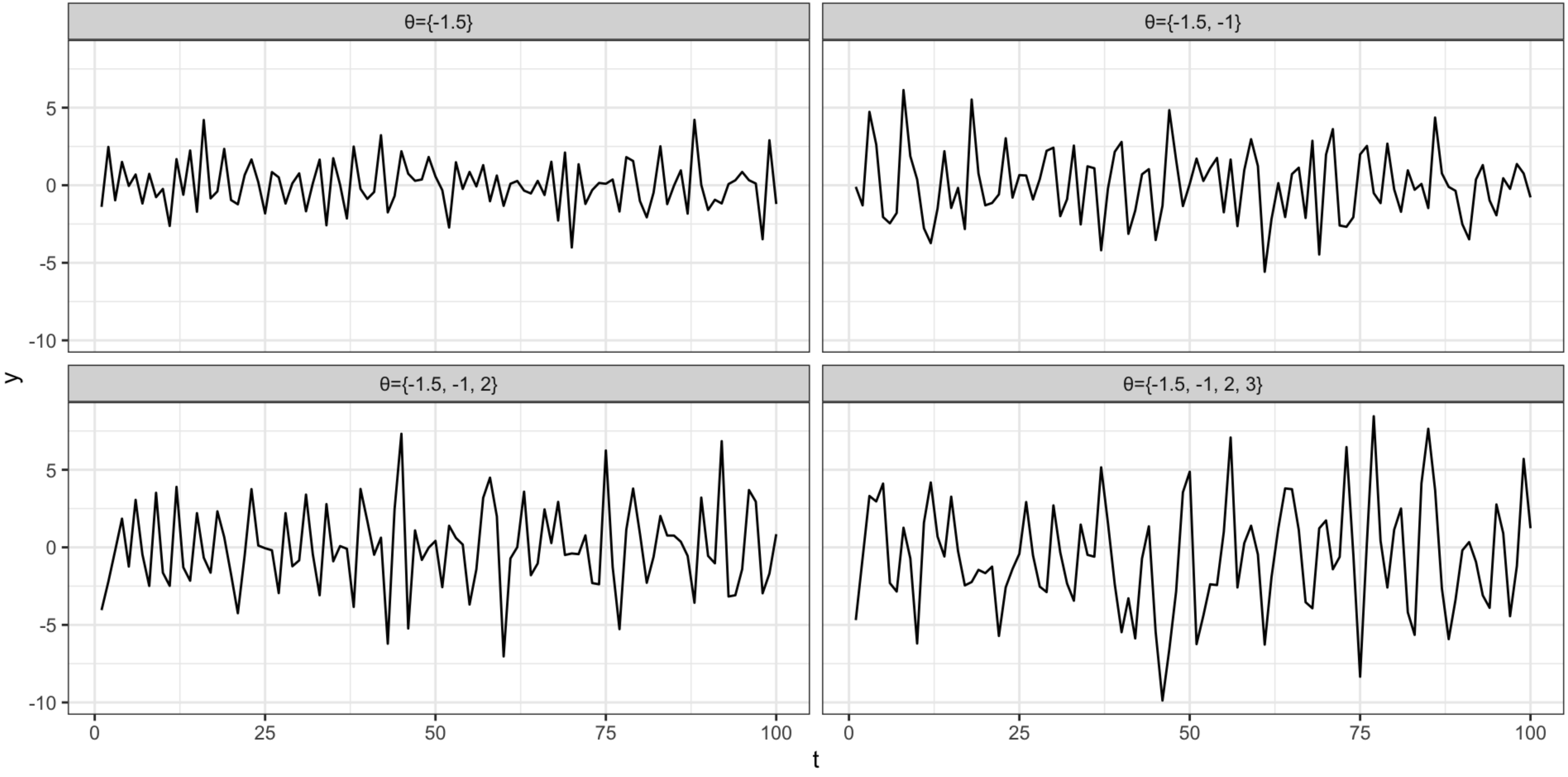
Properties:

