Lecture 22: Gaussian quadrature and Legendre polynomials

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Summary so far

To approximate $\int_{-1}^{1} f(x)dx$:

1. Choose n points $x_1, ..., x_n$ in $(-1, 1)$
2. Construct the interpolating polynomial: $q(x) = \sum_{i=1}^{n} f(x_i)L_{n,i}(x)$
3. Integrate q:

$$\int_{-1}^{1} q(x)dx = \sum_{i=1}^{n} w_i f(x_i) \qquad w_i = \int_{-1}^{1} L_{n,i}(x)dx$$

4. Approximate the integral of f:

$$\int_{-1}^{1} f(x)dx \approx \int_{-1}^{1} q(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

Today: Which points $x_1, ..., x_n$ do we use??

Warmup

Warmup activity to motivate importance of node choice:

https://sta379-s25.github.io/practice_questions/pq_22_warmup.html

- Work with your neighbors on the warmup activity
- In a bit, we will discuss key points as a class

Warmup

- If $x_1 = -0.1$, $x_2 = 0.5$, then $w_1 = 5/3$ and $w_2 = 1/3$
- Best two-point rule: $x_1 = -1\sqrt{3}$, $x_2 = 1\sqrt{3}$, $w_1 = w_2 = 1$

$$\int_{-1}^{1} (x^3 - 2x^2 + 3)dx = 14/3$$

$$\frac{5}{3} f(-0.1) + \frac{1}{3} f(0.5) =$$

$$f(-1/\sqrt{3}) + f(1/\sqrt{3}) =$$

Warmup

- If $x_1 = -0.1$, $x_2 = 0.5$, then $w_1 = 5/3$ and $w_2 = 1/3$
- Best two-point rule: $x_1 = -1\sqrt{3}$, $x_2 = 1\sqrt{3}$, $w_1 = w_2 = 1$

$$\int_{-1}^{1} (2x + 1) dx = 2$$

$$\frac{5}{3} f(-0.1) + \frac{1}{3} f(0.5) =$$

$$f(-1/\sqrt{3}) + f(1/\sqrt{3}) =$$

Summary so far

- Choose n points $x_1, ..., x_n$ in $(-1, 1)$

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i) \qquad w_i = \int_{-1}^{1} L_{n,i}(x)dx$$

- If $f(x)$ is a polynomial of degree $\leq n-1$, approximation is **exact**:

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

  for *any* choice of n distinct points $x_1, ..., x_n$ in $(-1, 1)$.

- If we are **clever** about choosing $x_1, ..., x_n$, we can get exact integrals for polynomials of degree $\leq 2n-1$

Next step: How should we be clever? Turns out the best nodes $x_1, ..., x_n$ are the roots of **Legendre polynomials**

Legendre polynomials

The **Legendre polynomials** are a set of polynomials $p_0, p_1, p_2, \ldots$
The first few Legendre polynomials are:

$$p_0(x) = 1 \quad p_1(x) = x \quad p_2(x) = \frac{1}{2}(3x^2 - 1) \quad p_3(x) = \frac{1}{2}(5x^3 - 3x^2)$$

Degree: Degree of p_n is n

Roots of Legendre polynomials

- $p_1(x) = x$. Root of p_1:

- $p_2(x) = \frac{1}{2}(3x^2 - 1)$. Roots of p_2:

- $p_3(x) = \frac{1}{3}(5x^3 - 3x^2)$. Roots of p_3:

Properties of Legendre polynomials

Let p_n be the nth Legendre polynomial

- p_n has degree n
- p_n has n distinct roots in $(-1, 1)$
- Let $g(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}$. Then

$$\int_{-1}^{1} g(x) p_n(x) dx = 0$$

Why the Legendre polynomials?

Theorem: Suppose $f(x)$ is a polynomial of degree $2n-1$. Let p_n be the nth Legendre polynomial, and let $x_1, ..., x_n$ be the n roots of p_n. Then

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i) \qquad w_i = \int_{-1}^{1} L_{n,i}(x)dx$$

Your turn

Practice questions with roots of Legendre polynomials and Gaussian quadrature:

https://sta379-s25.github.io/practice_questions/pq_22.html

- Start in class
- You are welcome and encouraged to work with your neighbors
- Solutions posted on course website