AR, MA, and ARMA Models

Lecture 09

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AR models

AR(1) models

From last time we derived the following properties for AR(1) models,

$$y_t = \delta + \phi \, y_{t-1} + w_t$$
$$w_t \overset{\text{iid}}{\sim} N(0, \sigma_w^2)$$

The process y_t is stationary iff $|\phi| < 1$, and if stationary then

AR(p) models

We can generalize from an AR(1) to an AR(p) model by simply adding additional autoregressive terms to the model.

$$
\begin{aligned}
AR(p): \quad y_t &= \delta + \phi_1\, y_{t-1} + \phi_2\, y_{t-2} + \cdots + \phi_p\, y_{t-p} + w_t \\
&= \delta + w_t + \sum_{i=1}^{p} \phi_i\, y_{t-i}
\end{aligned}
$$

What are the properities of $AR(p)$, specifically

1. Stationarity conditions?

2. Expected value?

3. Autocovariance / autocorrelation?

Lag operator

The lag operator is convenience notation for writing out AR (and other) time series models.

We define the lag operator L as follows,

$$L\,y_t = y_{t-1}$$

this can be generalized where,

$$\begin{aligned} L^2 y_t &= L\,(L\,y_t) \\ &= L\,y_{t-1} \\ &= y_{t-2} \end{aligned}$$

therefore,

$$L^k\,y_t = y_{t-k}$$

Lag polynomial

Lets rewrite the $AR(p)$ model using the lag operator,

$$
\begin{aligned}
y_t &= \delta + \phi_1\, y_{t-1} + \phi_2\, y_{t-2} + \ldots + \phi_p\, y_{t-p} + w_t \\
&= \delta + \phi_1\, L\, y_t + \phi_2\, L^2\, y_t + \ldots + \phi_p\, L^p\, y_t + w_t
\end{aligned}
$$

If we group all of the y_t terms, we get the following

$$
\begin{aligned}
\delta + w_t &= y_t - \phi_1\, L\, y_t - \phi_2\, L^2\, y_t - \ldots - \phi_p\, L^p\, y_t \\
&= (1 - \phi_1\, L - \phi_2\, L^2 - \ldots - \phi_p\, L^p)\, y_t
\end{aligned}
$$

This polynomial of lags

$$
\phi_p(L) = (1 - \phi_1\, L - \phi_2\, L^2 - \cdots - \phi_p\, L^p)
$$

Stationarity of $\text{AR}(p)$ processes

Claim: An $\text{AR}(p)$ process is stationary if the roots of the characteristic polynomial lay *outside* the complex unit circle

If we define $\lambda = 1/L$ then we can rewrite the characteristic polynomial as

$$(\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \cdots - \phi_{p-1} \lambda - \phi_p)$$

then as a corollary of our claim the $\text{AR}(p)$ process is stationary if the roots of this new polynomial are *inside* the complex unit circle, i.e. $|\lambda| < 1$.

Example AR(1)

Example AR(2)

AR(2) Stationarity Conditions

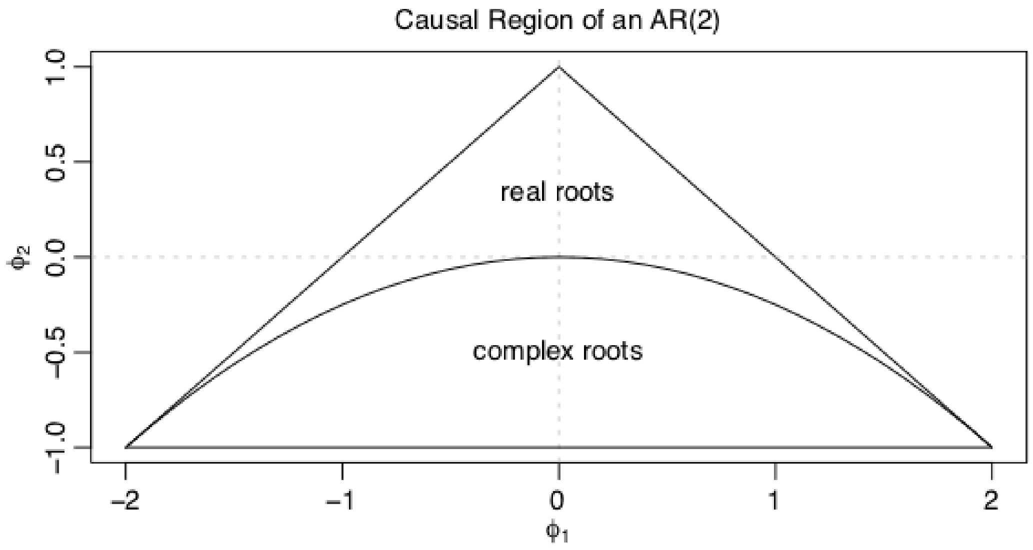


Fig. 3.3. *Causal region for an AR(2) in terms of the parameters.*

From Shumway & Stofer 4th ed.