$$E(y_t) = \delta$$

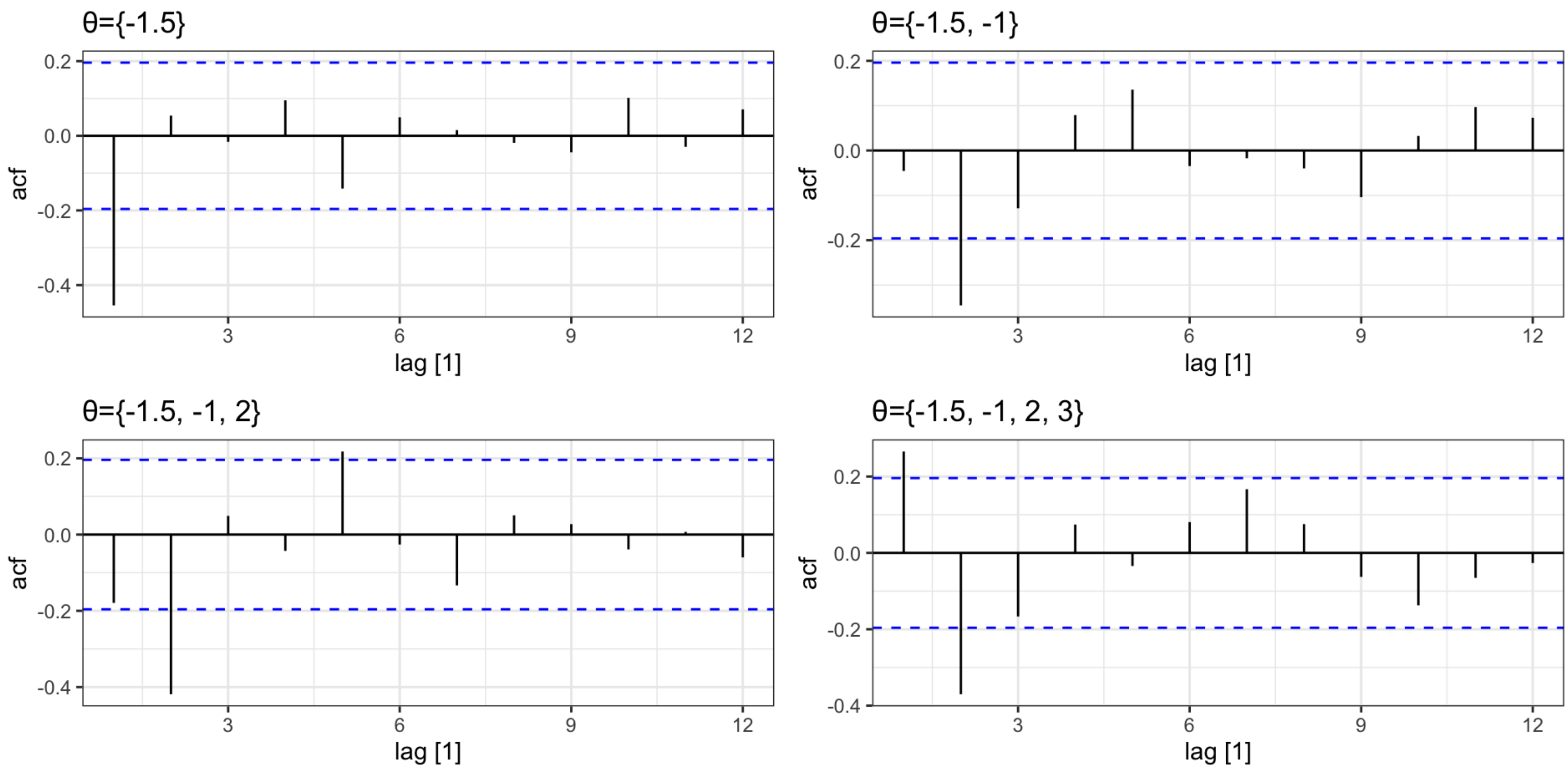
$$Var(y_t) = \gamma(0) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2) \, \sigma_w^2$$

$$Cov(y_t, y_{t+h}) = \gamma(h) = \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|} & \text{if } |h| \leq q \\ 0 & \text{if } |h| > q \end{cases}$$

