

AR, MA, and ARMA Models

Lecture 09

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AR models

AR(1) models

From last time we derived the following properties for AR(1) models,

$$y_t = \delta + \phi y_{t-1} + w_t$$
$$w_t \stackrel{\text{iid}}{\sim} N(0, \sigma_w^2)$$

The process y_t is stationary iff $|\phi| < 1$, and if stationary then

AR(p) models

We can generalize from an AR(1) to an AR(p) model by simply adding additional autoregressive terms to the model.

$$\begin{aligned}\text{AR}(p) : \quad y_t &= \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t \\ &= \delta + w_t + \sum_{i=1}^p \phi_i y_{t-i}\end{aligned}$$

What are the properties of AR(p), specifically

1. Stationarity conditions?
2. Expected value?
3. Autocovariance / autocorrelation?

Lag operator

The lag operator is convenience notation for writing out AR (and other) time series models.

We define the lag operator L as follows,

$$L y_t = y_{t-1}$$

this can be generalized where,

$$\begin{aligned} L^2 y_t &= L (L y_t) \\ &= L y_{t-1} \\ &= y_{t-2} \end{aligned}$$

therefore,

$$L^k y_t = y_{t-k}$$

Lag polynomial

Lets rewrite the AR(p) model using the lag operator,

$$\begin{aligned} y_t &= \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + w_t \\ &= \delta + \phi_1 L y_t + \phi_2 L^2 y_t + \dots + \phi_p L^p y_t + w_t \end{aligned}$$

If we group all of the y_t terms, we get the following

$$\begin{aligned} \delta + w_t &= y_t - \phi_1 L y_t - \phi_2 L^2 y_t - \dots - \phi_p L^p y_t \\ &= (1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p) y_t \end{aligned}$$

This polynomial of lags

$$\phi_p(L) = (1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p)$$

Stationarity of AR(p) processes

Claim: An AR(p) process is stationary if the roots of the characteristic polynomial lay *outside* the complex unit circle

If we define $\lambda = 1/L$ then we can rewrite the characteristic polynomial as

$$(\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \dots - \phi_{p-1} \lambda - \phi_p)$$

then as a corollary of our claim the AR(p) process is stationary if the roots of this new polynomial are *inside* the complex unit circle, i.e. $|\lambda| < 1$.

Example AR(1)

Example AR(2)

AR(2) Stationarity Conditions

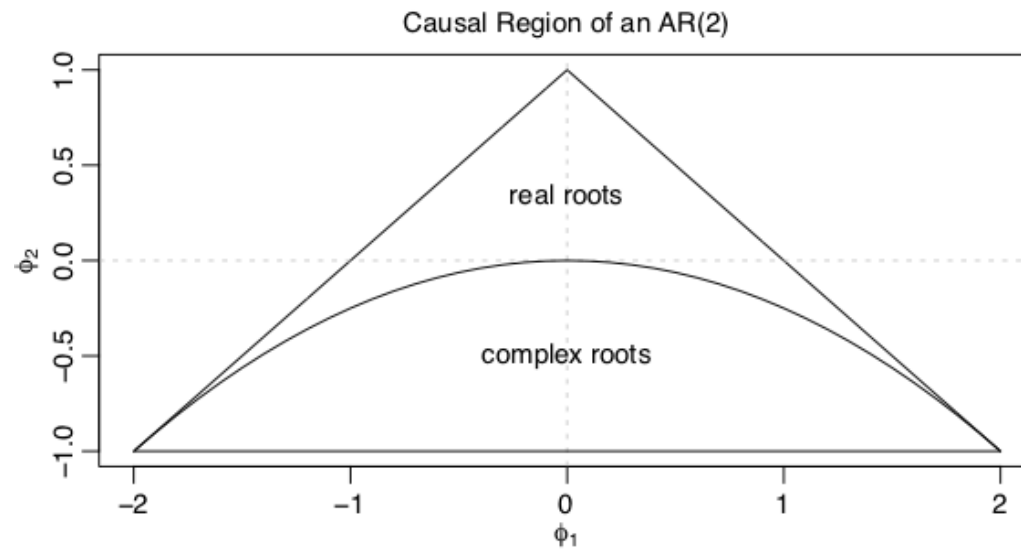


Fig. 3.3. Causal region for an AR(2) in terms of the parameters.

Proof Sketch

We can rewrite the AR(p) model into an AR(1) form using matrix notation

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t$$

$$\boldsymbol{\xi}_t = \boldsymbol{\delta} + \mathbf{F} \boldsymbol{\xi}_{t-1} + \mathbf{w}_t$$

where

$$\underset{p \times 1}{\boldsymbol{\xi}_t} = [y_t, y_{t-1}, y_{t-2}, \dots, y_{t-p+1}]'$$

$$\underset{p \times 1}{\boldsymbol{\delta}} = [\delta, 0, 0, \dots, 0]'$$

$$\underset{p \times 1}{\mathbf{w}_t} = [w_t, 0, 0, \dots, 0]'$$

$$\underset{p \times p}{\mathbf{F}} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Putting it together

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \delta \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \delta + w_t + \sum_{i=1}^p \phi_i y_{t-i} \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix}$$

Proof sketch (cont.)

So just like the original AR(1) we can expand out the autoregressive equation

$$\begin{aligned}\xi_t &= \delta + w_t + F \xi_{t-1} \\ &= \delta + w_t + F (\delta + w_{t-1}) + F^2 (\delta + w_{t-2}) + \cdots \\ &\quad + F^{t-1} (\delta + w_1) + F^t (\delta + w_0) \\ &= \left(\sum_{i=0}^t F^i \right) \delta + \sum_{i=0}^t F^i w_{t-i}\end{aligned}$$

and therefore we need $\lim_{t \rightarrow \infty} F^t \rightarrow 0$ so that $\lim_{t \rightarrow \infty} \sum_{i=0}^t F^i < \infty$.

