Fitting ARIMA Models

Lecture 12

Dr. Colin Rundel

Model Fitting

Fitting ARIMA

For an $\text{ARIMA}(p, d, q)$ model,

- Assumes that the data is stationary after differencing
- Handling d is straight forward, just difference the original data d times (leaving $n - d$ observations)

$$y'_t = \Delta^d \, y_t$$

- After differencing, fit an $\text{ARMA}(p, q)$ model to y'_t.
- To keep things simple we'll assume $w_t \overset{iid}{\sim} \quad (0, \sigma^2_w)$

MLE - Stationarity & iid normal errors

If both of these assumptions are met, then the time series y_t will also be normal.

In general, the vector $\boldsymbol{y} = (y_1, y_2, \ldots, y_t)'$ will have a multivariate normal distribution with mean $\{\boldsymbol{\mu}\}_i = E(y_i) = E(y_t)$ and covariance $\boldsymbol{\Sigma}$ where $\{\boldsymbol{\Sigma}\}_{ij} = \gamma(i - j)$.

The joint density of $\boldsymbol{y}$ is given by

$$f_{\boldsymbol{y}}(\boldsymbol{y}) = \frac{1}{(2\pi)^{t/2} \, \det(\boldsymbol{\Sigma})^{1/2}} \times \exp\left(-\frac{1}{2}(\boldsymbol{y} - \boldsymbol{\mu})' \, \Sigma^{-1} \, (\boldsymbol{y} - \boldsymbol{\mu})\right)$$

AR

Fitting AR(1)

$$y_t = \delta + \phi\, y_{t-1} + w_t$$

We need to estimate three parameters: δ, ϕ, and σ_w^2, we know

$$E(y_t) = \frac{\delta}{1-\phi} \qquad Var(y_t) = \frac{\sigma_w^2}{1-\phi^2}$$

$$\gamma(h) = \frac{\sigma_w^2}{1-\phi^2}\phi^{|h|}$$

Using these properties it is possible to write the distribution of $\boldsymbol{y}$ as a MVN but that does not make it easy to write down a (simplified) closed form for the MLE estimate for δ, ϕ, and σ_w^2.

Conditional Density

We can also rewrite the density as follows,

$$
\begin{aligned}
f(\boldsymbol{y}) &= f(y_t,\, y_{t-1},\, \ldots,\, y_2,\, y_1) \\
&= f(y_t \mid y_{t-1},\, \ldots,\, y_2,\, y_1) f(y_{t-1} \mid y_{t-2},\, \ldots,\, y_2,\, y_1) \cdots f(y_2 \mid y_1) f(y_1) \\
&= f(y_t \mid y_{t-1}) f(y_{t-1} \mid y_{t-2}) \cdots f(y_2 \mid y_1) f(y_1)
\end{aligned}
$$

where,

$$
\begin{aligned}
y_1 &\sim \left(\delta,\ \frac{\sigma_w^2}{1-\phi^2}\right) \\
y_t \mid y_{t-1} &\sim \left(\delta + \phi\, y_{t-1},\ \sigma_w^2\right) \\
f(y_t \mid y_{t-1}) &= \frac{1}{\sqrt{2\pi\, \sigma_w^2}} \exp\left(-\frac{1}{2} \frac{(y_t - \delta + \phi\, y_{t-1})^2}{\sigma_w^2}\right)
\end{aligned}
$$

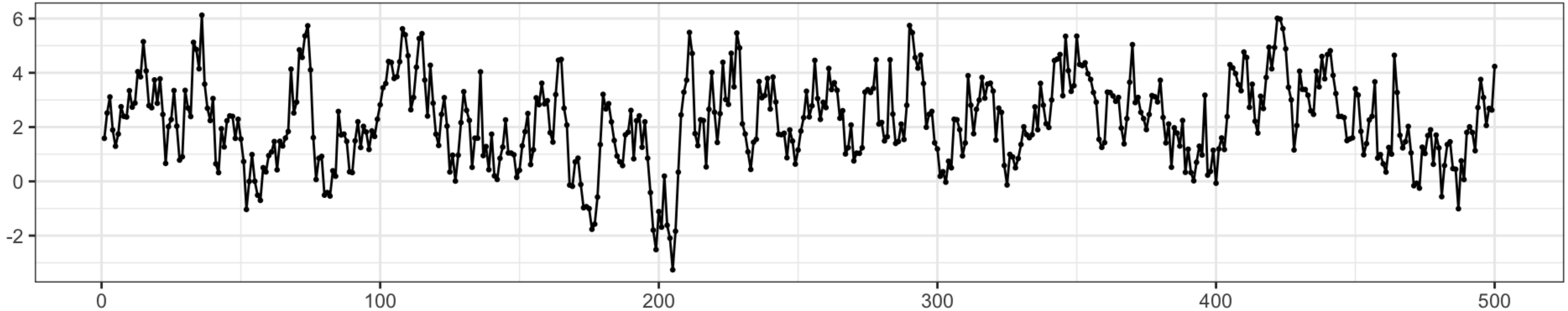
Log likelihood of AR(1)

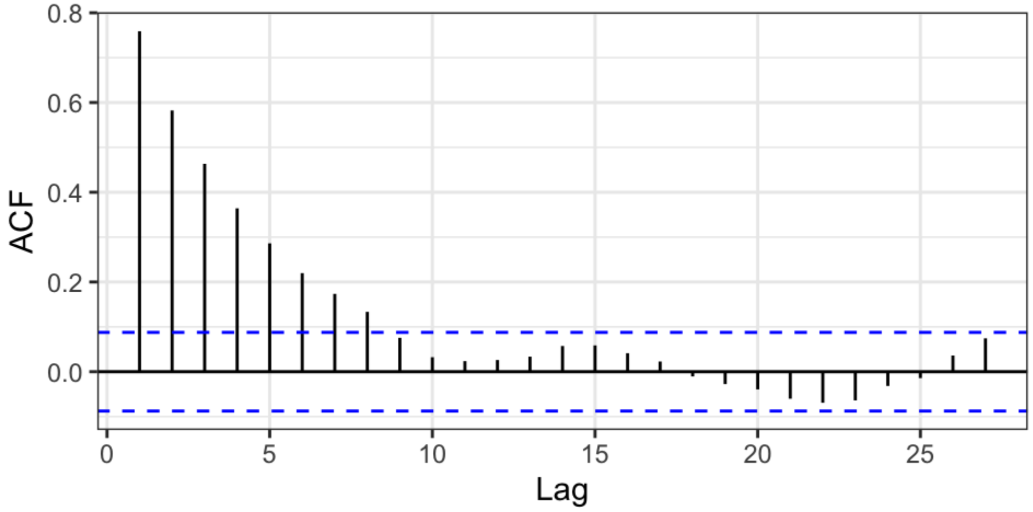
$$\log f(y_t \,|\, y_{t-1}) = -\frac{1}{2}\left(\log 2\pi + \log \sigma_w^2 + \frac{1}{\sigma_w^2}(y_t - \delta + \phi\, y_{t-1})^2\right)$$