Proof Sketch

We can rewrite the $\text{AR}(p)$ model into an $\text{AR}(1)$ form using matrix notation

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t$$
$$\boldsymbol{\xi}_t = \boldsymbol{\delta} + \boldsymbol{F}\,\boldsymbol{\xi}_{t-1} + \boldsymbol{w}_t$$

where

$$\underset{p\times 1}{\boldsymbol{\xi}_t} = [y_t, y_{t-1}, y_{t-2}, \ldots, y_{t-p+1}]'$$
$$\underset{p\times 1}{\boldsymbol{\delta}} = [\delta, 0, 0, \ldots, 0]'$$
$$\underset{p\times 1}{\boldsymbol{w}_t} = [w_t, 0, 0, \ldots, 0]'$$

$$\underset{p\times p}{\boldsymbol{F}} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Putting it together

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \delta \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \delta + w_t + \sum_{i=1}^{p} \phi_i \, y_{t-i} \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix}$$

Proof sketch (cont.)

So just like the original $\text{AR}(1)$ we can expand out the autoregressive equation

$$
\begin{aligned}
\boldsymbol{\xi}_t &= \boldsymbol{\delta} + \boldsymbol{w}_t + \boldsymbol{F}\,\boldsymbol{\xi}_{t-1} \\
&= \boldsymbol{\delta} + \boldsymbol{w}_t + \boldsymbol{F}\,(\boldsymbol{\delta} + \boldsymbol{w}_{t-1}) + \boldsymbol{F}^2\,(\boldsymbol{\delta} + \boldsymbol{w}_{t-2}) + \cdots \\
&\qquad\qquad + \boldsymbol{F}^{t-1}\,(\boldsymbol{\delta} + \boldsymbol{w}_1) + \boldsymbol{F}^t\,(\boldsymbol{\delta} + \boldsymbol{w}_0) \\
&= \left(\sum_{i=0}^{t} F^i\right)\boldsymbol{\delta} + \sum_{i=0}^{t} F^i\, w_{t-i}
\end{aligned}
$$

and therefore we need $\lim_{t\to\infty} F^t \to 0$ so that $\lim_{t\to\infty} \sum_{i=0}^{t} F^i < \infty$.

Proof sketch (cont.)

We can find the eigen decomposition such that $\boldsymbol{F} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^{-1}$ where the columns of $\boldsymbol{Q}$ are the eigenvectors of $\boldsymbol{F}$ and $\boldsymbol{\Lambda}$ is a diagonal matrix of the corresponding eigenvalues.

A useful property of the eigen decomposition is that

$$\boldsymbol{F}^{\mathrm{i}} = \boldsymbol{Q}\boldsymbol{\Lambda}^{\mathrm{i}}\boldsymbol{Q}^{-1}$$

Using this property we can rewrite our equation from the previous slide as

$$\begin{aligned}
\boldsymbol{\xi}_{\mathrm{t}} &= \Big(\sum_{\mathrm{i}=0}^{\mathrm{t}} \mathrm{F}^{\mathrm{i}}\Big)\boldsymbol{\delta} + \sum_{\mathrm{i}=0}^{\mathrm{t}} \mathrm{F}^{\mathrm{i}}\, \mathrm{w}_{\mathrm{t-i}} \\
&= \left(\sum_{\mathrm{i}=0}^{\mathrm{t}} \boldsymbol{Q}\boldsymbol{\Lambda}^{\mathrm{i}}\boldsymbol{Q}^{-1}\right)\boldsymbol{\delta} + \sum_{\mathrm{i}=0}^{\mathrm{t}} \boldsymbol{Q}\boldsymbol{\Lambda}^{\mathrm{i}}\boldsymbol{Q}^{-1}\, \mathrm{w}_{\mathrm{t-i}}
\end{aligned}$$

Proof sketch (cont.)

$$\boldsymbol{\Lambda}^i = \begin{bmatrix} \lambda_1^i & 0 & \cdots & 0 \\ 0 & \lambda_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p^i \end{bmatrix}$$

Therefore, $\lim_{t\to\infty} F^t \to 0$ when $\lim_{t\to\infty} \Lambda^t \to 0$ which requires that

$$|\lambda_i| < 1 \quad \text{for all } i$$

Proof sketch (cont.)

Eigenvalues are defined such that for $\boldsymbol{\lambda}$,

$$\det(\boldsymbol{F} - \boldsymbol{\lambda}\,\boldsymbol{I}) = 0$$

based on our definition of $\boldsymbol{F}$ our eigenvalues will therefore be the roots of

$$\lambda^p - \phi_1\,\lambda^{p-1} - \phi_2\,\lambda^{p-2} - \cdots - \phi_{p_1}\,\lambda^1 - \phi_p = 0$$

which if we multiply by $1/\lambda^p$ where $L = 1/\lambda$ gives

$$1 - \phi_1\,L - \phi_2\,L^2 - \cdots - \phi_{p_1}\,L^{p-1} - \phi_p\,L^p = 0$$

Properties of $\text{AR}(2)$

For a *stationary* $\text{AR}(2)$ process,

Properties of $\text{AR}(2)$ (cont.)