Proof sketch (cont.)

We can find the eigen decomposition such that $F = Q\Lambda Q^{-1}$ where the columns of Q are the eigenvectors of F and Λ is a diagonal matrix of the corresponding eigenvalues.

A useful property of the eigen decomposition is that

$$F^i = Q\Lambda^i Q^{-1}$$

Using this property we can rewrite our equation from the previous slide as

$$\begin{aligned}\xi_t &= \left(\sum_{i=0}^t F^i \right) \delta + \sum_{i=0}^t F^i w_{t-i} \\ &= \left(\sum_{i=0}^t Q\Lambda^i Q^{-1} \right) \delta + \sum_{i=0}^t Q\Lambda^i Q^{-1} w_{t-i}\end{aligned}$$

Proof sketch (cont.)

$$\Lambda^i = \begin{bmatrix} \lambda_1^i & 0 & \cdots & 0 \\ 0 & \lambda_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p^i \end{bmatrix}$$

Therefore, $\lim_{t \rightarrow \infty} F^t \rightarrow 0$ when $\lim_{t \rightarrow \infty} \Lambda^t \rightarrow 0$ which requires that

$$|\lambda_i| < 1 \quad \text{for all } i$$

Proof sketch (cont.)

Eigenvalues are defined such that for λ ,

$$\det(\mathbf{F} - \lambda \mathbf{I}) = 0$$

based on our definition of \mathbf{F} our eigenvalues will therefore be the roots of

$$\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \dots - \phi_{p-1} \lambda - \phi_p = 0$$

which if we multiply by $1/\lambda^p$ where $L = 1/\lambda$ gives

$$1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_{p-1} L^{p-1} - \phi_p L^p = 0$$

Properties of AR(2)

For a *stationary* AR(2) process,

Properties of $AR(2)$ (cont.)

Properties of AR(p)

For a *stationary* AR(p) process,

$$E(Y_t) = \frac{\delta}{1 - \phi_1 - \phi_2 - \dots - \phi_p}$$

$$\text{Var}(y_t) = \gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \dots + \phi_p \gamma(p) + \sigma_w^2$$

$$\text{Cov}(y_t, y_{t+h}) = \gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2) + \dots + \phi_p \gamma(h-p)$$

$$\text{Corr}(y_t, y_{t+h}) = \rho(h) = \phi_1 \rho(h-1) + \phi_2 \rho(h-2) + \dots + \phi_p \rho(h-p)$$

Moving Average (MA) Processes

MA(1)

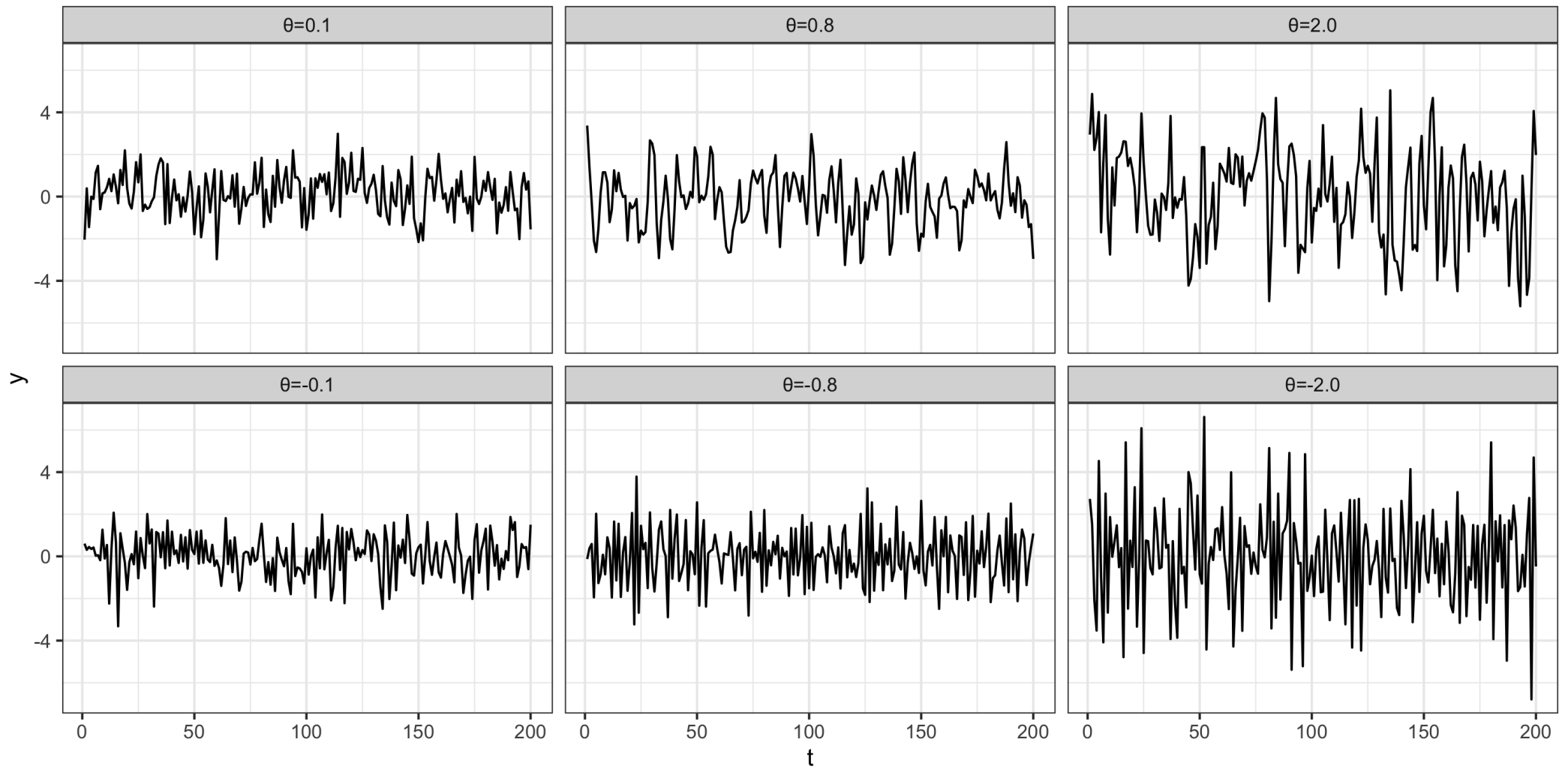
A moving average process is similar to an AR process, except that the autoregression is on the error term.