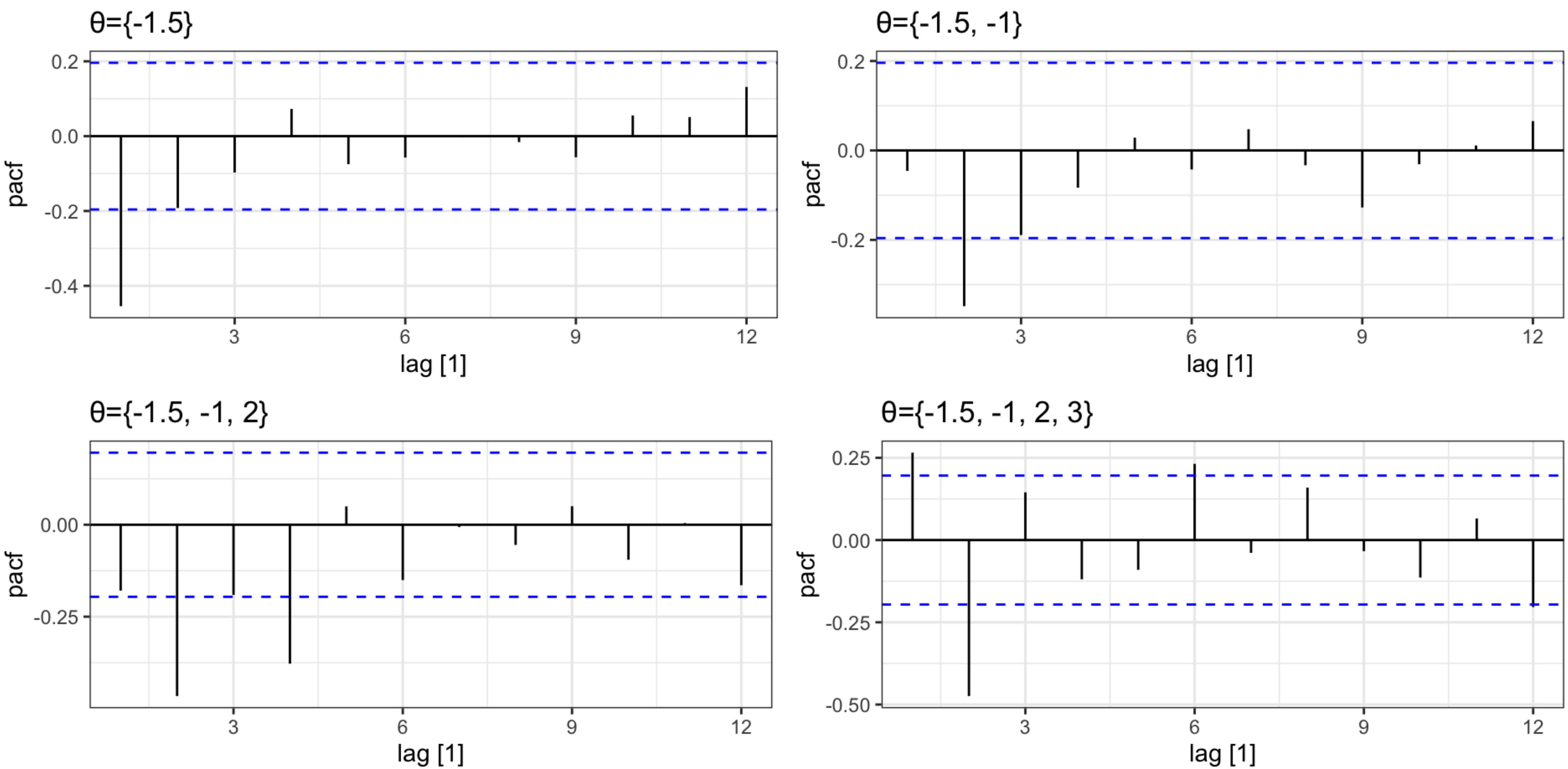
# Example series



# ACF



# PACF


θ={-1.5}
0.2
0.0
-0.2
-0.4
pacf
3
6
9
12
lag [1]
θ={-1.5, -1}
0.2
0.0
-0.2
pacf
3
6
9
12
lag [1]
θ={-1.5, -1, 2}
0.00
-0.25
pacf
3
6
9
12
lag [1]
θ={-1.5, -1, 2, 3}
0.25
0.00
-0.25
-0.50
pacf
3
6
9
12
lag [1]