Properties of AR(p)

For a *stationary* AR(p) process,

$$E(Y_t) = \frac{\delta}{1 - \phi_1 - \phi_2 - \ldots - \phi_p}$$

$$Var(y_t) = \gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \ldots + \phi_p\gamma(p) + \sigma_w^2$$

$$Cov(y_t, y_{t+h}) = \gamma(h) = \phi_1\gamma(h-1) + \phi_2\gamma(h-2) + \ldots + \phi_p\gamma(h-p)$$

$$Corr(y_t, y_{t+h}) = \rho(h) = \phi_1\, \rho(h-1) + \phi_2\, \rho(h-2) + \ldots + \phi_p\, \rho(h-p)$$

Moving Average (MA) Processes

MA(1)

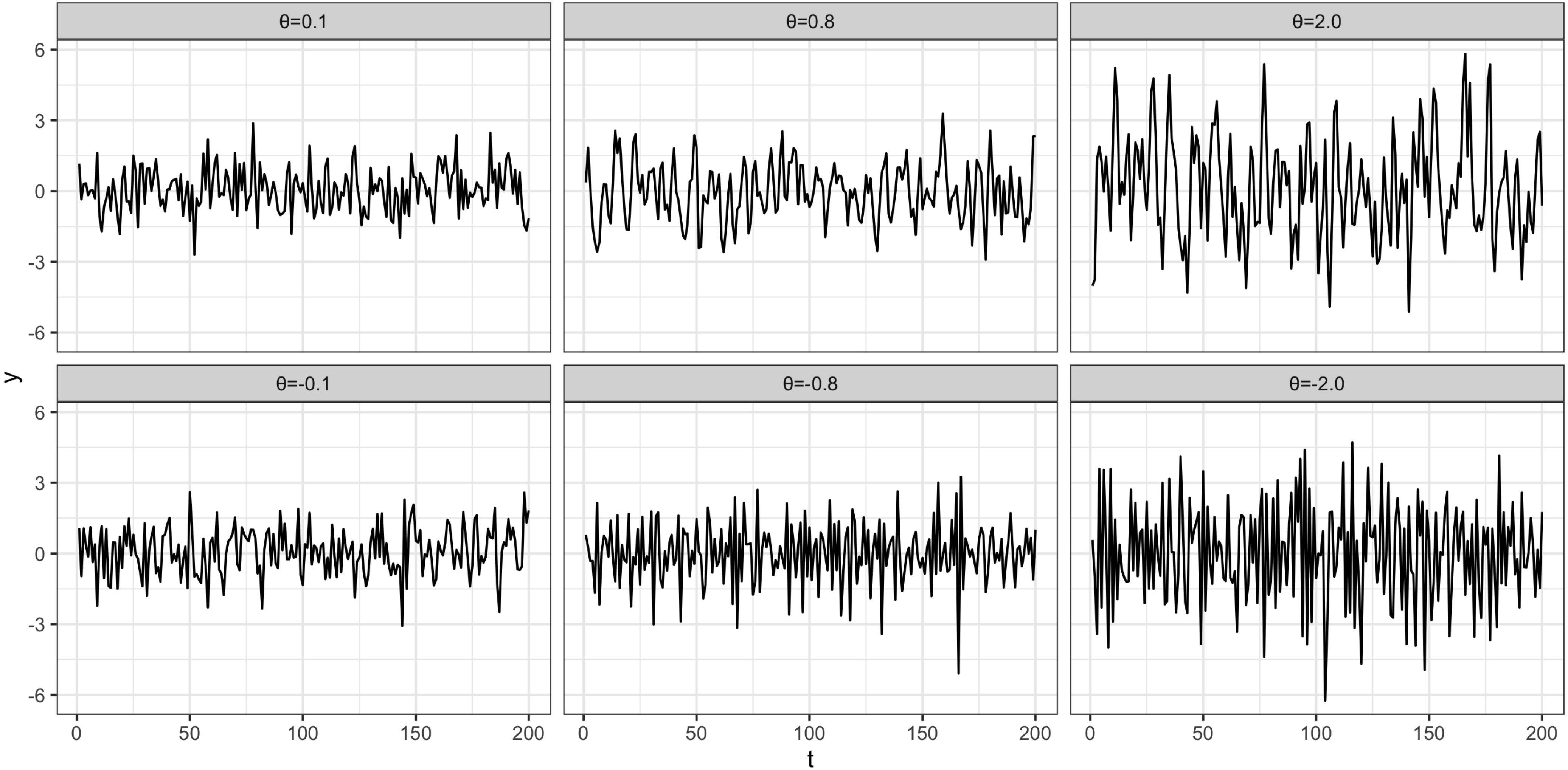
A moving average process is similar to an AR process, except that the autoregression is on the error term.

$$\text{MA}(1): \qquad y_t = \delta + w_t + \theta\, w_{t-1}$$

Properties:

MA(1) - properties (cont.)