$$\text{MA}(1) : \quad y_t = \delta + w_t + \theta w_{t-1}$$

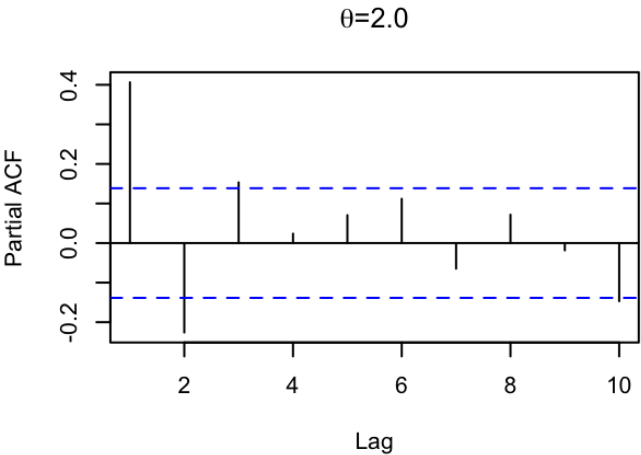
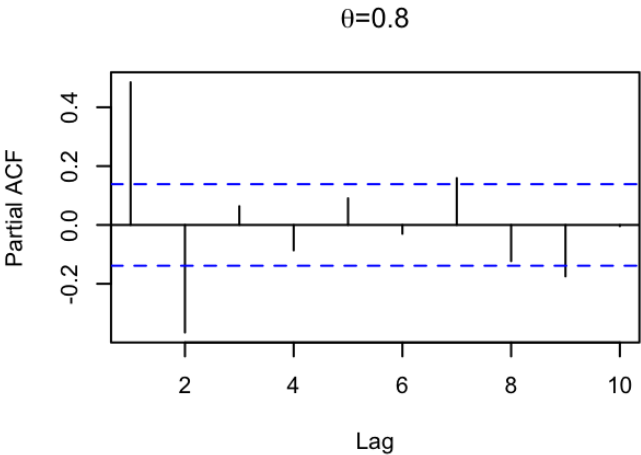
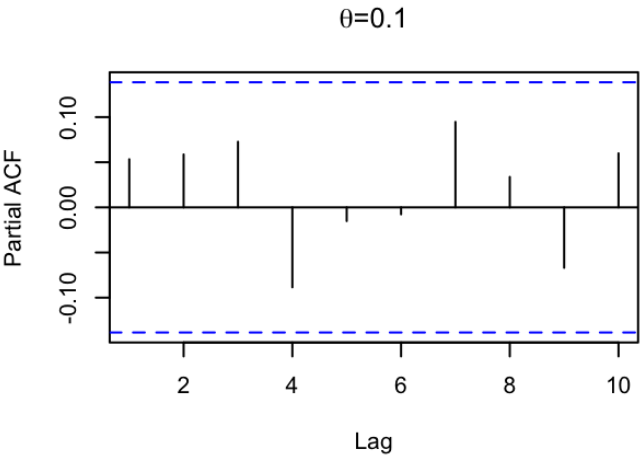
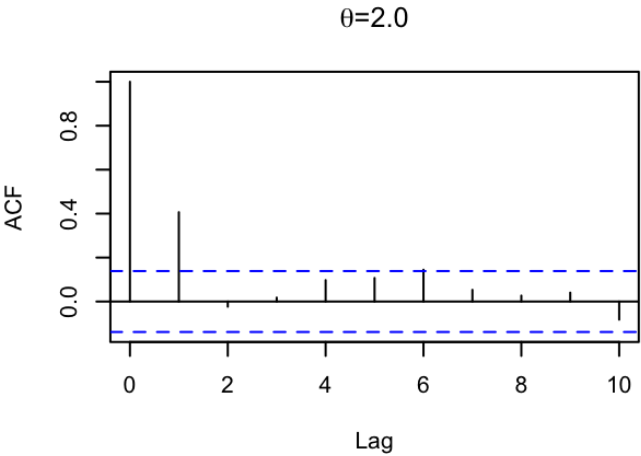
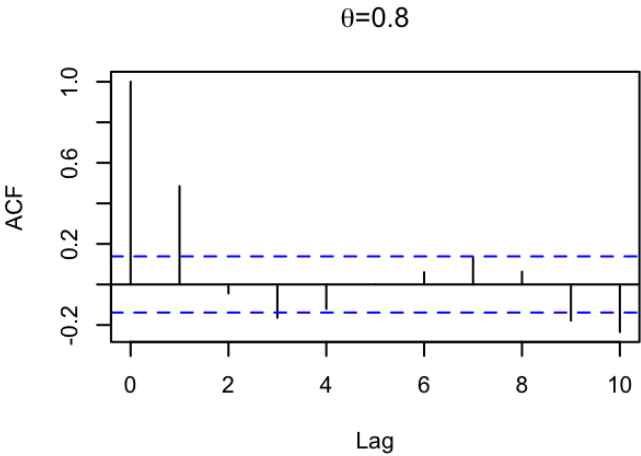
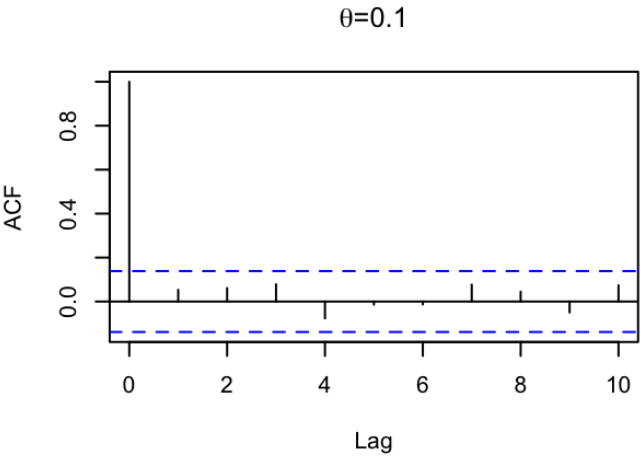
Properties:

MA(1) - properties (cont.)

Time series



ACF



MA(q)

$$\text{MA}(q) : y_t = \delta + w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}$$

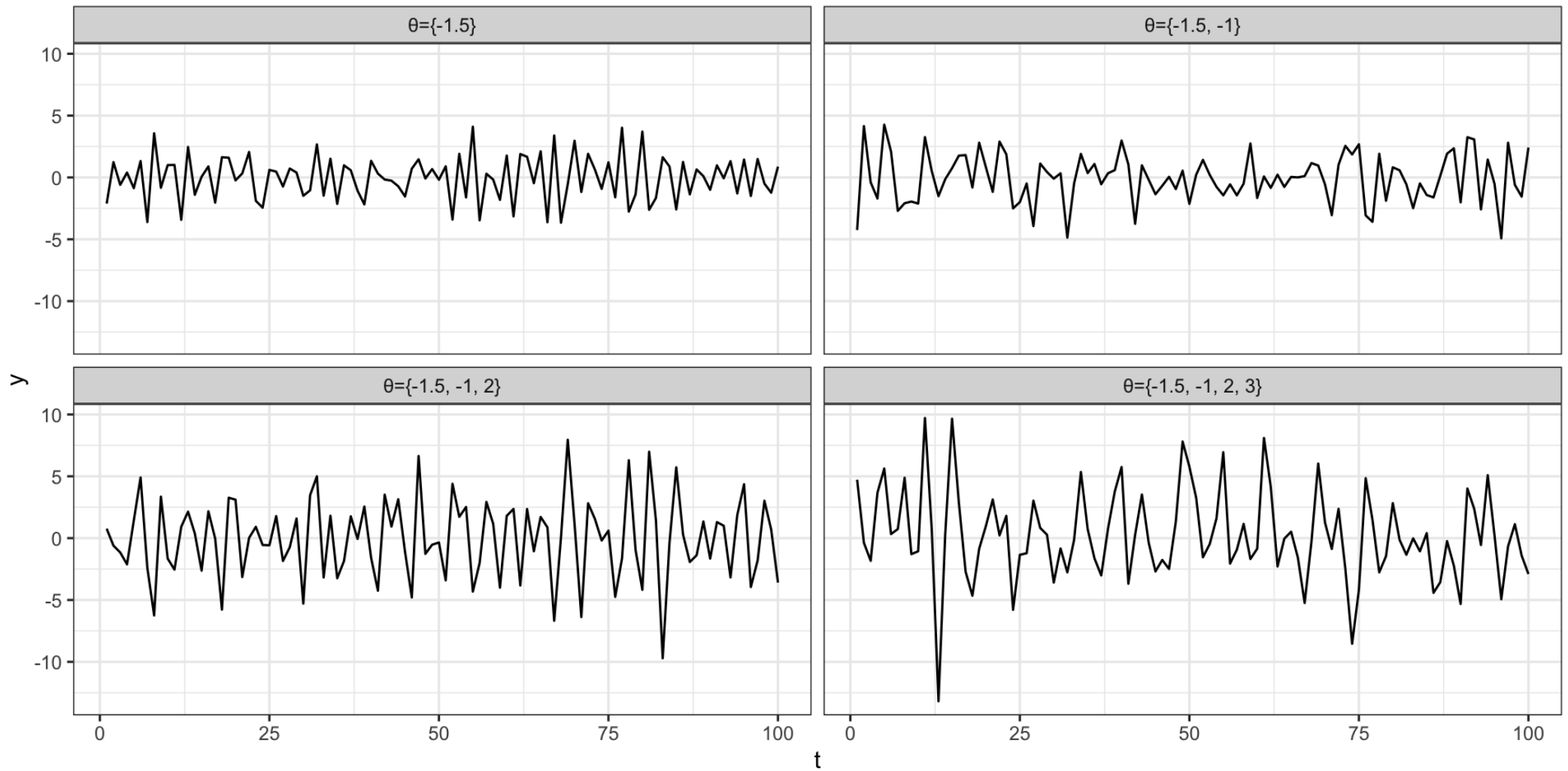
Properties:

$$E(y_t) = \delta$$

$$\text{Var}(y_t) = \gamma(0) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2) \sigma_w^2$$