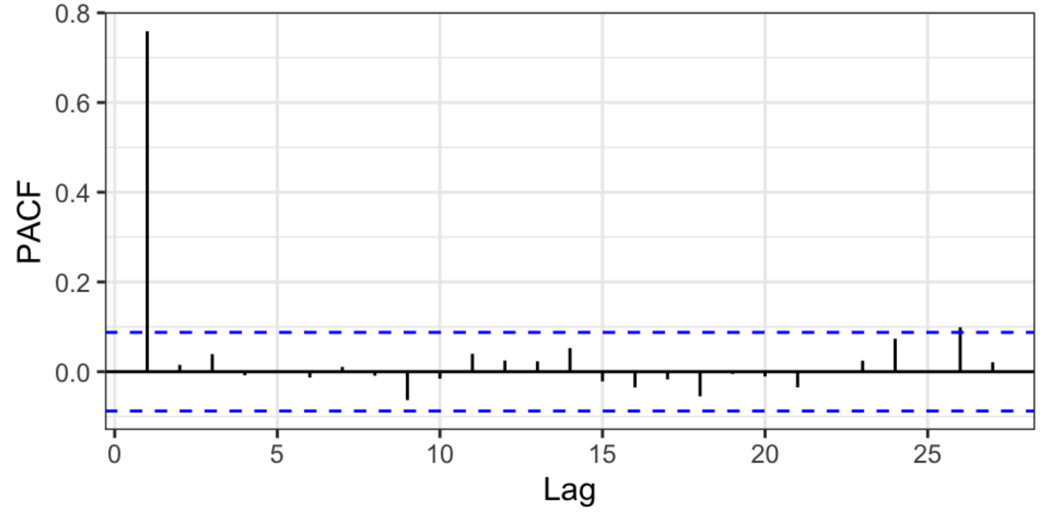
$$\begin{aligned}
\ell(\delta, \phi, \sigma_w^2) = \log f(\boldsymbol{y}) &= \log f(y_1) + \sum_{i=2}^{t} \log f(y_i \,|\, y_{i-1}) \\
&= -\frac{1}{2}\left(\log 2\pi + \log \sigma_w^2 - \log(1 - \phi^2) + \frac{(1-\phi^2)}{\sigma_w^2}(y_1 - \delta)^2 \right) \\
&\quad - \frac{1}{2}\left((n-1)\log 2\pi + (n-1)\log \sigma_w^2 + \frac{1}{\sigma_w^2} \sum_{i=2}^{n} (y_i - \delta + \phi\, y_{i-1})^2 \right) \\
&= -\frac{1}{2}\Bigg(n \log 2\pi + n \log \sigma_w^2 - \log(1-\phi^2) \\
&\qquad + \frac{1}{\sigma_w^2}\left((1-\phi^2)(y_1 - \delta)^2 + \sum_{i=2}^{n} (y_i - \delta + \phi\, y_{i-1})^2 \right) \Bigg)
\end{aligned}$$

AR(1) Example

with $\phi = 0.75$, $\delta = 0.5$, and $\sigma_w^2 = 1$,







ARIMA

```r
( ar1_arima = forecast::Arima(ar1, order = c(1,0,0)) )
```

```
Series: ar1
ARIMA(1,0,0) with non-zero mean

Coefficients:
         ar1    mean
      0.7601  2.2178
s.e.  0.0290  0.1890

sigma^2 = 1.045:  log likelihood = -719.84
AIC=1445.67   AICc=1445.72   BIC=1458.32
```

mean vs δ?

The reported mean value from the ARIMA model is $E(y_t)$ and not δ - for an ARIMA(1,0,0)

$$E(y_t) = \frac{\delta}{1-\phi} \quad \Rightarrow \quad \delta = E(y_t) * (1-\phi)$$

True $E(y_t)$:

```r
0.5 / (1-0.75)
```

```
[1] 2
```

Sample δ:

```r
ar1_arima$coef[2] *
  (1 - ar1_arima$model$phi)
```

```
intercept
0.5319962
```

lm

```
d = tsibble::as_tsibble(ar1) %>%
  as_tibble() %>%
  rename(y = value)
summary({ ar1_lm = lm(y~lag(y), data=d) })
```

```

Call:
lm(formula = y ~ lag(y), data = d)

Residuals:
    Min      1Q  Median      3Q     Max 
-2.7194 -0.6991 -0.0139  0.6323  3.3518 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  0.53138    0.07898   6.728 4.74e-11 ***
lag(y)       0.76141    0.02918  26.090  < 2e-16 ***
```

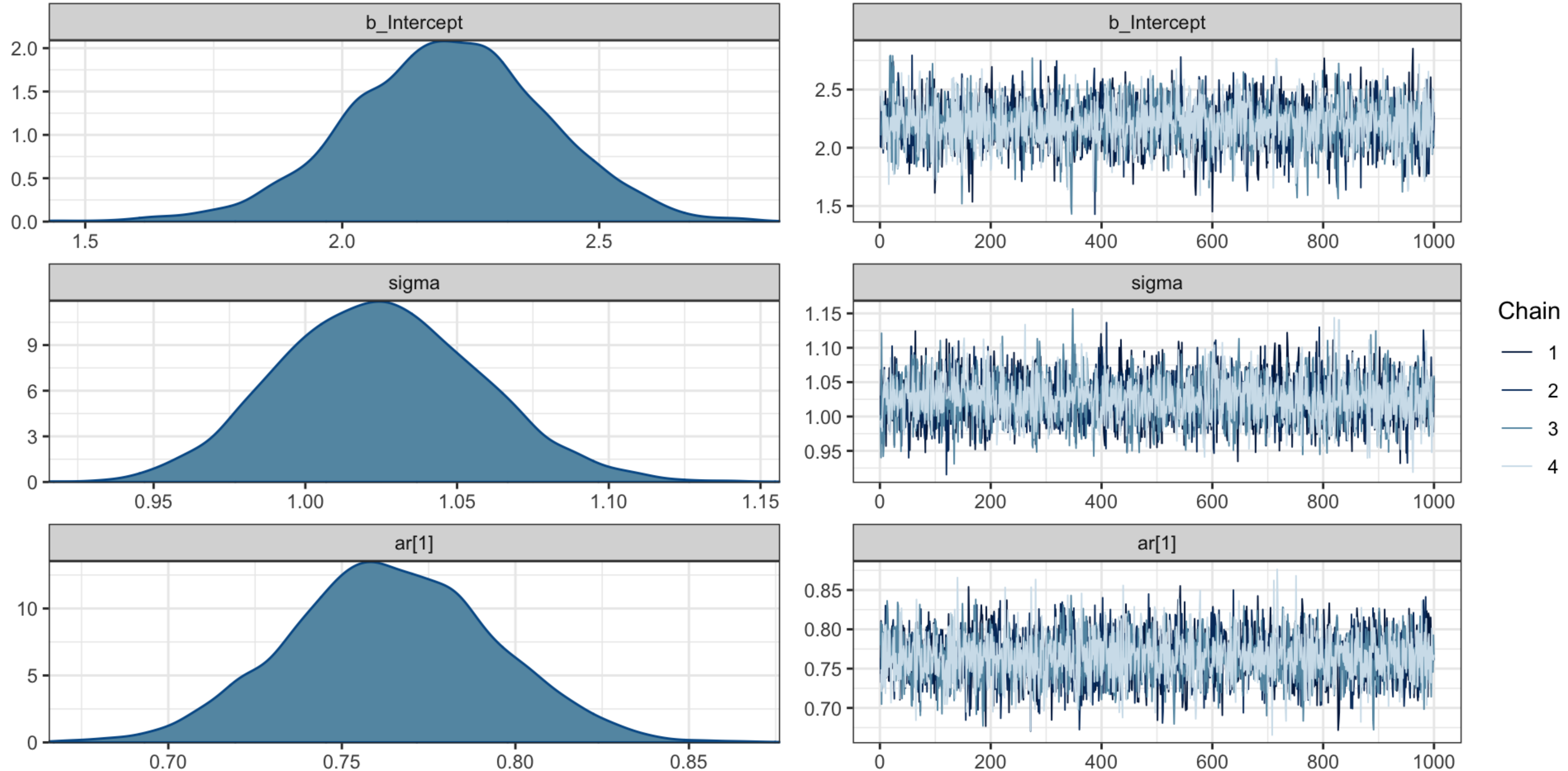
Bayesian AR(1) Model

```r
library(brms) # must be loaded for arma to work
( ar1_brms = brm(y ~ arma(p = 1, q = 0), data=d, refresh=0) )
```

```
 Family: gaussian
  Links: mu = identity; sigma = identity
Formula: y ~ arma(p = 1, q = 0)
   Data: d (Number of observations: 500)
  Draws: 4 chains, each with iter = 2000; warmup = 1000; thin = 1;
         total post-warmup draws = 4000

Correlation Structures:
      Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
ar[1]     0.76      0.03     0.71     0.82 1.00     3820     2893

Population-Level Effects:
          Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
Intercept     2.20      0.19     1.82     2.58 1.00     3728     2811

Family Specific Parameters:
```

