Lec 9

AR(1)

$$y_t = \delta + \phi y_{t-1} + w_t$$

$$y_t = \delta + \phi L y_t + v_t$$

$$(1 - \phi L) y_t = \delta + v_t$$

$$(\lambda - \phi) y_t = \delta + v_t$$

Roots: $\lambda - \phi = 0$

$$\lambda = \phi$$

Claim: $|\phi| < 1$

AR(2) $(\lambda^2 - \phi_1 \lambda - \phi_2) y_t = \delta + v_t$

Roots: $\lambda^2 - \phi_1 \lambda - \phi_2 \Rightarrow \lambda = \dfrac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$

AR(2) $\quad (\lambda^2 - \phi_1 \lambda - \phi_2) y_t = \delta + u_t$

Roots: $\quad \lambda^2 - \phi_1 \lambda - \phi_2 \Rightarrow \lambda = \dfrac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$

If $\phi_1^2 + 4\phi_2 > 0$ then:
(real soln)

$$\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1$$

$$\sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1$$

$$\cancel{\phi_1^2} + 4\phi_2 < 4 - 4\phi_1 + \cancel{\phi_1^2}$$

$$\phi_1 + \phi_2 < 1$$

$$\frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} > -1$$

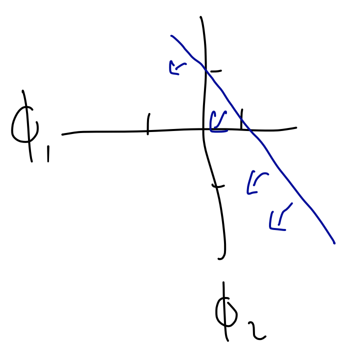
$$\phi_1 + 2 > \sqrt{\phi_1^2 + 4\phi_2}$$

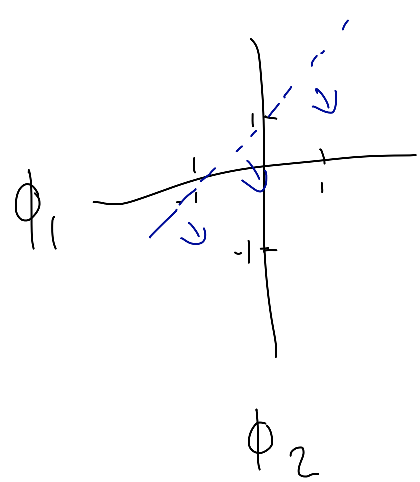
$$\cancel{\phi_1^2} + 4\phi_1 + 4 > \cancel{\phi_1^2} + 4\phi_2$$

$$\phi_1 - \phi_2 > 1$$

or

$$\phi_2 - \phi_1 < 1$$





if $\phi_1^2 + 4\phi_2 < 0$

(img roots)

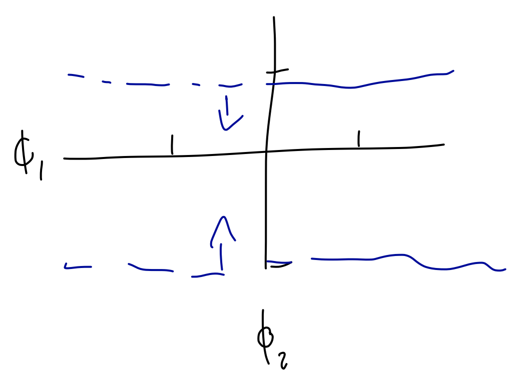
$$\left| \frac{\phi_1}{2} \pm \frac{\sqrt{-(\phi_1^2 + 4\phi_2)}}{2} i \right| < 1$$

$$\sqrt{\left(\frac{\phi_1}{2}\right)^2 + \left(\frac{\sqrt{-(\phi_1^2+4\phi_2)}}{2}\right)^2} < 1$$

$$\sqrt{\cancel{\frac{\phi_1^2}{4}} + \frac{\cancel{-\phi_1^2} + \cancel{4}\phi_2}{\cancel{4}}} < 1$$

$$\sqrt{\phi_2} < 1$$

$$-1 < \phi_2 < 1$$



$\boxed{AR(2)}$ $\quad y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + v_t$

If stationary:

① $E(y_t) = \delta + \phi_1 E(y_{t-1}) + \phi_2 E(y_{t-2})$

$(1-\phi_1-\phi_2) E(y_t) = \delta$

$E(y_t) = \dfrac{\delta}{1-\phi_1-\phi_2}$

$\boxed{\text{Let } \tilde{y}_t = y_t - E(y_t)}$ $\quad \boxed{Var(\tilde{y}_t) = Var(y_t)}$ $\quad \boxed{Cov(\tilde{y}_t, \tilde{y}_{t-1}) = Cov(y_t, y_{t-1})}$

② $\gamma(h) = E(\tilde{y}_t \cdot \tilde{y}_{t-h}) = E\left(\phi_1 \tilde{y}_{t-1}\tilde{y}_{t-h} + \phi_2 \tilde{y}_{t-2}\tilde{y}_{t-h} + w_t \tilde{y}_{t-h}\right)$

$= \phi_1 E(\tilde{y}_{t-1}\tilde{y}_{t-h}) + \phi_2 E(\tilde{y}_{t-2}\tilde{y}_{t-h}) + E(v_t \tilde{y}_{t-h})$

$= \phi_1 \gamma(h-1) + \phi_2 (h-2) + \sigma_v^2 \mathbb{1}_{h=0}$

$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma_v^2$

$\gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1) \quad \Rightarrow \quad \gamma(1) = \dfrac{\phi_1}{1-\phi_2}\gamma(0)$

$\gamma(2) = \phi_1 \gamma(1) + \phi_2 \gamma(0) \quad \Rightarrow \gamma(2) = \dfrac{\phi_1^2 + \phi_1(1-\phi_2)}{1-\phi_2}\gamma(0)$

MA(1)

$$Y_t = \delta + w_t + \theta w_{t-1}$$

① $E(Y_t) = \delta + E(w_t) + \theta E(w_{t-1})$

$= \delta$

② $\gamma(0) = Var(Y_t) = Var(w_t) + \theta^2 Var(w_{t-1})$

$= (1+\theta^2)\sigma_w^2$

$$\gamma(h) = E(\tilde{Y}_t \tilde{Y}_{t-h})$$

$= E\left((w_t + \theta w_{t-1})(w_{t-h} + \theta w_{t-h-1}) \right)$

$= E(w_t w_{t-h}) + E(w_t \theta w_{t-h-1})$

$+ E(\theta w_{t-1} w_{t-h}) + E(\theta w_{t-1} \theta w_{t-h-1})$

$$= \begin{cases} \theta^2 \sigma_w^2 + \sigma_w^2 & \text{if } h = 0 \\ \theta \sigma_w^2 & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$