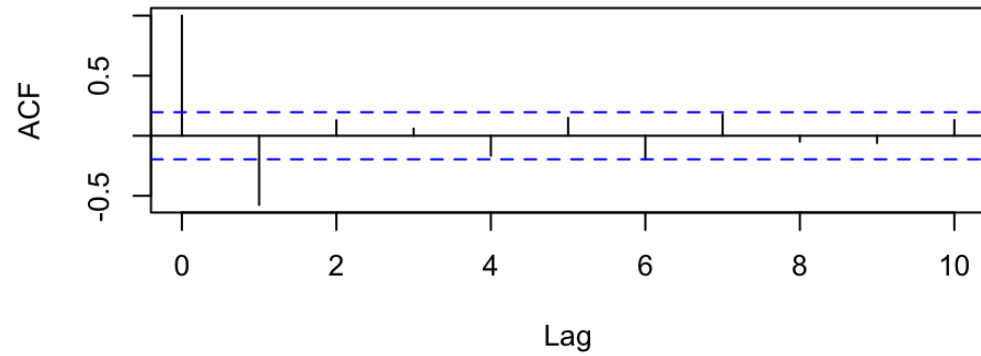
$$\text{Cov}(y_t, y_{t+h}) = \gamma(h) = \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|} & \text{if } |h| \leq q \\ 0 & \text{if } |h| > q \end{cases}$$

Example series

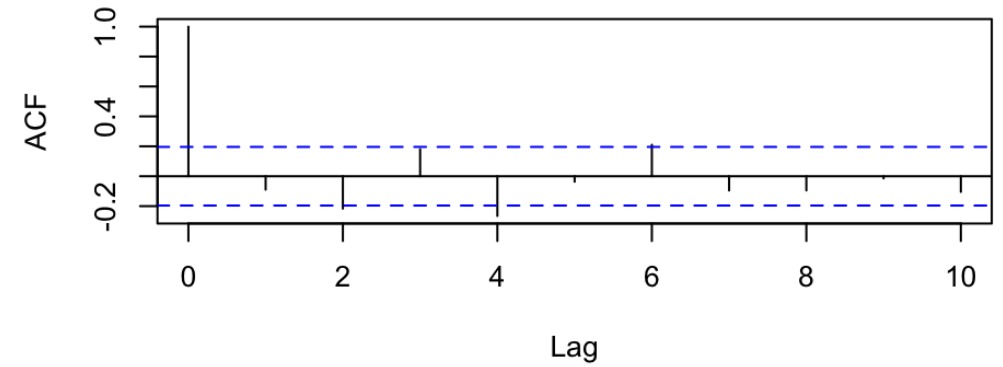


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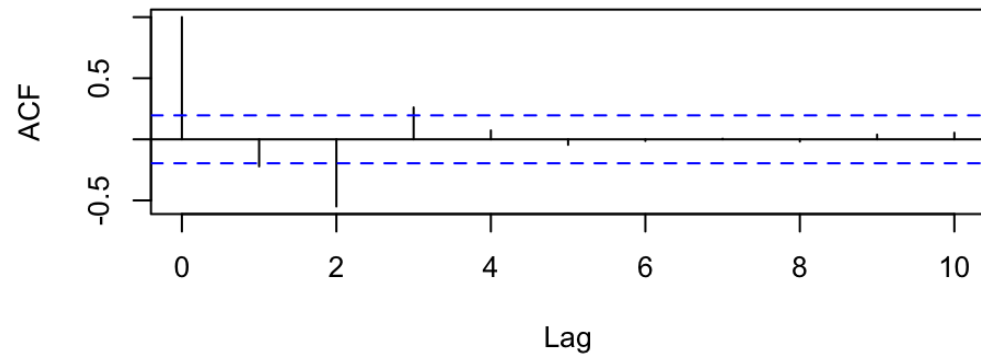
$\theta=\{-1.5\}$