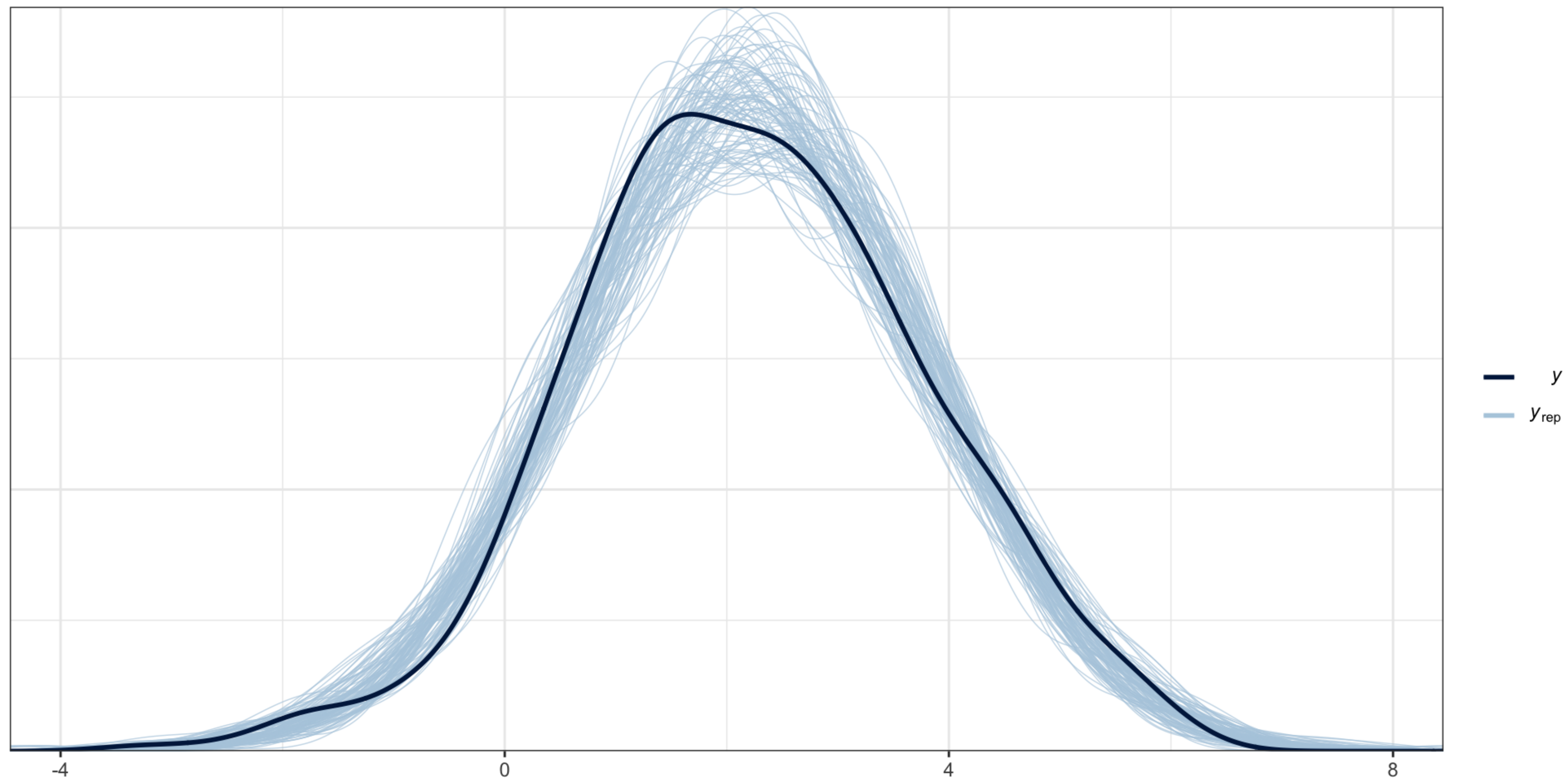
Chains

```
plot(ar1_brms)
```

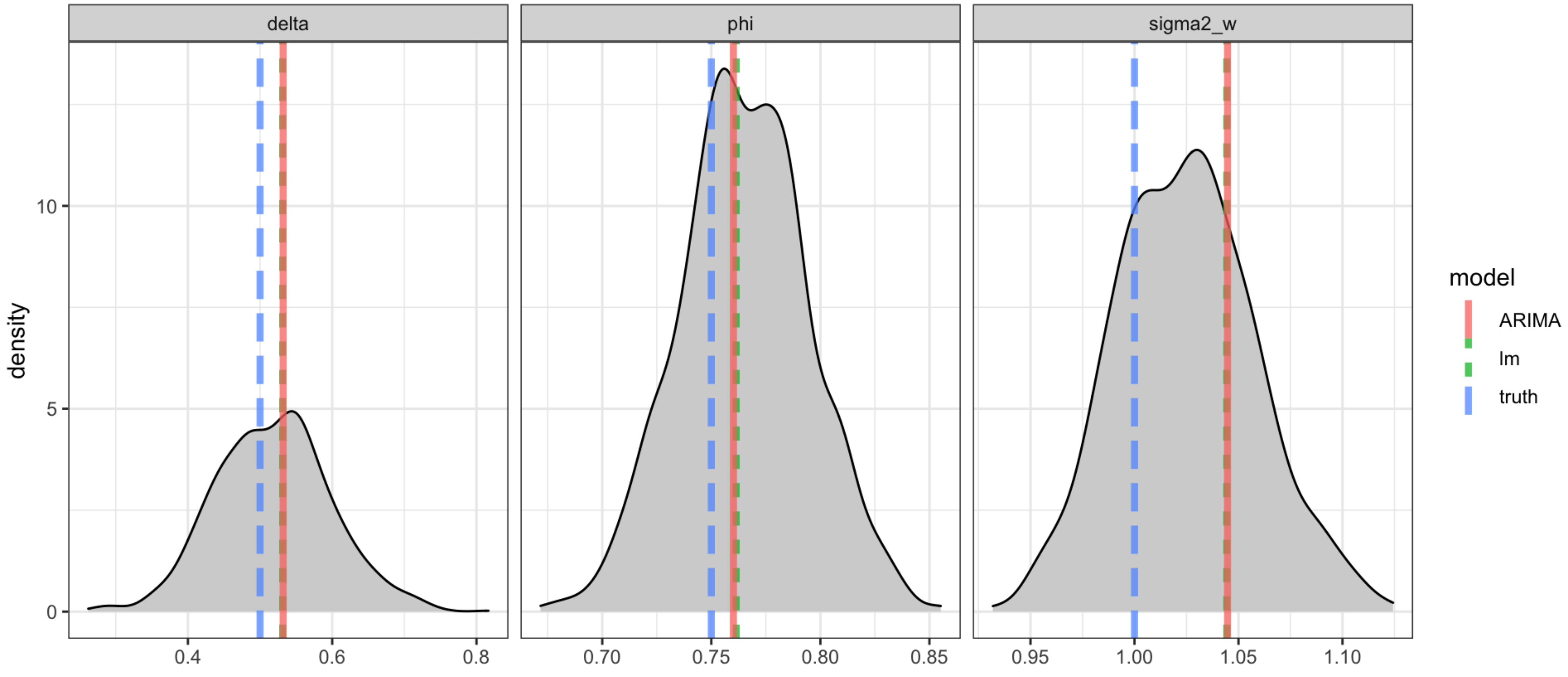


PP Checks

```r
pp_check(ar1_brms, ndraws=100)
```

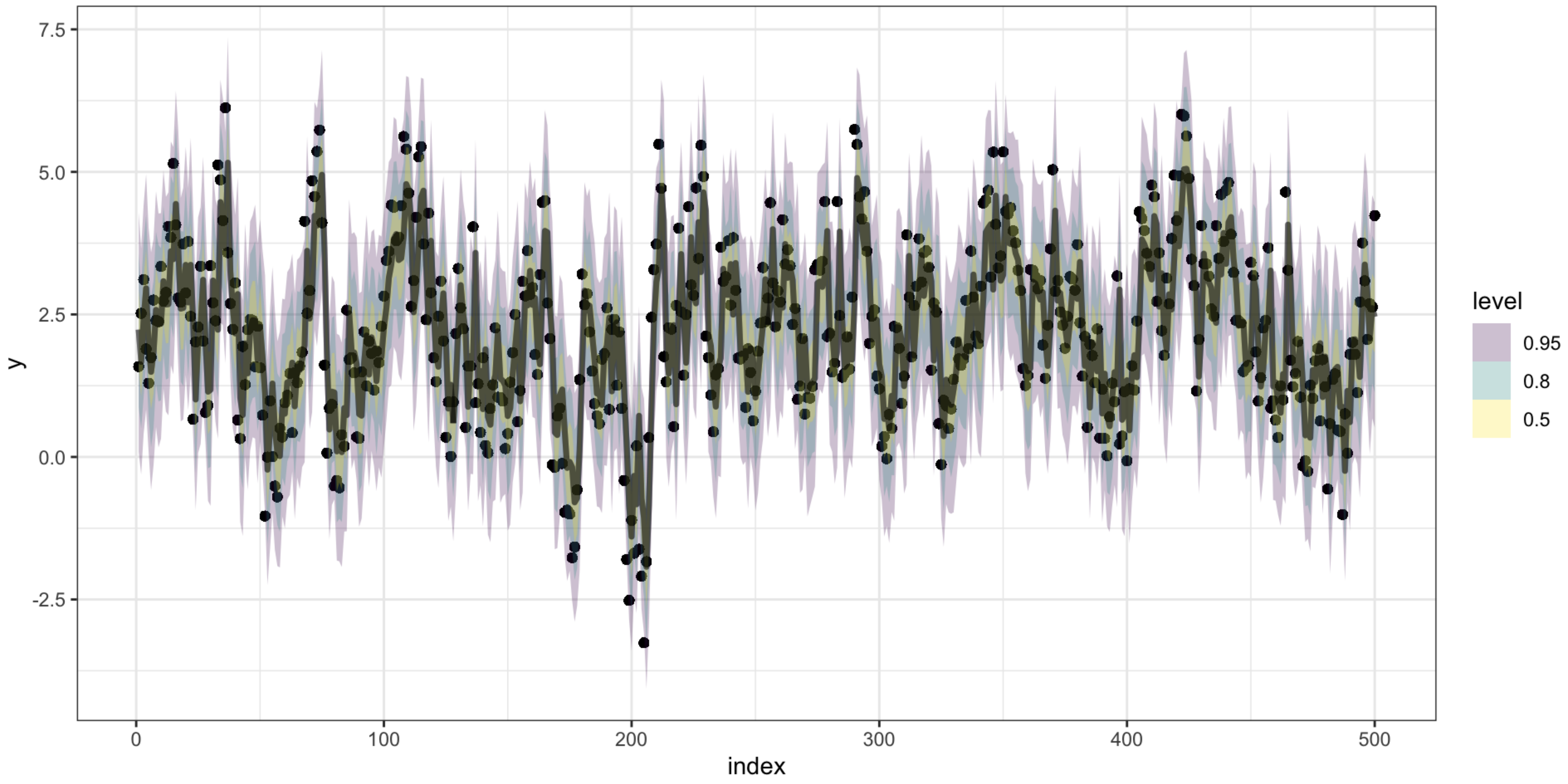


Posteriors



delta
phi
sigma2_w
density
10
5
0
0.4
0.6
0.8
0.70
0.75
0.80
0.85
0.95
1.00
1.05
1.10
model
ARIMA
lm
truth

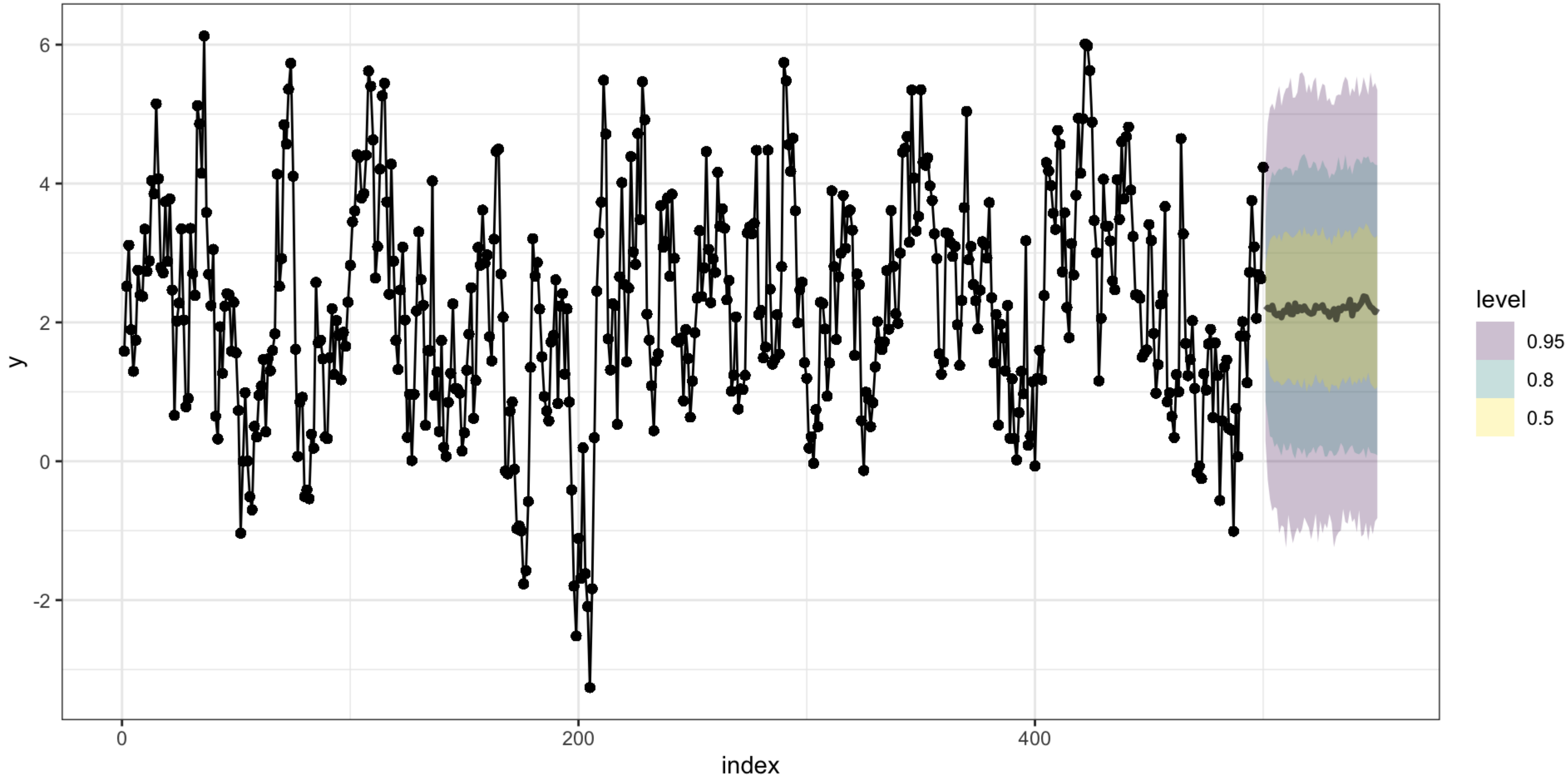
Predictions



Forecasting

```r
ar1_brms_fc = ar1_brms %>%
  predicted_draws_fix(
    newdata = tibble(index=501:550, y=NA)
  ) %>%
  filter(.chain == 1)
```

```r
ar1_brms %>%
  predicted_draws_fix(newdata = d) %>%
  filter(.chain == 1) %>%
  ggplot(aes(y=y, x=index)) +
    geom_point() +
    geom_line() +
    tidybayes::stat_lineribbon(
      data = ar1_brms_fc,
      aes(y=.prediction), alpha=0.25
    )
```



6
4
2
0
-2
y
0
200
400
index
level
0.95
0.8
0.5

Fitting AR(p)

Lagged Regression

As with the AR(1), we can rewrite the density using conditioning,

$$\begin{aligned} f(\boldsymbol{y}) &= f(y_t, y_{t-1}, \ldots, y_2, y_1) \\ &= f(y_n | y_{n-1}, \ldots, y_{n-p}) \cdots f(y_{p+1} | y_p, \ldots, y_1) f(y_p, \ldots, y_1) \end{aligned}$$

Regressing y_t on $y_{t-1}, \ldots, y_{t-p}$ gets us an approximate solution, but it ignores the $f(y_1, y_2, \ldots, y_p)$ part of the likelihood.