# ARMA Model

# ARMA Model

An ARMA model is a composite of AR and MA processes,

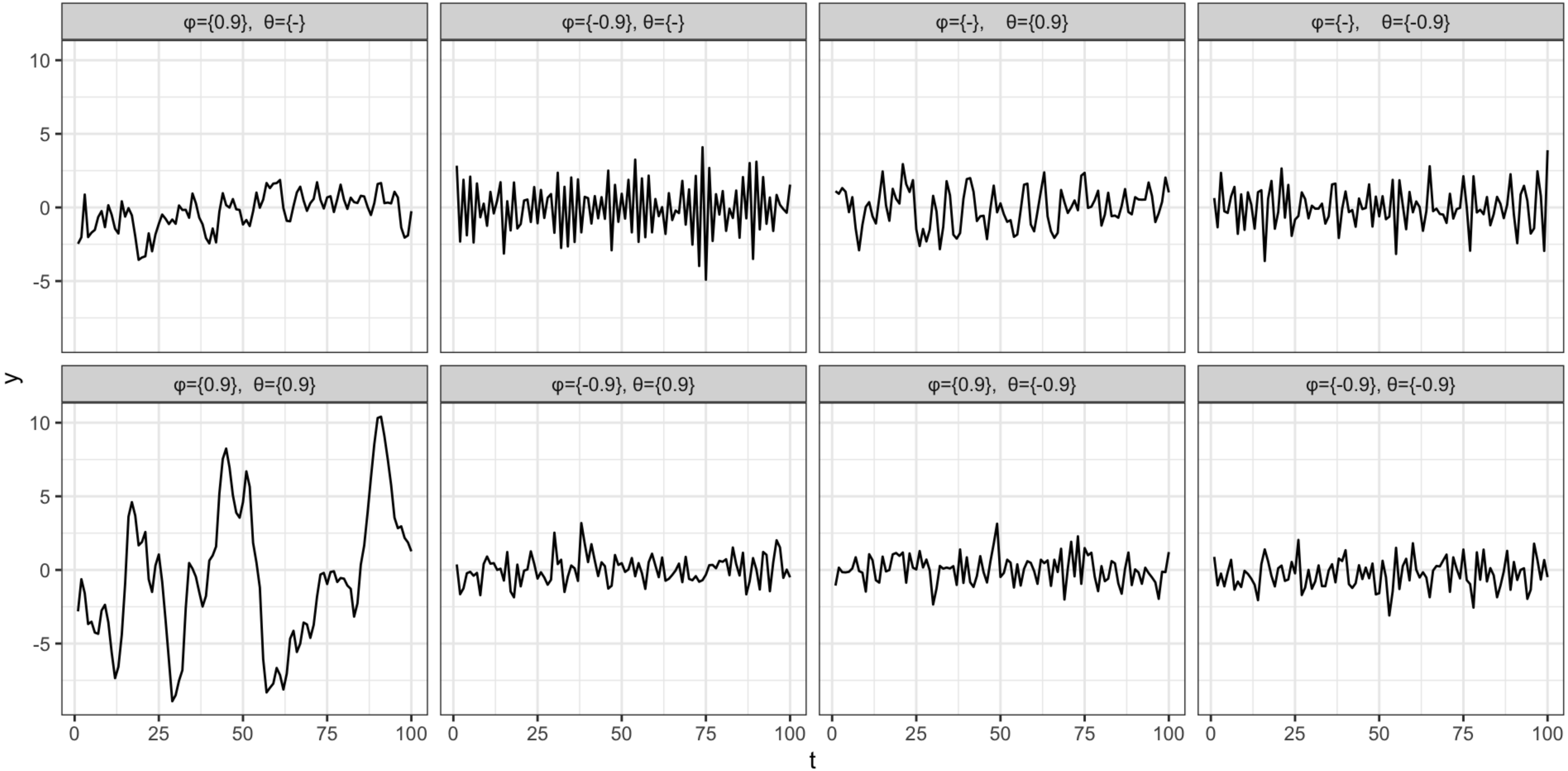
ARMA(p, q):

$$y_t = \delta + \phi_1 y_{t-1} + \cdots \phi_p y_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t_q}$$

