Lecture 7: Generating random variables – acceptance-rejection sampling

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Recap: Inverse transform method

Suppose X is a random variable with cdf F. Let

$$F^{-1}(u) = \inf\{t : F(t) \geq u\}$$

(If F invertible, this is the usual inverse)

1. Generate $U \sim Uniform(0, 1)$
2. Let $X = F^{-1}(U)$

Then, $X \sim F$

Today: How else can we generate random variables?

Example

Suppose we would like to generate $X \sim Beta(\alpha, \beta)$

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad x \in (0,1)$$

- Inverse transform method: $F_X(t) = \,?$

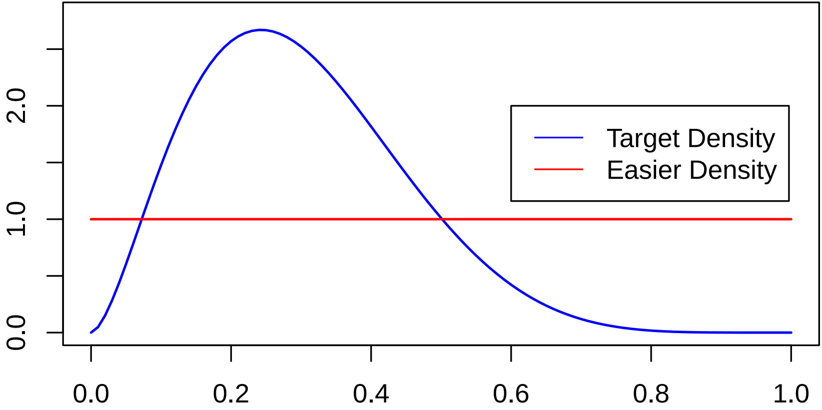
Acceptance-rejection sampling: motivation

- **Want:** $X \sim Beta(\alpha, \beta)$, $f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

However, it may be difficult to directly simulate from this distribution. Can you think of another distribution on $(0, 1)$ which is *easier* to simulate?

Acceptance-rejection sampling: motivation

- **Want:** $X \sim Beta(\alpha, \beta)$, $f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$
- **Can get:** $Y \sim Uniform(0, 1)$



Suppose we sample $Y \sim Uniform(0, 1)$ and observe $y = 0.9$. Is it likely we would observe that draw from the Beta distribution shown here?

Acceptance-rejection sampling: motivation

- **Want:** $X \sim Beta(\alpha, \beta)$, $f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$
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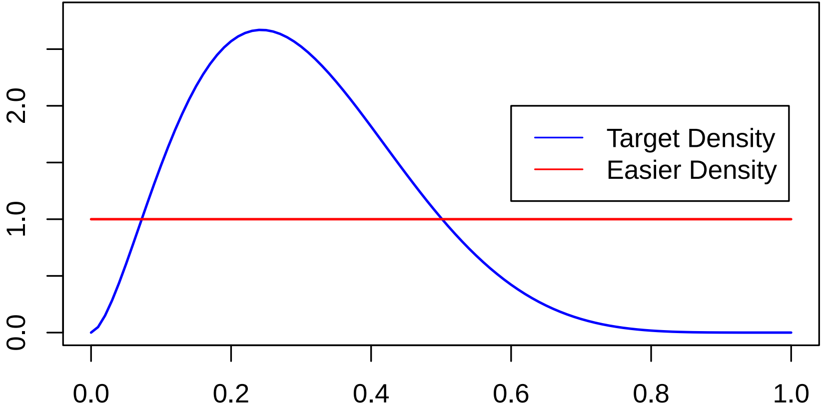


Suppose we sample $Y \sim Uniform(0, 1)$ and observe $y = 0.25$. Is it likely we would observe that draw from the Beta distribution shown here?