How much does this matter (vs. using the full likelihood)?

- If p is near to n then probably a lot
- If $p << n$ then probably not much

Method of Moments

Recall for an AR(p) process,

$$
\begin{aligned}
\gamma(0) &= \sigma_w^2 + \phi_1\gamma(1) + \phi_2\gamma(2) + \ldots + \phi_p\gamma(p) \\
\gamma(h) &= \phi_1\gamma(h-1) + \phi_2\gamma(h-2) + \ldots \phi_p\gamma(h-p)
\end{aligned}
$$

We can rewrite the first equation in terms of σ_w^2,

$$
\sigma_w^2 = \gamma(0) - \phi_1\gamma(1) - \phi_2\gamma(2) - \ldots - \phi_p\gamma(p)
$$

these are called the Yule-Walker equations.

Yule-Walker

These equations can be rewritten into matrix notation as follows

$$\underset{p\times p}{\boldsymbol{\Gamma}_p}\,\underset{p\times 1}{\boldsymbol{\phi}} = \underset{p\times 1}{\boldsymbol{\gamma}_p} \qquad\qquad \underset{1\times 1}{\sigma_w^2} = \underset{1\times 1}{\gamma(0)} - \underset{1\times p}{\boldsymbol{\phi}'}\,\underset{p\times 1}{\boldsymbol{\gamma}_p}$$

where

$$\begin{aligned}\underset{p\times p}{\boldsymbol{\Gamma}_p} &= \{\gamma(j-k)\}_{j,k} \\ \underset{p\times 1}{\boldsymbol{\phi}} &= (\phi_1, \phi_2, \ldots, \phi_p)' \\ \underset{p\times 1}{\boldsymbol{\gamma}_p} &= (\gamma(1), \gamma(2), \ldots, \gamma(p))'\end{aligned}$$

If we estimate the covariance structure from the data we obtain $\hat{\boldsymbol{\gamma}}_p$ and $\hat{\boldsymbol{\Gamma}}_p$ which we can plug in and solve for $\boldsymbol{\phi}$ and σ_w^2,

$$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\Gamma}}_p^{-1}\,\hat{\boldsymbol{\gamma}}_p \qquad\qquad \hat{\sigma}_w^2 = \gamma(0) - \hat{\boldsymbol{\gamma}}_p'\,\hat{\boldsymbol{\Gamma}}_p^{-1}\,\hat{\boldsymbol{\gamma}}_p$$

ARMA

Fitting $ARMA(2,2)$

$$y_t = \delta + \phi_1\, y_{t-1} + \phi_2\, y_{t-2} + \theta_1 w_{t-1} + \theta_2 w_{t-2} + w_t$$

We now need to estimate six parameters: δ, ϕ_1, ϕ_2, θ_1, θ_2 and σ_w^2.

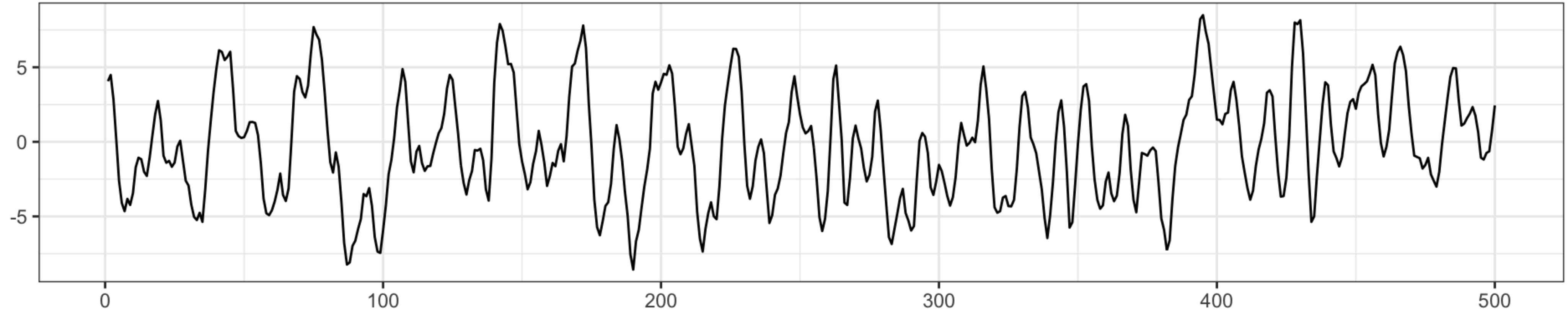
We could figure out $E(y_t)$, $Var(y_t)$, and $Cov(y_t, y_{t+h})$, but the last two are going to be pretty nasty and the full MVN likehood is similarly going to be unpleasant to work with.

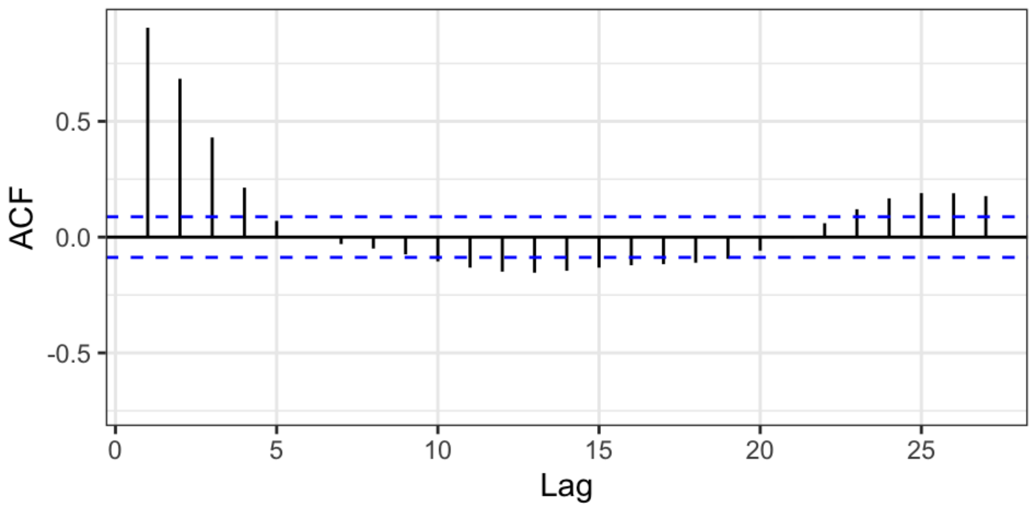
Like the AR(1) and AR(p) processes we want to use conditioning to simplify things.

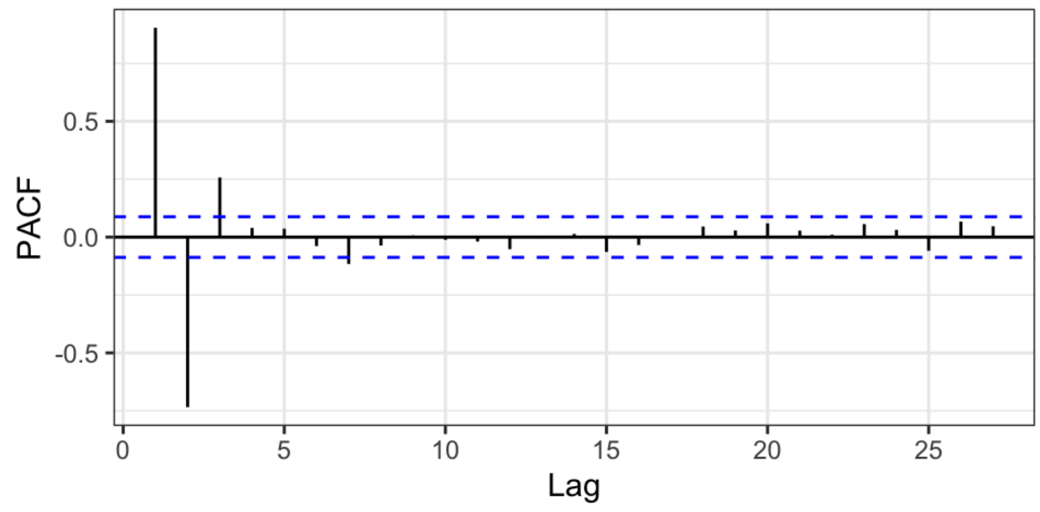
$$\begin{aligned} y_t \mid \delta, y_{t-1}, y_{t-2}, w_{t-1}, w_{t-2} \\ \sim \quad (\delta + \phi_1\, y_{t-1} + \phi_2\, y_{t-2} + \theta_1 w_{t-1} + \theta_2 w_{t-2},\ \sigma_w^2) \end{aligned}$$