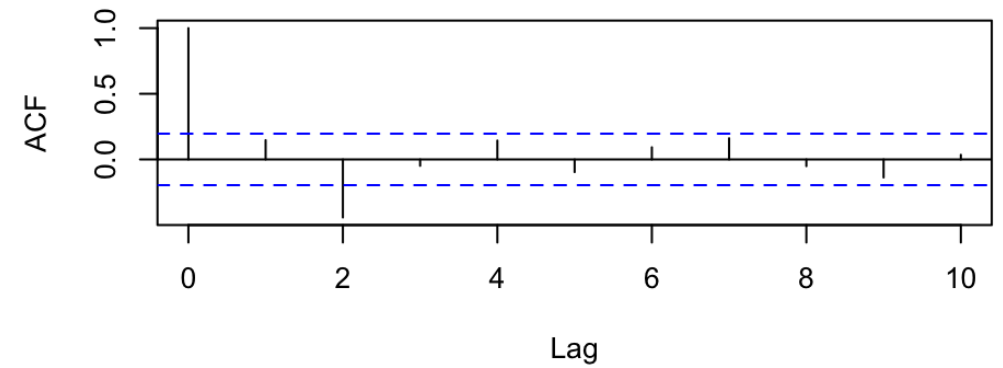
$\theta=\{-1.5, -1\}$



$\theta=\{-1.5, -1, 2\}$

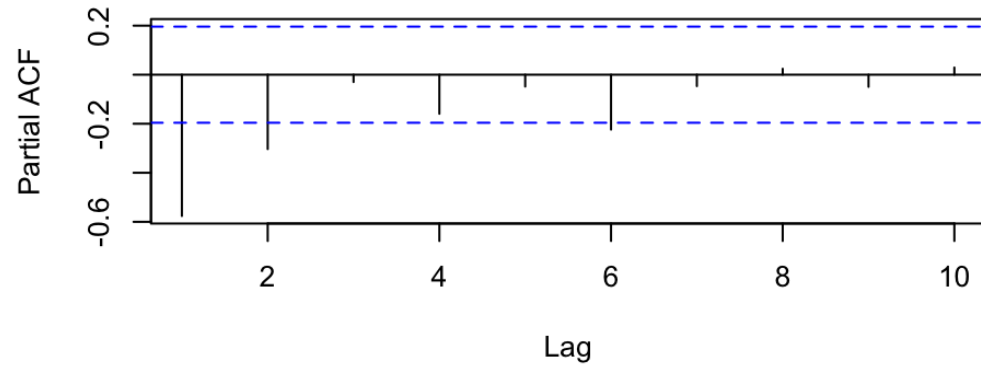


$\theta=\{-1.5, -1, 2, 3\}$

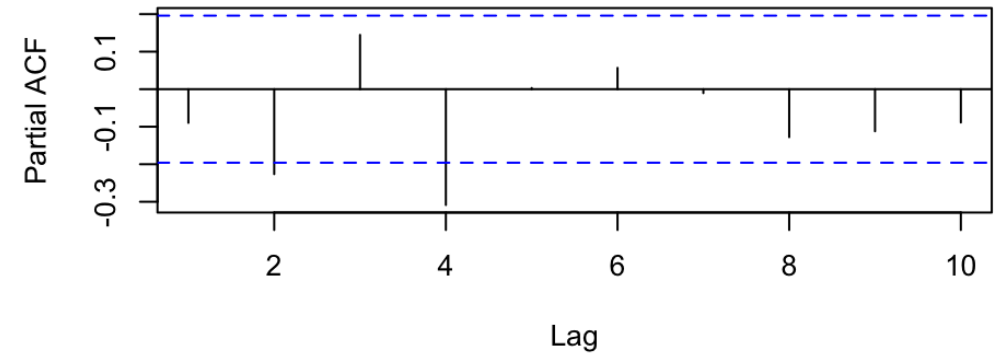


PACF

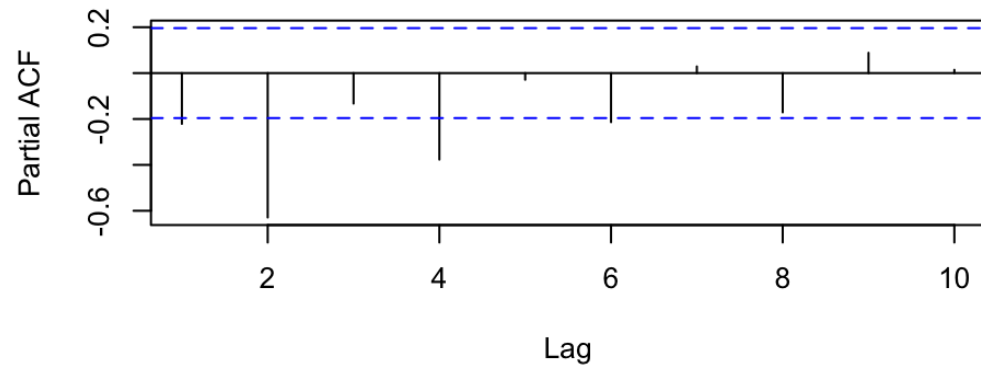
$$\theta = \{-1.5\}$$



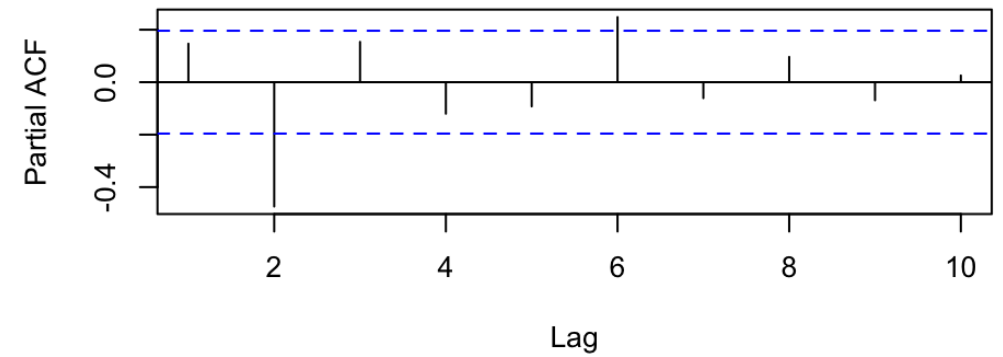
$$\theta = \{-1.5, -1\}$$



$$\theta = \{-1.5, -1, 2\}$$



$$\theta = \{-1.5, -1, 2, 3\}$$



ARMA Model

ARMA Model

An ARMA model is a composite of AR and MA processes,