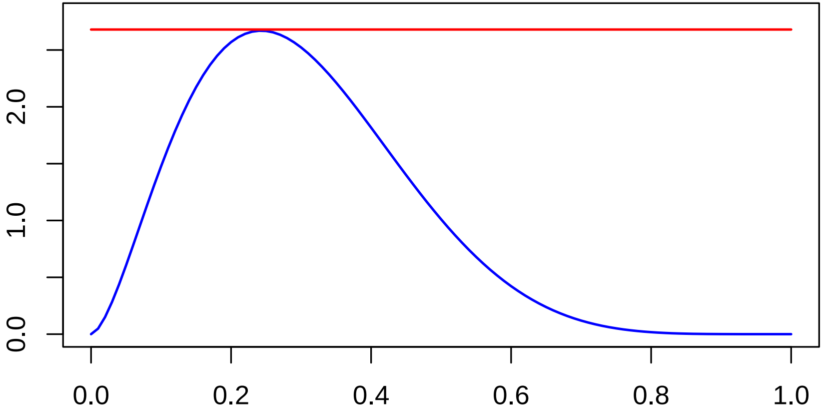
Acceptance-rejection sampling

Suppose we would like to generate a continuous random variable X with pdf f.

Illustration

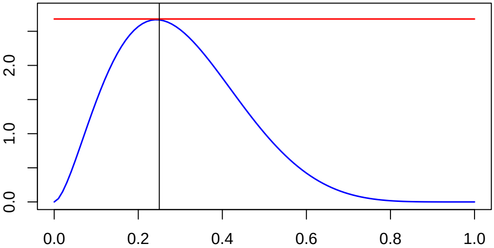
Target density: $f(t) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1},\ t \in (0,1)$

Candidate density: $g(t) = 1,\ t \in (0,1)$



Illustration

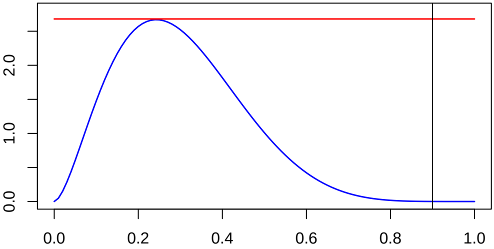
Now sample $Y \sim g$ and $U \sim Uniform(0,1)$. *Accept* Y if $U \leq \frac{f(Y)}{cg(Y)}$



Suppose we observe $Y = 0.25$. Are we likely to *accept* or *reject* 0.25 as a sample from f?

Illustration

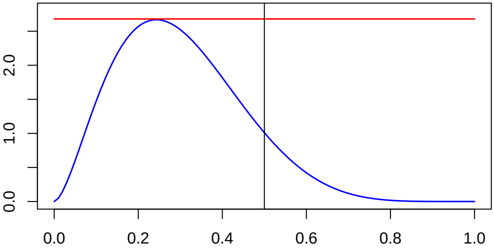
Now sample $Y \sim g$ and $U \sim Uniform(0,1)$. *Accept* Y if $U \leq \frac{f(Y)}{cg(Y)}$



Suppose we observe $Y = 0.9$. Are we likely to *accept* or *reject* 0.9 as a sample from f?

Illustration

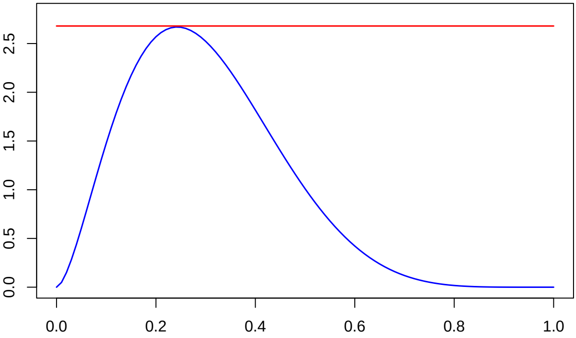
Now sample $Y \sim g$ and $U \sim Uniform(0, 1)$. *Accept* Y if $U \leq \frac{f(Y)}{cg(Y)}$



Suppose we observe $Y = 0.5$. Are we likely to *accept* or *reject* 0.5 as a sample from f?

Illustration

- $Y \sim g$ and $U \sim Uniform(0,1)$
- *Accept* Y if $U \leq \frac{f(Y)}{cg(Y)}$



Your turn

Practice questions on the course website:

https://sta379-s25.github.io/practice_questions/pq_7.html

- Implement acceptance-rejection sampling for the beta example
- Start in class. You are welcome to work with others
- Practice questions are to help you practice. They are not submitted and not graded
- Solutions are posted on the course website

Next time: formal proof that this sampling procedure works!