ARMA(2,2) Example

with $\phi = (1.3, -0.5)$, $\theta = (0.5, 0.2)$, $\delta = 0$, and $\sigma_w^2 = 1$ using the same models







ARIMA

```r
forecast::Arima(y, order = c(2,0,2), include.mean = FALSE) %>% summary()
```

```
Series: y
ARIMA(2,0,2) with zero mean

Coefficients:
         ar1      ar2     ma1     ma2
      1.3702  -0.5634  0.4324  0.1274
s.e.  0.0652   0.0597  0.0740  0.0593

sigma^2 = 0.928:  log likelihood = -690.59
AIC=1391.18   AICc=1391.3   BIC=1412.25

Training set error measures:
```

AR only lm

```r
lm(y ~ lag(y,1) + lag(y,2)) %>% summary()
```

```

Call:
lm(formula = y ~ lag(y, 1) + lag(y, 2))

Residuals:
    Min      1Q  Median      3Q     Max 
-2.5507 -0.6394  0.0231  0.6421  2.8795 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -0.04934    0.04521  -1.091    0.276    
lag(y, 1)    1.58595    0.02972  53.360   <2e-16 ***
lag(y, 2)   -0.75056    0.02968 -25.288   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Hannan-Rissanen Algorithm

1. Estimate a high order AR (remember AR $\Leftrightarrow$ MA when stationary + invertible)
2. Use AR to estimate values for unobserved w_t via `lm` with `lag`s
3. Regress y_t onto $y_{t-1}, \ldots, y_{t-p}, \hat{w}_{t-1}, \ldots \hat{w}_{t-q}$
4. Update $\hat{w}_{t-1}, \ldots \hat{w}_{t-q}$ based on current model,
5. Goto 3, repeat until convergence

Hannan-Rissanen - Step 1 & 2

```r
(ar = ar.mle(y, order.max = 10))
```

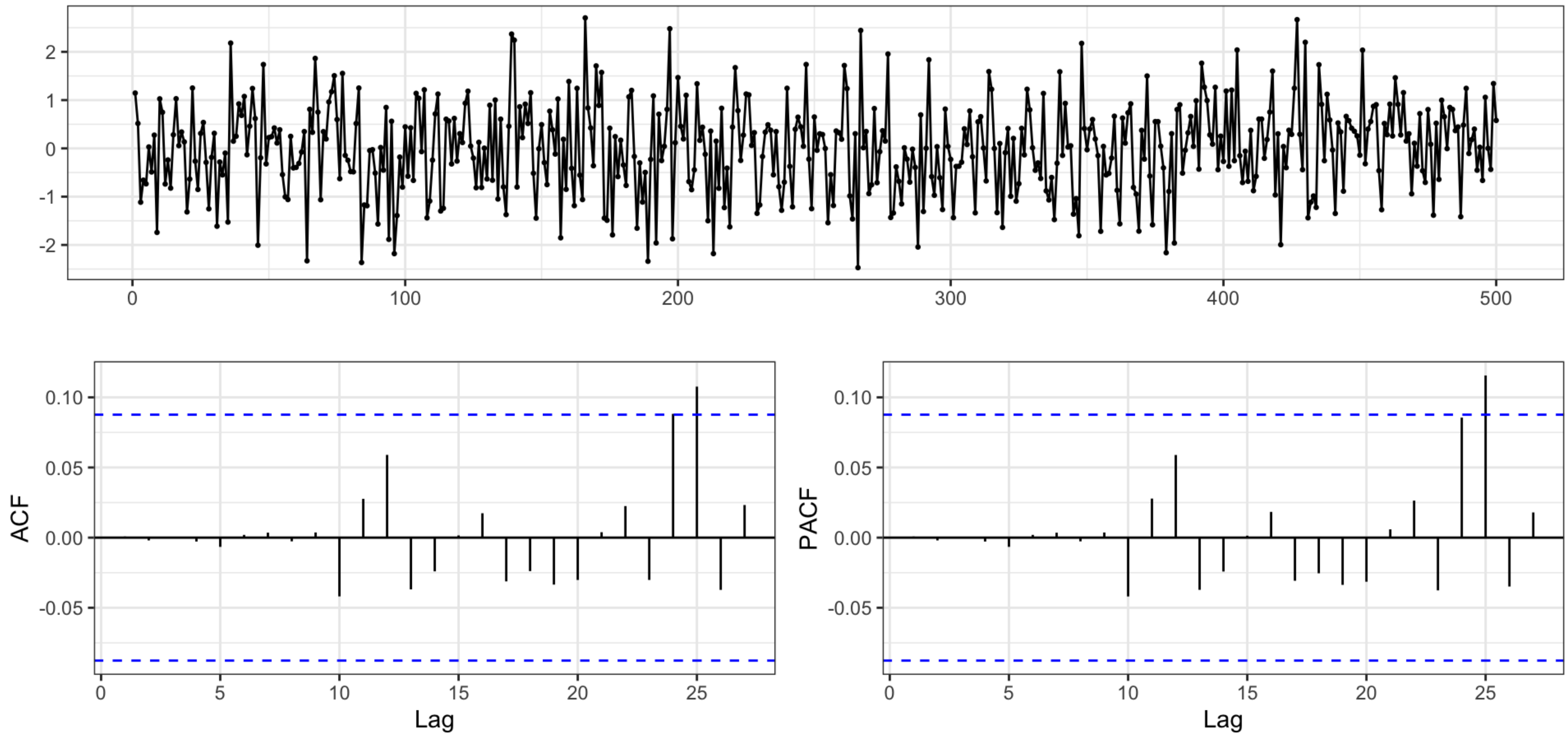
```
Call:
ar.mle(x = y, order.max = 10)

Coefficients:
      1        2        3        4        5  
 1.7977  -1.1882   0.2859  -0.0282  -0.0585  
      6        7  
 0.1857  -0.1212  

Order selected 7  sigma^2 estimated as  0.8989
```

```r
(ar = forecast::Arima(y, order = c(10,0,0)))
```

```
Series: y 
ARIMA(10,0,0) with non-zero mean 

Coefficients:
         ar1      ar2     ar3      ar4      ar5
      1.7940  -1.1824  0.2840  -0.0291  -0.0497
s.e.  0.0447   0.0919  0.1061   0.1069   0.1066
         ar6      ar7      ar8      ar9    ar10
      0.1492  -0.0657  -0.0298  -0.0028  0.0020
s.e.  0.1067   0.1072   0.1068   0.0925  0.0449
         mean
      -0.2661
s.e.   0.3251

sigma^2 = 0.9183:  log likelihood = -684.48
AIC=1392.97   AICc=1393.61   BIC=1443.54
```

Residuals

```
forecast::ggtsdisplay(ar$resid)
```



Hannan-Rissanen - Step 3

```
d = tibble(
  y = y %>% strip_attr(),
  index = seq_along(y),
  w_hat1 = ar$resid %>% strip_attr()
)

(lm1 = lm(y ~ lag(y,1) + lag(y,2) + lag(w_hat1,1) + lag(w_hat1,2), data=d)) %>%
  summary()
```

```
Call:
lm(formula = y ~ lag(y, 1) + lag(y, 2) + lag(w_hat1, 1) + lag(w_hat1,
    2), data = d)

Residuals:
     Min       1Q   Median       3Q      Max
-2.60703 -0.64222  0.05338  0.61697  2.65558

Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)    -0.05660    0.04374  -1.294   0.1962
```

Hannan-Rissanen - Step 4

```r
d = modelr::add_residuals(d,lm1,"w_hat2")

(lm2 = lm(y ~ lag(y,1) + lag(y,2) + lag(w_hat2,1) + lag(w_hat2,2), data=d)) %>%
  summary()
```

```

Call:
lm(formula = y ~ lag(y, 1) + lag(y, 2) + lag(w_hat2, 1) + lag(w_hat2,
    2), data = d)