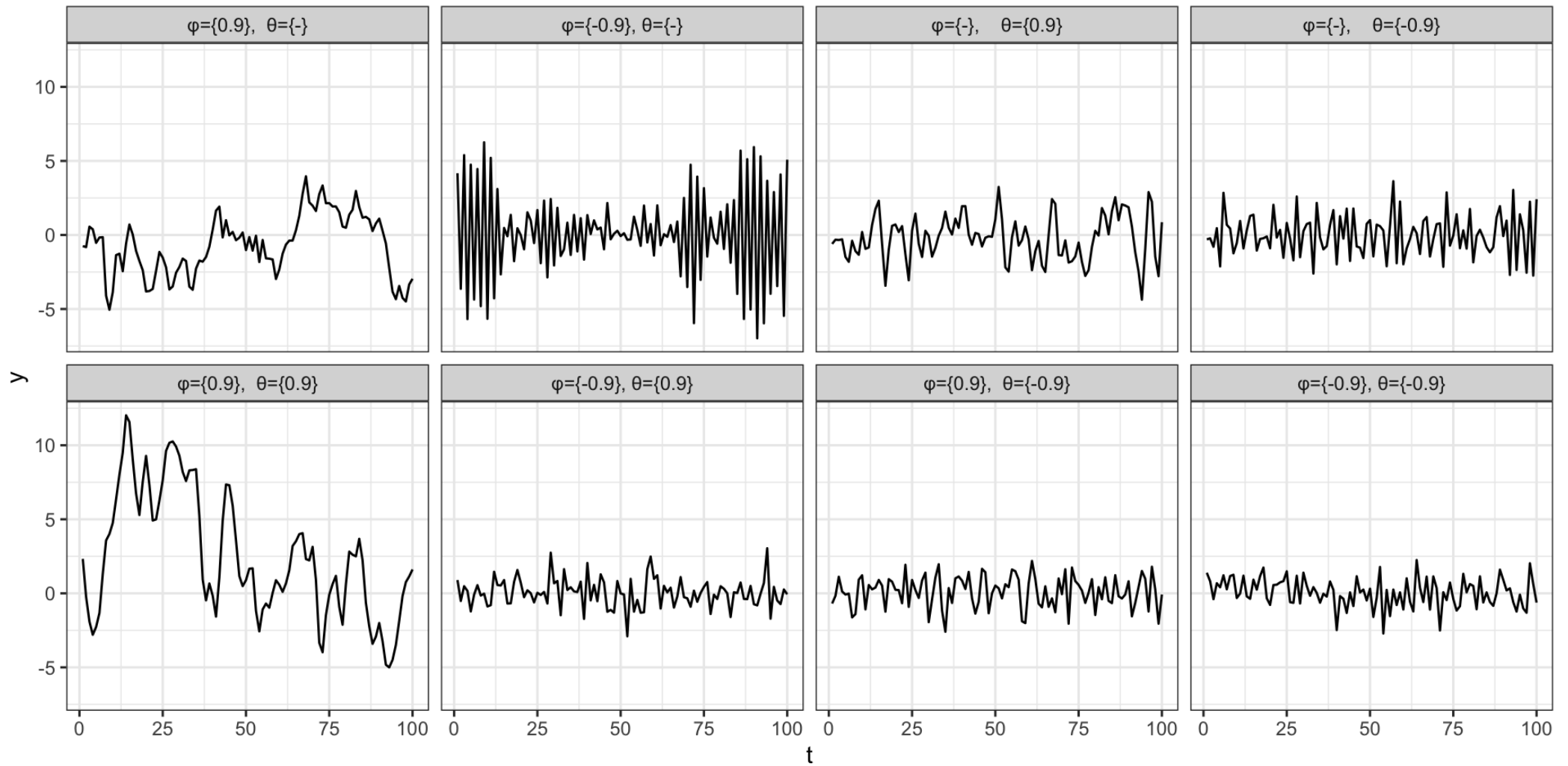
ARMA(p, q):

$$y_t = \delta + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q}$$

$$\phi_p(L)y_t = \delta + \theta_q(L)w_t$$

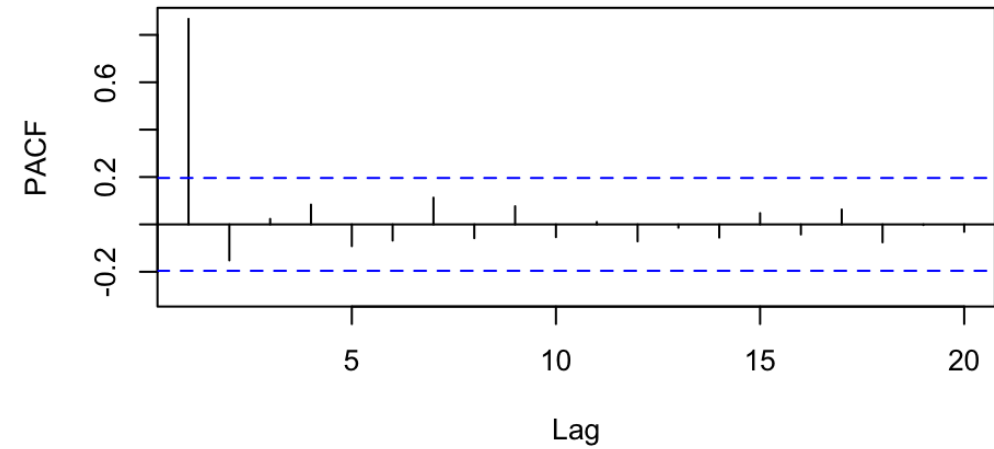
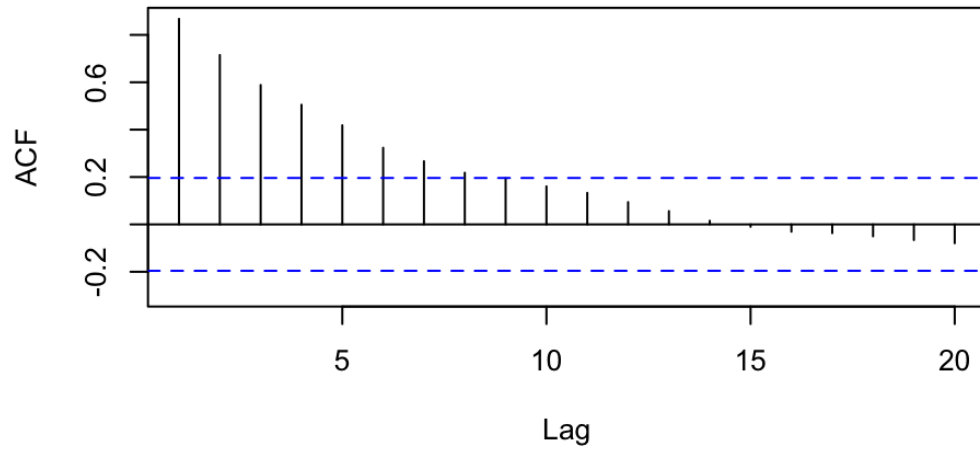
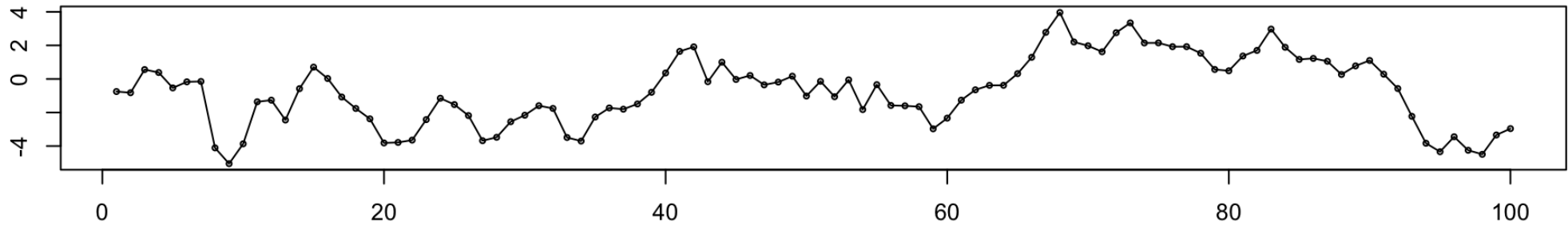
Since all MA processes are stationary, we only need to examine the AR component to determine stationarity, i.e. check roots of $\phi_p(L)$ lie outside the complex unit circle.