Time series



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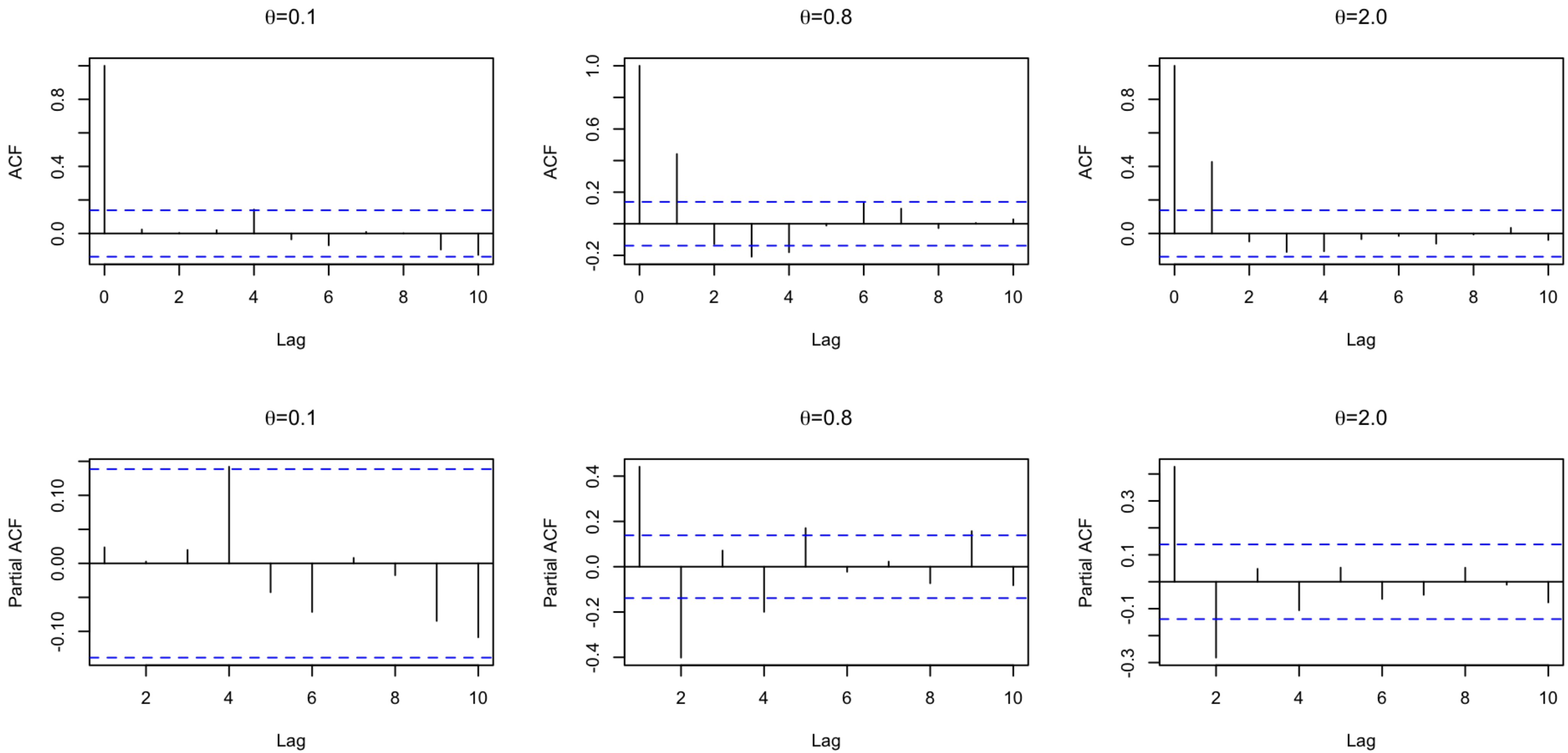


MA(q)

$$\text{MA(q)}: \qquad y_t = \delta + w_t + \theta_1\, w_{t-1} + \theta_2\, w_{t-2} + \cdots + \theta_q\, w_{t-q}$$

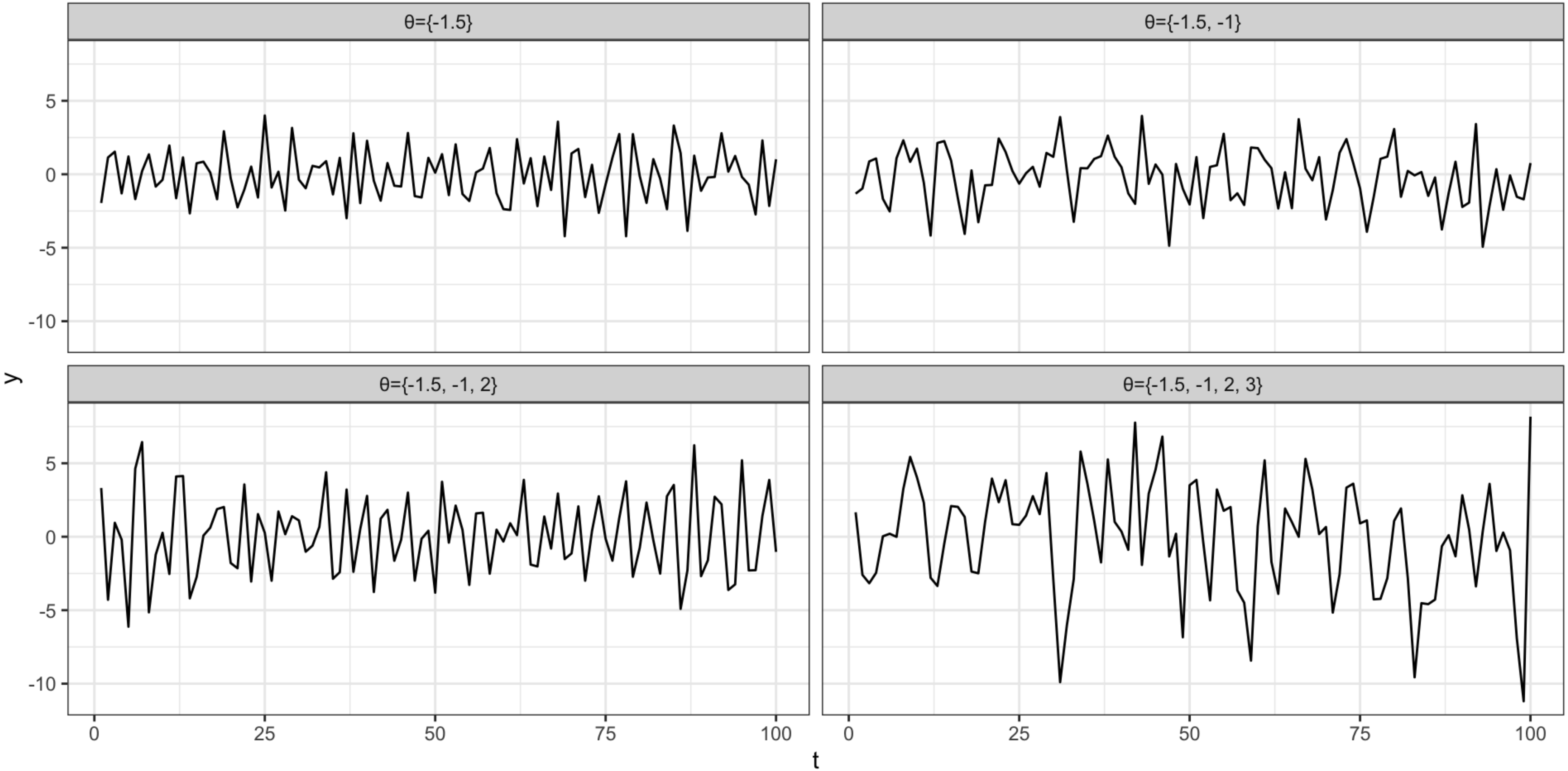
Properties:

$$E(y_t) = \delta$$

$$Var(y_t) = \gamma(0) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\, \sigma_w^2$$

$$Cov(y_t, y_{t+h}) = \gamma(h) = \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|} & \text{if } |h| \le q \\ 0 & \text{if } |h| > q \end{cases}$$

Example series



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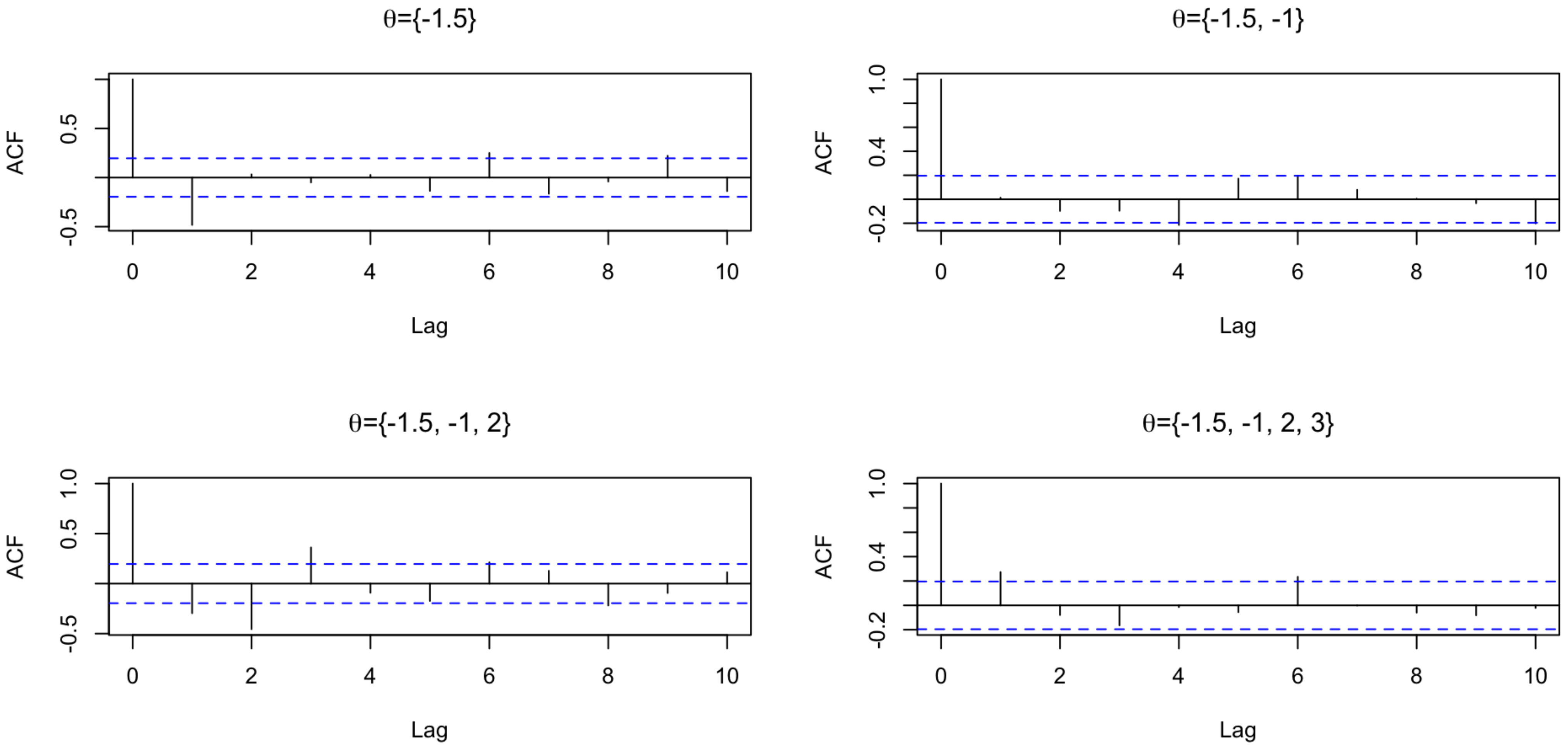