Residuals:
     Min       1Q   Median       3Q      Max
-2.56613 -0.62523  0.04716  0.60059  2.75031

Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)    -0.05595    0.04367  -1.281   0.2007
lag(y, 1)       1.35622    0.05582  24.296  < 2e-16 ***
```

Hannan-Rissanen - Step 3.2 + 4.2

```r
d = modelr::add_residuals(d,lm2,"w_hat3")

(lm3 = lm(y ~ lag(y,1) + lag(y,2) + lag(w_hat3,1) + lag(w_hat3,2), data=d)) %>%
  summary()
```

```

Call:
lm(formula = y ~ lag(y, 1) + lag(y, 2) + lag(w_hat3, 1) + lag(w_hat3, 
    2), data = d)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.65130 -0.61871  0.05274  0.59498  2.79882 

Coefficients:
               Estimate Std. Error t value Pr(>|t|)    
(Intercept)    -0.05273    0.04369  -1.207    0.228    
lag(y, 1)       1.33721    0.05702  23.453  < 2e-16 ***
```

Hannan-Rissanen - Step 3.3 + 4.3

```r
d = modelr::add_residuals(d,lm3,"w_hat4")

(lm4 = lm(y ~ lag(y,1) + lag(y,2) + lag(w_hat4,1) + lag(w_hat4,2), data=d)) %>%
  summary()
```

```

Call:
lm(formula = y ~ lag(y, 1) + lag(y, 2) + lag(w_hat4, 1) + lag(w_hat4, 
    2), data = d)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.63962 -0.61993  0.05707  0.60544  2.76777 

Coefficients:
               Estimate Std. Error t value Pr(>|t|)    
(Intercept)    -0.05192    0.04385  -1.184   0.2370    
lag(y, 1)       1.33603    0.05786  23.089  < 2e-16 ***
```

Hannan-Rissanen - Step 3.4 + 4.4

```
d = modelr::add_residuals(d,lm4,"w_hat5")

(lm5 = lm(y ~ lag(y,1) + lag(y,2) + lag(w_hat5,1) + lag(w_hat5,2), data=d)) %>%
  summary()
```

```
Call:
lm(formula = y ~ lag(y, 1) + lag(y, 2) + lag(w_hat5, 1) + lag(w_hat5,
    2), data = d)

Residuals:
     Min       1Q   Median       3Q      Max
-2.64145 -0.61486  0.04973  0.60549  2.78622

Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)    -0.05057    0.04383  -1.154    0.249
lag(y, 1)       1.33475    0.05789  23.056  < 2e-16 ***
```

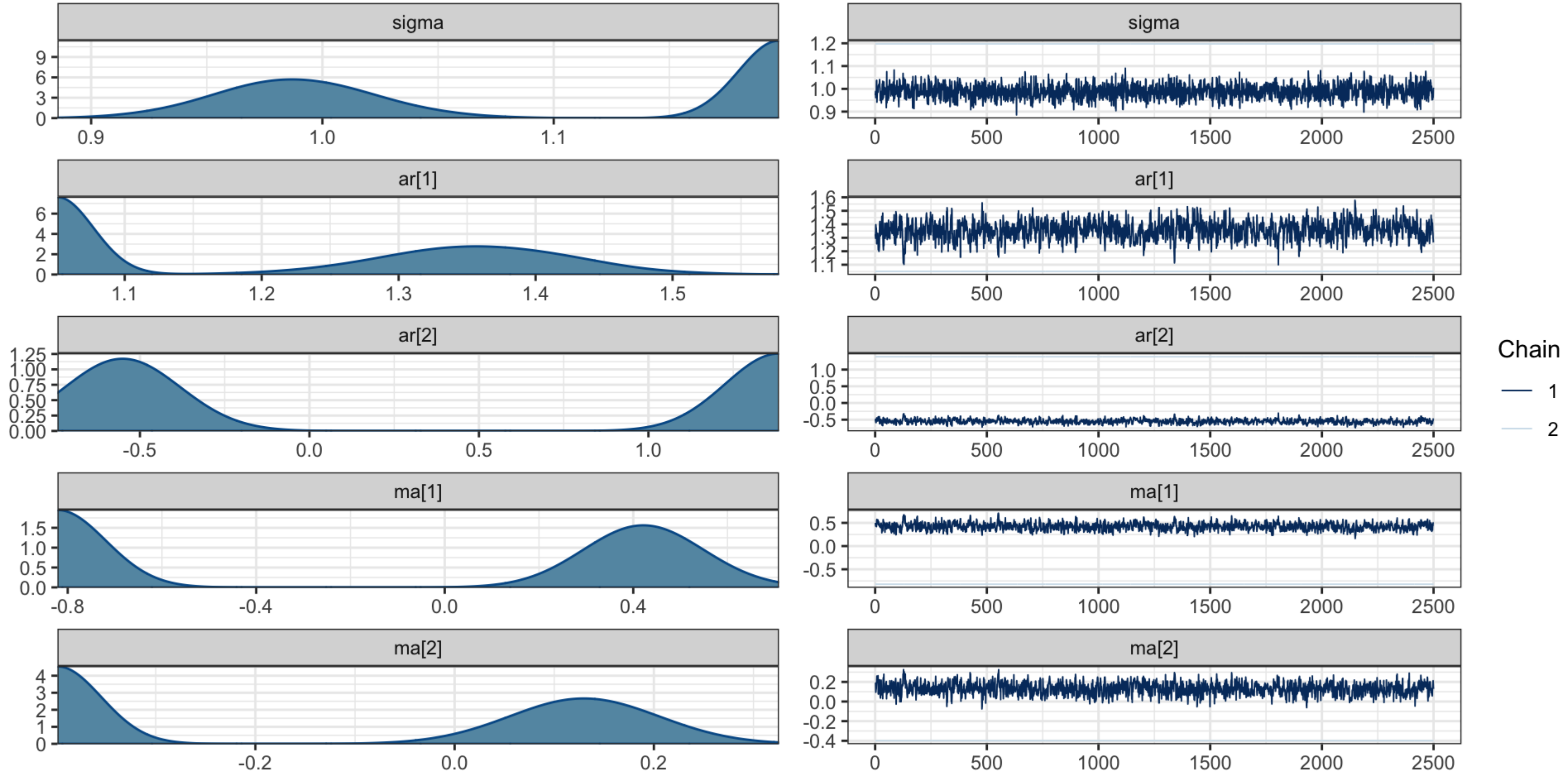
BRMS