Time series



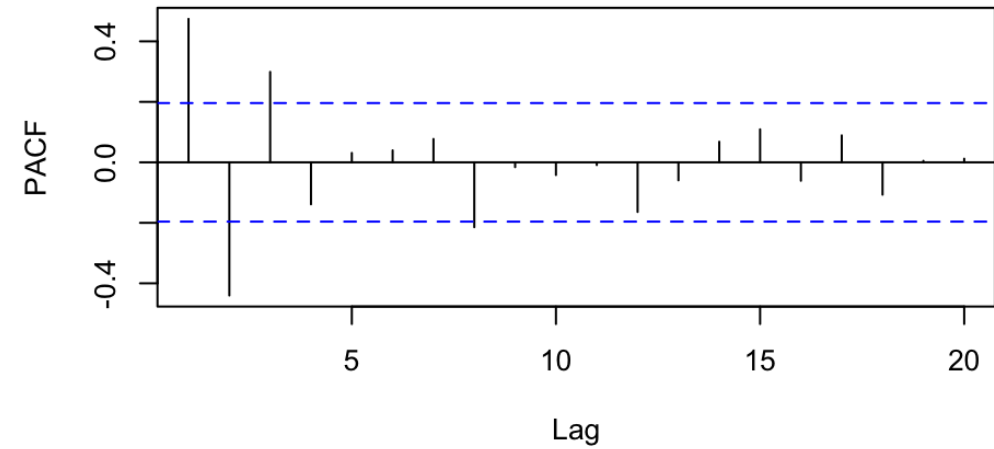
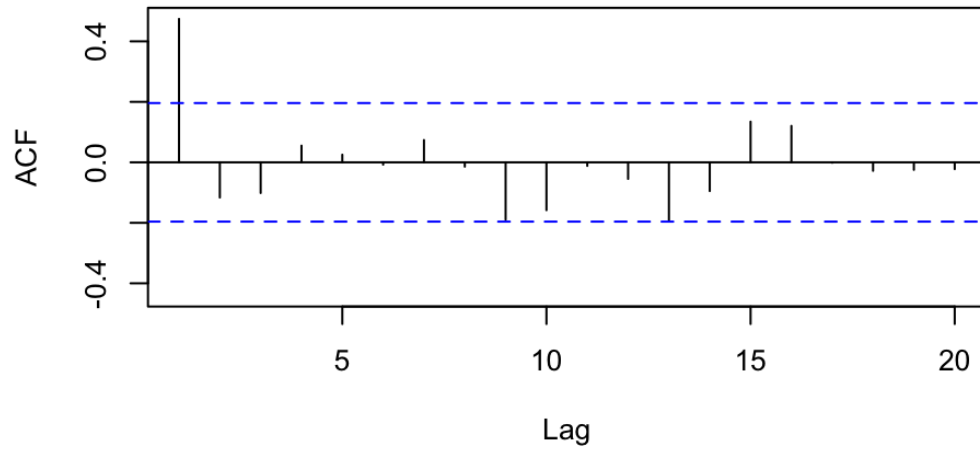
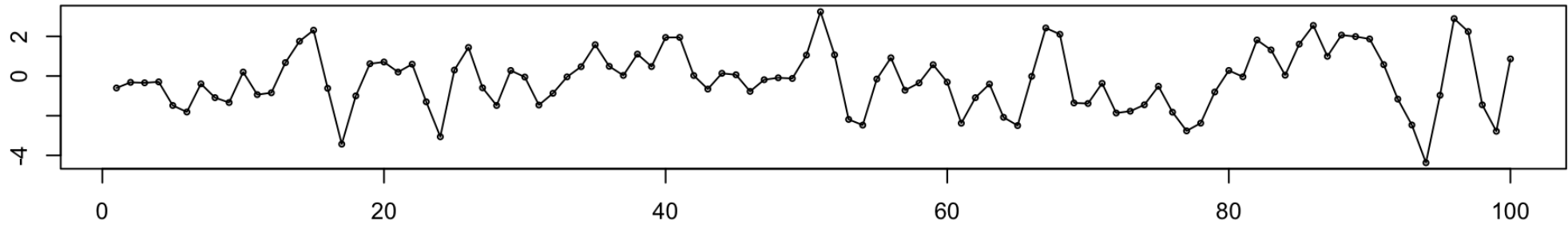
$$\phi = 0.9, \theta = 0$$

$\phi=\{0.9\}, \theta=\{0\}$



$$\phi = 0, \theta = 0.9$$

$\phi=\{0\}, \theta=\{0.9\}$



$$\phi = 0.9, \theta = 0.9$$

$\phi=\{0.9\}, \theta=\{0.9\}$