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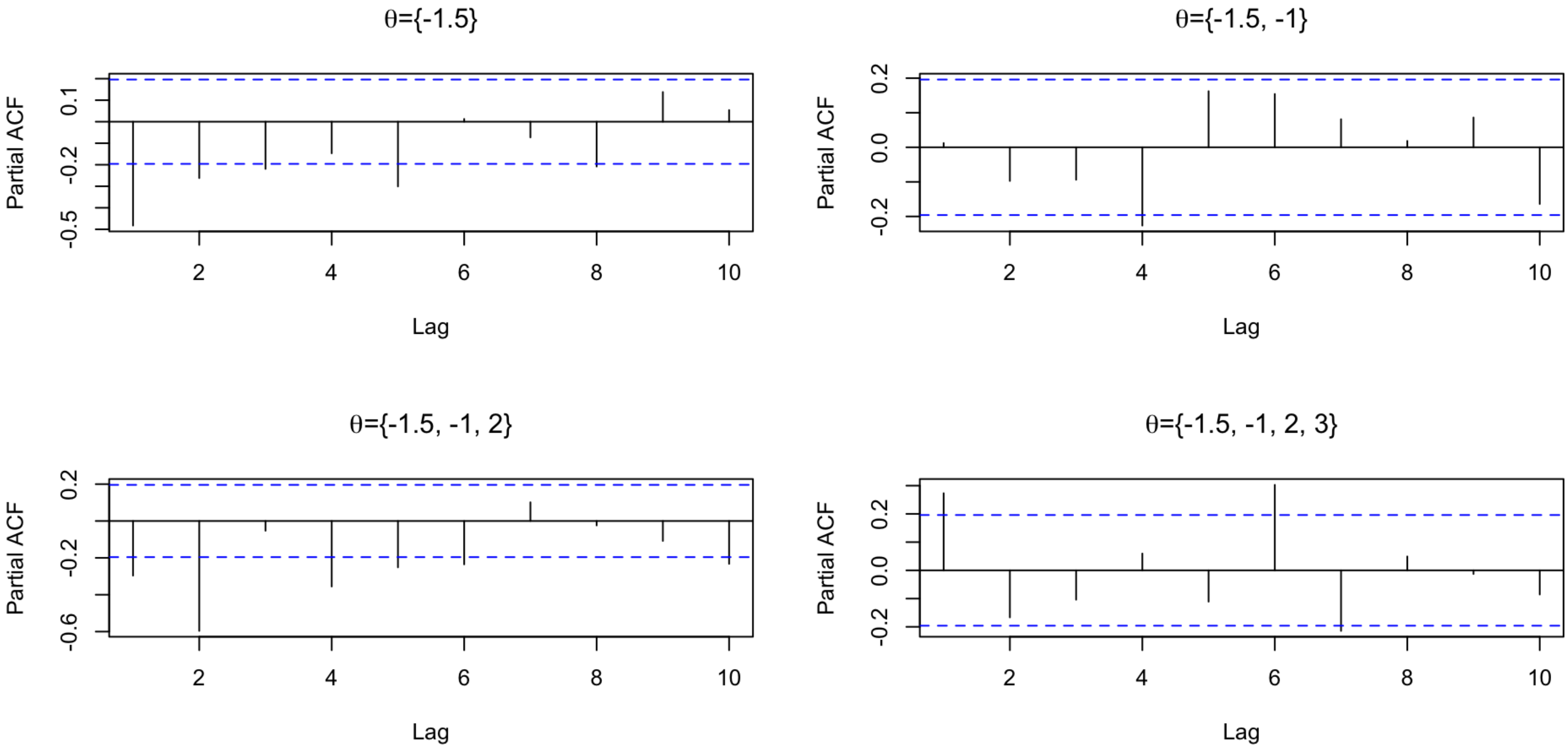