$$\phi_p(L) y_t = \delta + \theta_q(L) w_t$$

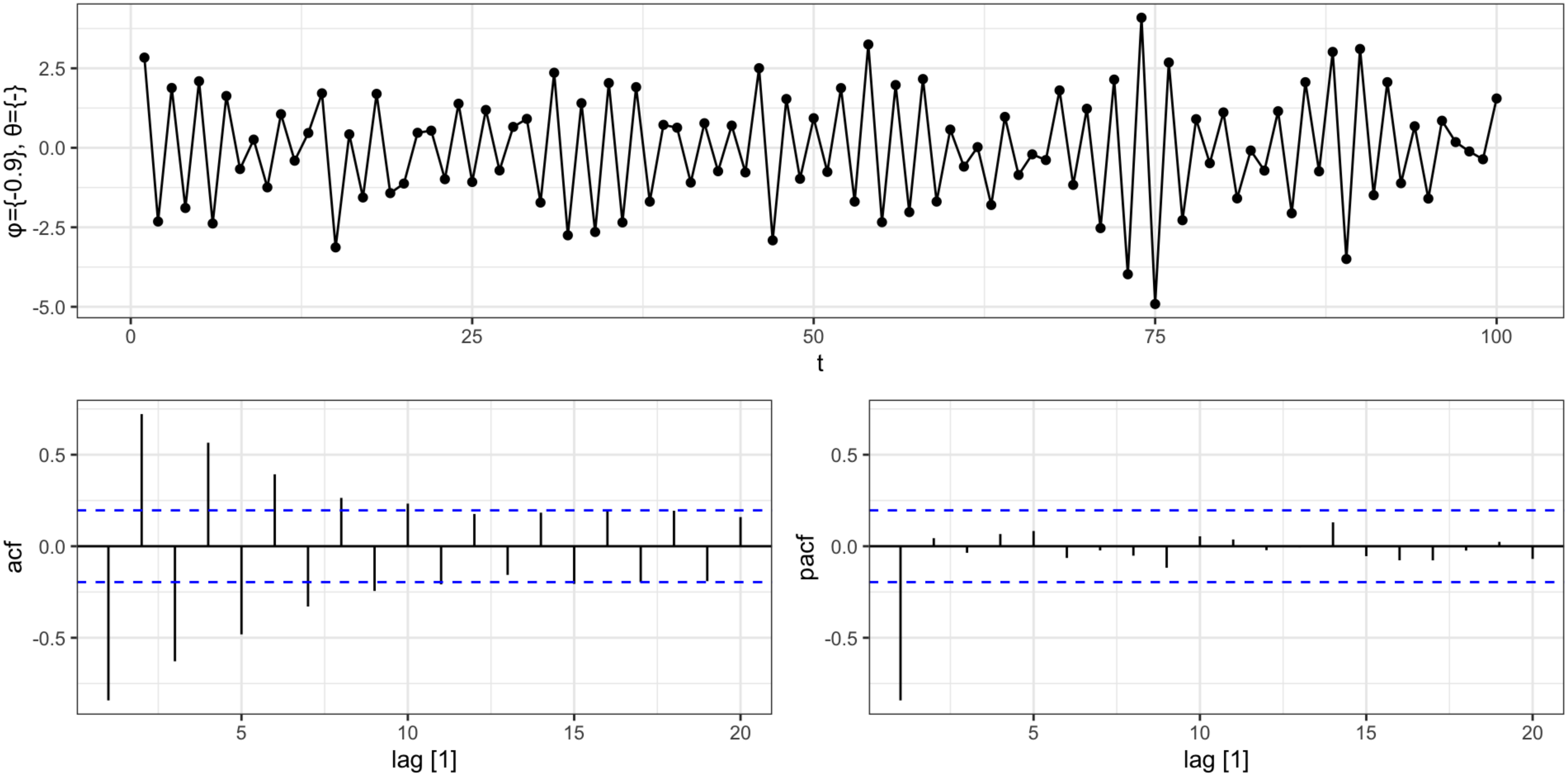
Since all MA processes are stationary, we only need to examine the AR component to determine stationarity, i.e. check roots of $\phi_p(L)$ lie outside the complex unit circle.