```r
( arma22_brms = brm(y~arma(p=2,q=2)-1, data=d, chains=2, refresh=0, iter = 5000))
```

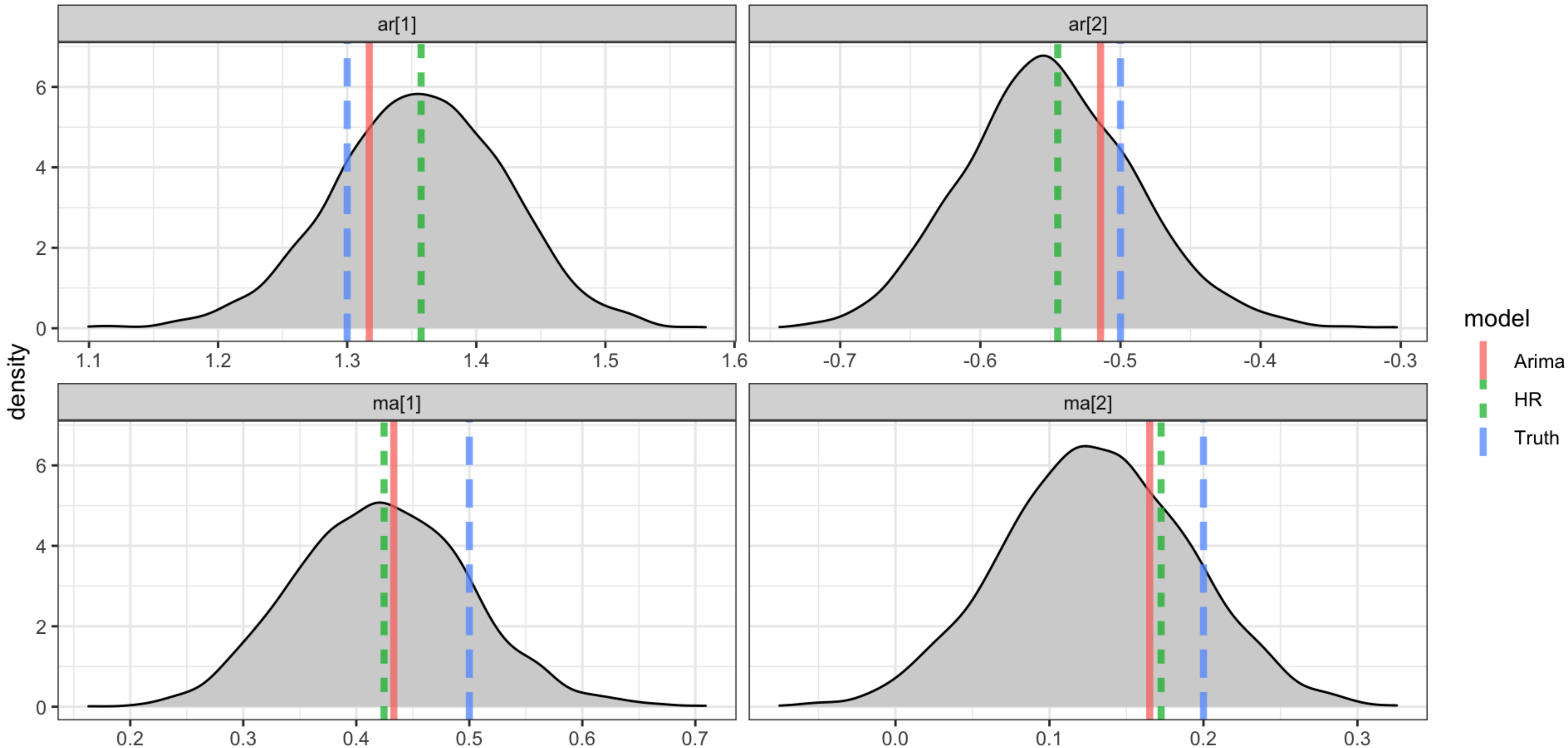
```
 Family: gaussian 
  Links: mu = identity; sigma = identity 
Formula: y ~ arma(p = 2, q = 2) - 1 
   Data: d (Number of observations: 500) 
  Draws: 2 chains, each with iter = 5000; warmup = 2500; thin = 1;
         total post-warmup draws = 5000

Correlation Structures:
      Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
ar[1]     1.20      0.16     1.05     1.46 2.24        3        2
ar[2]     0.42      0.97    -0.65     1.38 2.24        3       NA
ma[1]    -0.20      0.62    -0.82     0.55 2.24        3        2
ma[2]    -0.13      0.27    -0.40     0.23 2.24        3        2
```

Chains

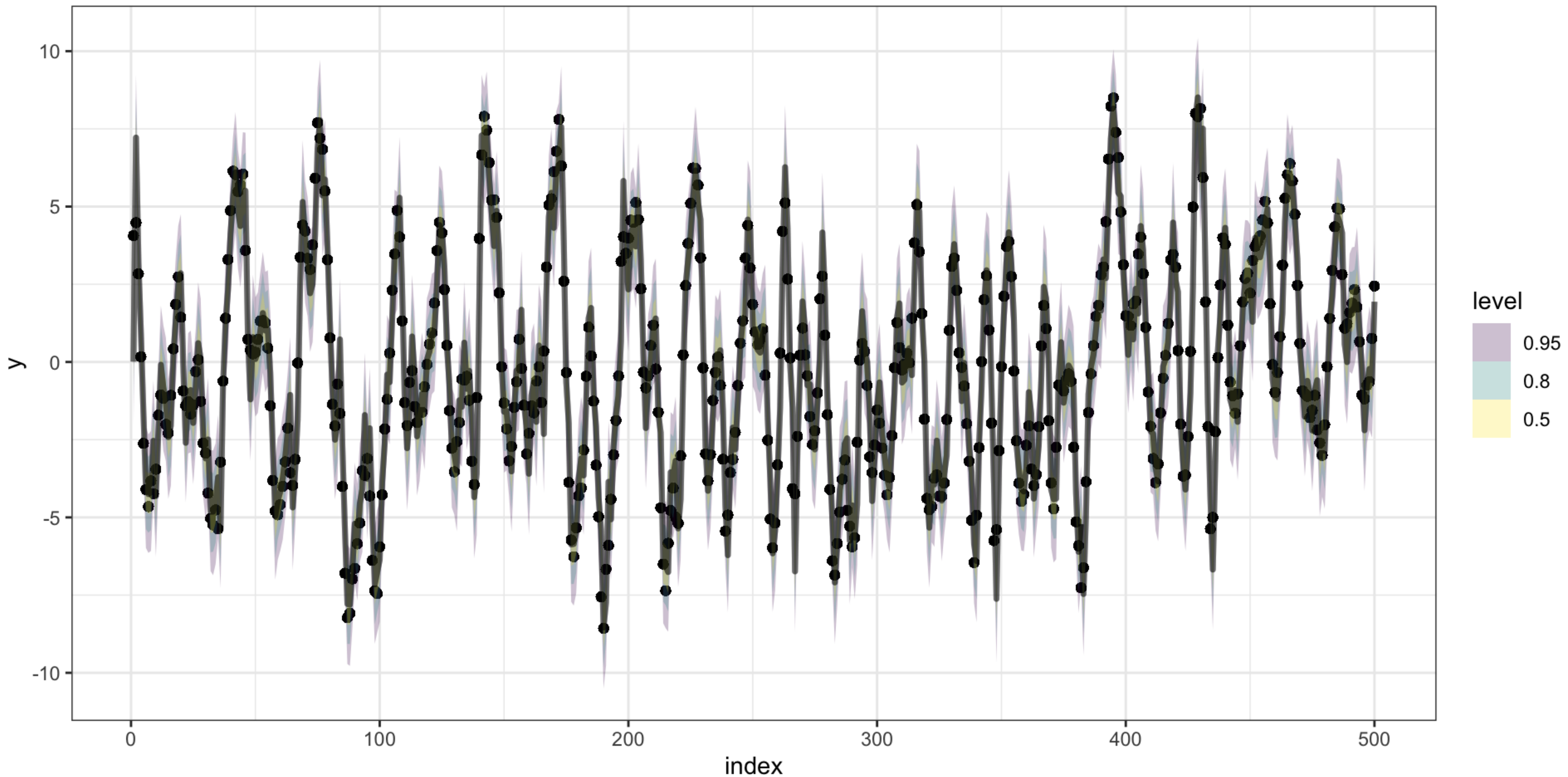
```
plot(arma22_brms)
```



Comparison


ar[1]
ar[2]
ma[1]
ma[2]
density
model
Arima
HR
Truth
0
2
4
6
1.1
1.2
1.3
1.4
1.5
1.6
-0.7
-0.6
-0.5
-0.4
-0.3
0.2
0.3
0.4
0.5
0.6
0.7
0.0
0.1
0.2
0.3

Predictions



Forecasting

:::{.small}

```
arma22_brms_fc = arma22_brms %>%
  predicted_draws_fix(
    newdata = tibble(index=501:550, y=NA)
  ) %>%
  filter(.chain == 1)
```

```
arma22_brms %>%
  predicted_draws_fix(newdata = d) %>%
  filter(.chain == 1) %>%
  ggplot(aes(y=y, x=index)) +
    geom_point() +
    geom_line() +
    tidybayes::stat_lineribbon(
      data = arma22_brms_fc,
      aes(y=.prediction), alpha=0.25
    )
```

Stan Code

```r
arma22_brms %>% stancode()
```

```
// generated with brms 2.18.0
functions {
}
data {
  int<lower=1> N;  // total number of observations
  vector[N] Y;  // response variable
  // data needed for ARMA correlations
  int<lower=0> Kar;  // AR order
  int<lower=0> Kma;  // MA order
  // number of lags per observation
  int<lower=0> J_lag[N];
  int prior_only;  // should the likelihood be ignored?
}
transformed data {
  int max_lag = max(Kar, Kma);
}
```