Lecture 24: Gaussian quadrature wrap-up

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Gaussian quadrature so far

Gauss-Legendre quadrature: $\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$

- $w_i = \int_{-1}^{1} L_{n,i}(x)dx$
- $x_1, ..., x_n$ are the roots of the nth **Legendre** polynomial p_n. Legendre polynomials satisfy
  $\int_{-1}^{1} (c_0 + c_1 x + \cdots + c_{n-1}x^{n-1})p_n(x)dx$

Gauss-Hermite quadrature: $\int_{-\infty}^{\infty} f(x)e^{-\frac{1}{2}x^2}dx \approx \sum_{i=1}^{n} w_i f(x_i)$

- $w_i = \int_{-\infty}^{\infty} L_{n,i}(x)e^{-\frac{1}{2}x^2}dx$
- $x_1, ..., x_n$ are the roots of the nth **Hermite** polynomial h_n. Hermite polynomials satisfy
  $\int_{-\infty}^{\infty} (c_0 + c_1 x + \cdots + c_{n-1}x^{n-1})h_n(x)e^{-\frac{1}{2}x^2}dx = 0$

Other types of integrals

Here is another type of integral that often comes up in statistics:

$$\int_0^\infty f(x)e^{-x}dx$$

Question: When might we see this type of integral?

Examples

$$\int_0^\infty f(x) e^{-x} dx$$

- Suppose $X \sim Gamma(\alpha, \beta)$. Then
$$\mathbb{E}[g(X)] = \int_0^\infty g(x) \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$$
- $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$
- (HW 7) pdf of sample correlation coefficient involves the integral
$$\int_0^\infty (\cosh t - \rho r)^{1-n} \, dt$$

Gauss-Laguerre quadrature

$$\int_0^\infty f(x)e^{-x}dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

- $w_i = \int_0^\infty L_{n,i}(x)e^{-x}dx$
- $x_1, \ldots, x_n$ are the roots of the nth **Laguerre** polynomial
- The nth Laguerre polynomial ℓ_n satisfies

$$\int_0^\infty (c_0 + c_1 x + \cdots + c_{n-1}x^{n-1})\ell_n(x)e^{-x}dx = 0$$

Gaussian quadrature in general

$$\int_a^b f(x)\omega(x)dx$$

where $\omega(x)$ is some weighting function.

- Gauss-Legendre quadrature: $a = -1$, $b = 1$, $\omega(x) = 1$ (Uniform density)
- Gauss-Hermite quadrature: $a = -\infty$, $b = \infty$, $\omega(x) = e^{-\frac{1}{2}x^2}$ (Normal density)
- Gauss-Laguerre quadrature: $a = 0$, $b = \infty$, $\omega(x) = e^{-x}$ (Gamma density)

In general: $\int_a^b f(x)\omega(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$

- Choose nodes $x_1, ..., x_n$ as roots of family of polynomials corresponding to ω

HW 7

https://sta379-s25.github.io/homework/hw7.html

- Derive a few Laguerre polynomials
- Use Gauss-Laguerre quadrature to approximate density of sample correlation coefficient