ARMA Model

ARMA Model

An ARMA model is a composite of AR and MA processes,

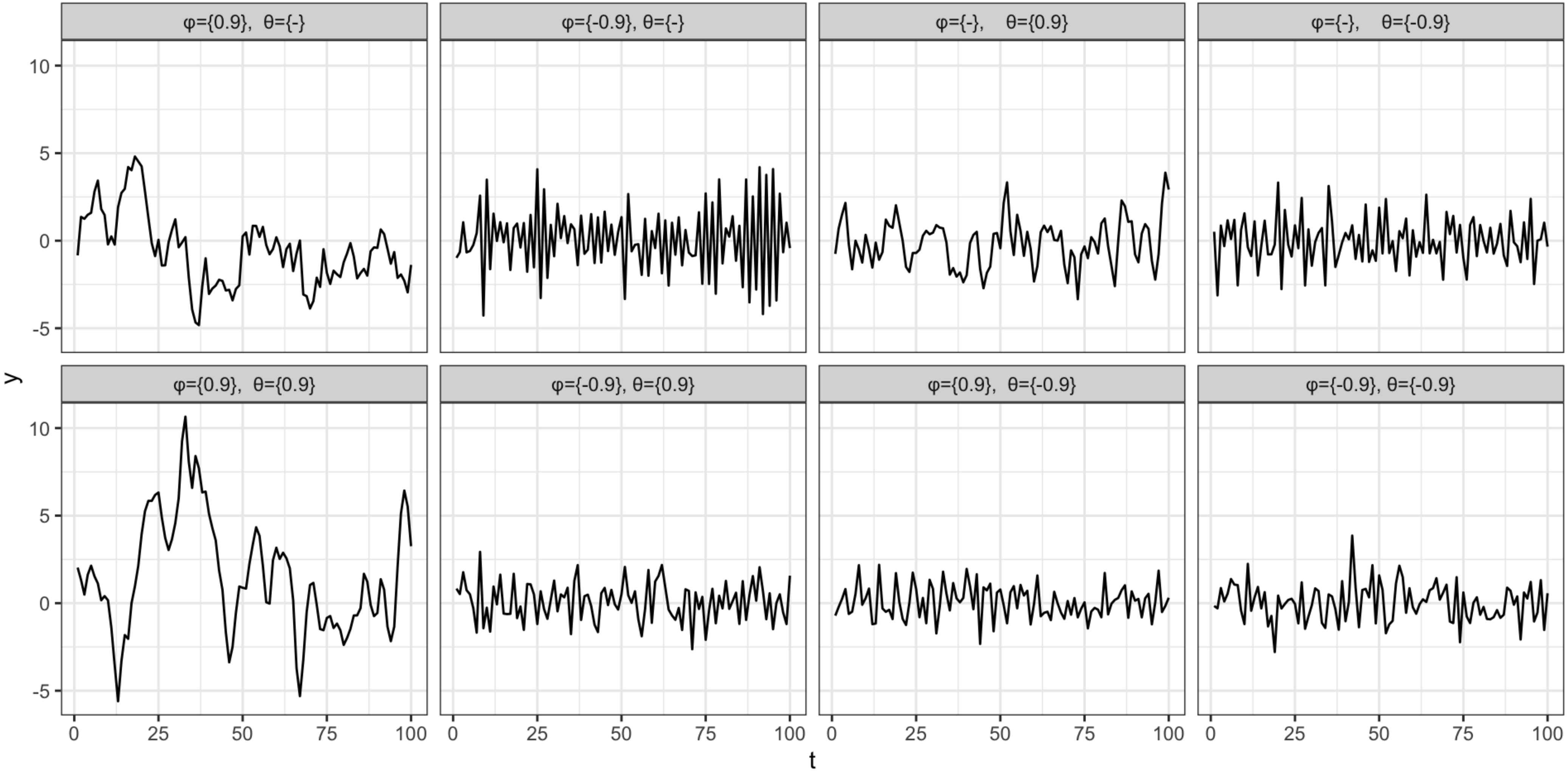
$\text{ARMA}(p, q)$:

$$y_t = \delta + \phi_1 \, y_{t-1} + \cdots \phi_p \, y_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t_q}$$

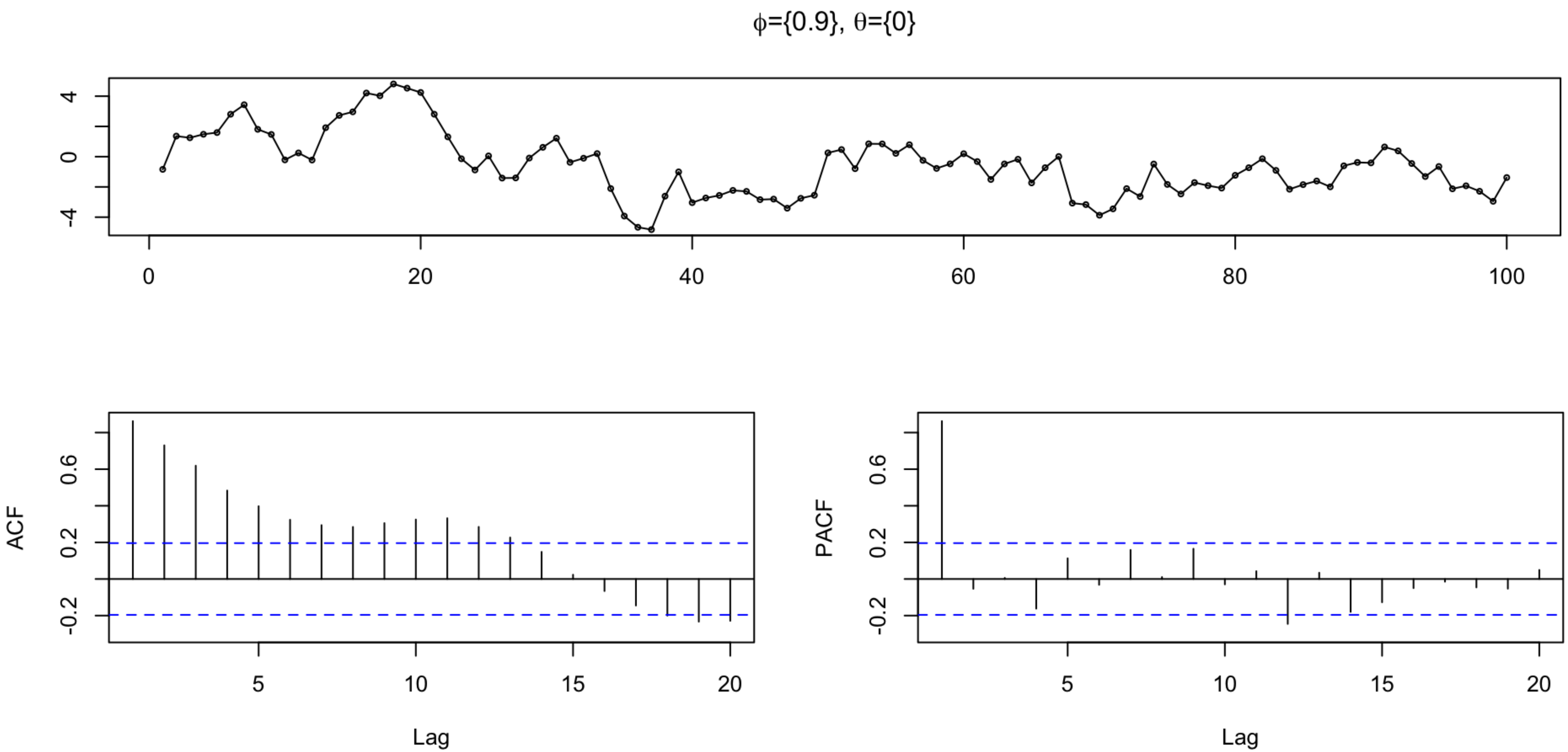
$$\phi_p(L) y_t = \delta + \theta_q(L) w_t$$