# Example time series

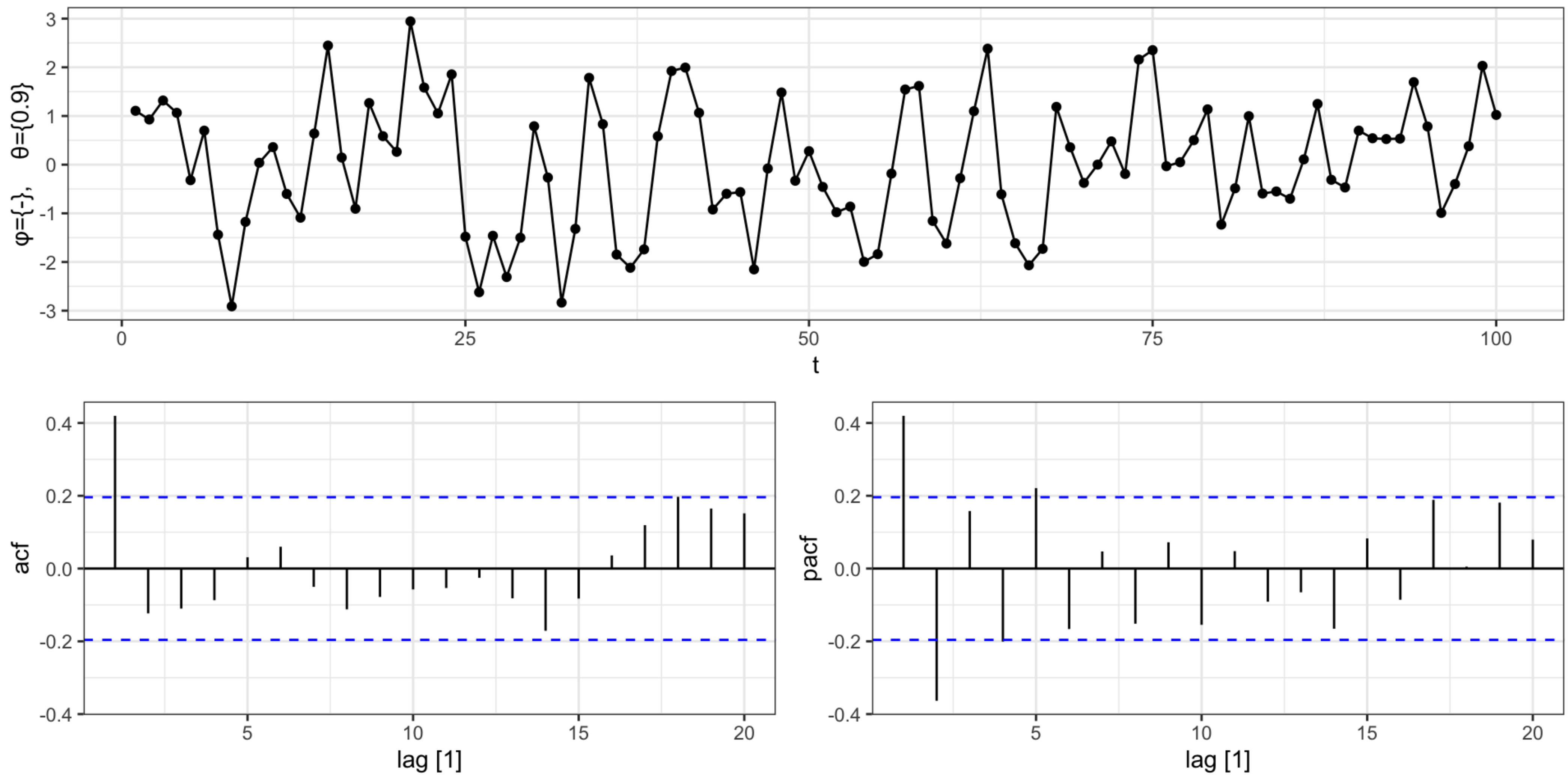



“φ={-0.9}, θ={-}” “” “”

# $\phi = 0.9, \theta = 0$



# $\phi = 0, \theta = 0.9$



# $\phi = 0.9, \theta = 0.9$