Since all MA processes are stationary, we only need to examine the AR component to determine stationarity, i.e. check roots of $\phi_p(L)$ lie outside the complex unit circle.

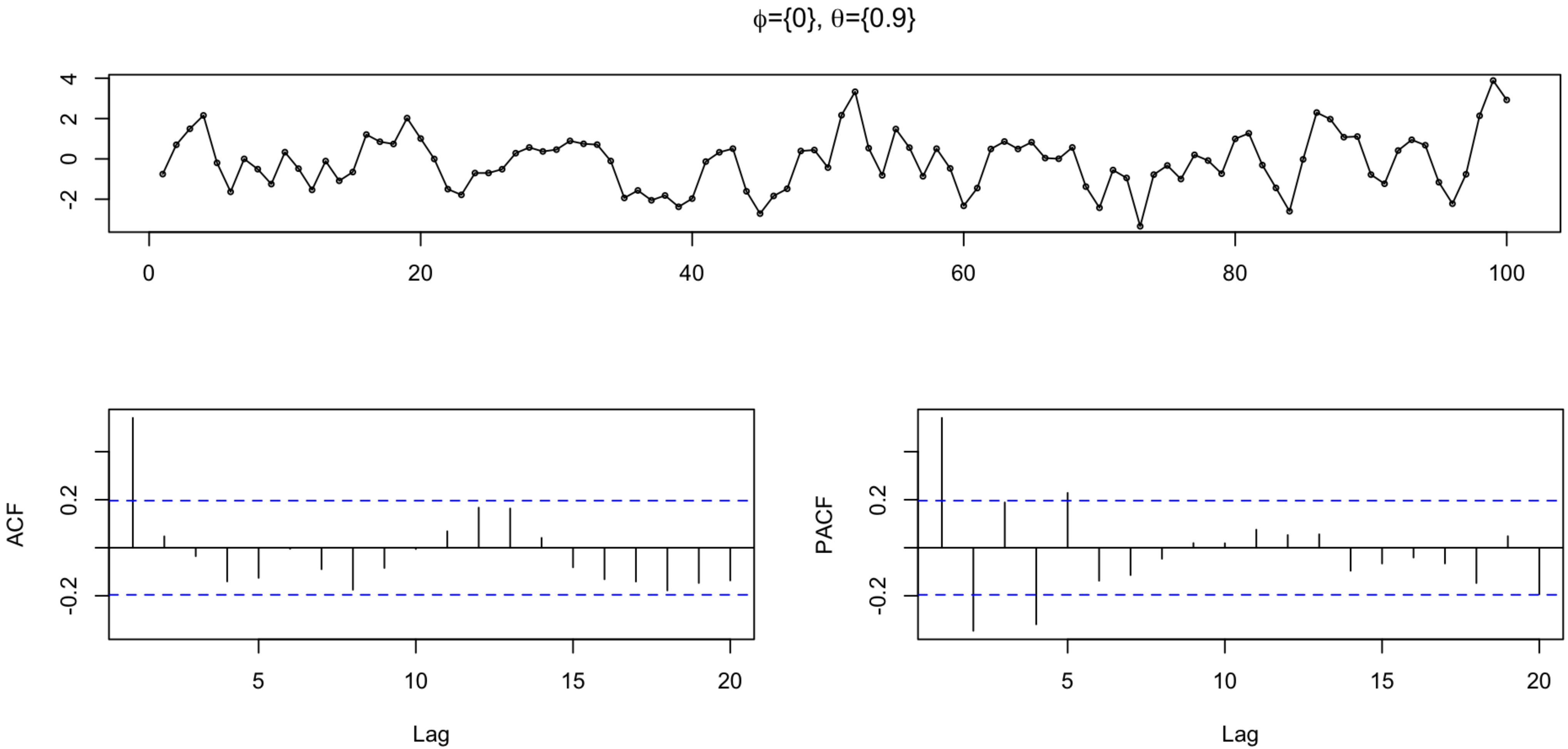
Time series


φ={0.9}, θ={-}
φ={-0.9}, θ={-}
φ={-}, θ={0.9}
φ={-}, θ={-0.9}
φ={0.9}, θ={0.9}
φ={-0.9}, θ={0.9}
φ={0.9}, θ={-0.9}
φ={-0.9}, θ={-0.9}
y
t
10
5
0
-5
0
25
50
75
100

$\phi = 0.9, \theta = 0$



$\phi = 0, \theta = 0.9$



$\phi = 0.9, \theta = 0.9$