Lecture 22: Gaussian quadrature and Legendre polynomials

Ciaran Evans

Course logistics

- Project 2 released, due April 18
  - No HW due that week or the week before
  - We will have several project work days in class
- Challenge 6 released (inverse variance weighting)

Summary so far

To approximate $\int\limits_{-1}^{1} f(x)dx$:

1. Choose n points $x_1, ..., x_n$ in $(-1, 1)$
2. Construct the interpolating polynomial: $q(x) = \sum\limits_{i=1}^{n} f(x_i)L_{n,i}(x)$
3. Integrate q:

$$\int_{-1}^{1} q(x)dx = \sum_{i=1}^{n} w_i f(x_i) \qquad w_i = \int_{-1}^{1} L_{n,i}(x)dx$$

4. Approximate the integral of f:

$$\int_{-1}^{1} f(x)dx \approx \int_{-1}^{1} q(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

Today: Which points $x_1, ..., x_n$ do we use??

Warmup

Warmup activity to motivate importance of node choice:

https://sta379-
s25.github.io/practice_questions/pq_22_warmup.html

- Work with your neighbors on the warmup activity
- In a bit, we will discuss key points as a class

Warmup

- If $x_1 = -0.1$, $x_2 = 0.5$, then $w_1 = 5/3$ and $w_2 = 1/3$
- Best two-point rule: $x_1 = -1/\sqrt{3}$, $x_2 = 1/\sqrt{3}$, $w_1 = w_2 = 1$

$$\int_{-1}^{1} (x^3 - 2x^2 + 3)dx = 14/3$$

$$\frac{5}{3}f(-0.1) + \frac{1}{3}f(0.5) = \quad 5.84 \quad \neq \frac{14}{3} \quad \times$$

$$f(-1/\sqrt{3}) + f(1/\sqrt{3}) = \quad \frac{14}{3} \quad \checkmark$$

Warmup

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degree 1 $= 2 - 1$

$$\int_{-1}^{1} (2x+1)dx = 2$$

$$\frac{5}{3} f(-0.1) + \frac{1}{3} f(0.5) = \quad \frac{5}{3}(2(-0.1)+1) + \frac{1}{3}(2(0.5)+1)$$

$$= 2 \quad ✓$$

$$f(-1/\sqrt{3}) + f(1/\sqrt{3}) = \quad (2(-\tfrac{1}{\sqrt{3}})+1) + (2(\tfrac{1}{\sqrt{3}})+1)$$

$$= 2 \quad ✓$$

Summary so far

- Choose n points $x_1, ..., x_n$ in $(-1, 1)$

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i) \qquad w_i = \int_{-1}^{1} L_{n,i}(x)dx$$

- If $f(x)$ is a polynomial of degree $\leq n-1$, approximation is **exact**:

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

  for *any* choice of n distinct points $x_1, ..., x_n$ in $(-1, 1)$.

- If we are **clever** about choosing $x_1, ..., x_n$, we can get exact integrals for polynomials of degree $\leq 2n-1$

Next step: How should we be clever? Turns out the best nodes $x_1, ..., x_n$ are the roots of **Legendre polynomials**

Legendre polynomials

The **Legendre polynomials** are a set of polynomials $p_0, p_1, p_2, ...$
The first few Legendre polynomials are:

$$p_0(x) = 1 \quad p_1(x) = x \quad p_2(x) = \frac{1}{2}(3x^2 - 1) \quad p_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Degree: Degree of p_n is n

Roots of Legendre polynomials

- $p_1(x) = x$. Root of p_1: $x_1 = 0$

  Best one point rule: $2f(0)$

- $p_2(x) = \frac{1}{2}(3x^2 - 1)$. Roots of p_2:

  $3x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x_1 = -\frac{1}{\sqrt{3}},\ x_2 = \frac{1}{\sqrt{3}}$

  Best two point rule: $f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$

- $p_3(x) = \frac{1}{3}(5x^3 - 3x)$. Roots of p_3:

  $5x^3 - 3x = 0 \Rightarrow x(5x^2 - 3) = 0$

  $x = 0$ or $x^2 = \frac{3}{5}$

  $\Rightarrow x_1 = -\sqrt{\frac{3}{5}},\ x_2 = 0,\ x_3 = \sqrt{\frac{3}{5}}$

Best 3-point rule: $\frac{5}{9} f(-\sqrt{\frac{3}{5}}) + \frac{8}{9} f(0) + \frac{5}{9} f(\sqrt{\frac{3}{5}})$

Properties of Legendre polynomials

Let p_n be the nth Legendre polynomial

- p_n has degree n
- p_n has n distinct roots in $(-1, 1)$
- Let $g(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}$. Then

i.e., deg $g \le n-1$

$$\int_{-1}^{1} g(x)p_n(x)dx = 0$$

In fact, making this true is how the Legendre polynomials are constructed

Example: $g(x) = x^2$, $p_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$\int_{-1}^{1} x^2 \cdot \frac{1}{2}(5x^3 - 3x)dx = \frac{1}{2}\int_{-1}^{1}(5x^5 - 3x^3)dx$$

$$= \frac{1}{2}\left[\frac{5x^6}{6} - \frac{3x^4}{4}\right]_{-1}^{1} = 0 \checkmark$$

Why the Legendre polynomials?

Theorem: Suppose $f(x)$ is a polynomial of degree $2n-1$. Let p_n be the nth Legendre polynomial, and let $x_1, \ldots, x_n$ be the n roots of p_n. Then

exact

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i) \qquad w_i = \int_{-1}^{1} L_{n,i}(x)dx$$

Proof: By polynomial division, there exist polynomials g, r s.t. $f(x) = g(x)p_n(x) + r(x)$

$\deg g \le n-1 \qquad \deg r \le n-1$

$$\sum_i w_i f(x_i) = \sum_i w_i (g(x_i)\underbrace{p_n(x_i)}_{=0}) + \sum_i w_i r(x_i)$$

(these are the roots of p_n!)

$$= \sum_i w_i r(x_i)$$

$$= \int_{-1}^{1} r(x)dx$$

(equality b/c $\deg r \le n-1$)

Why the Legendre polynomials?

Theorem: Suppose $f(x)$ is a polynomial of degree $2n-1$. Let p_n be the nth Legendre polynomial, and let $x_1, ..., x_n$ be the n roots of p_n. Then

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i) \qquad w_i = \int_{-1}^{1} L_{n,i}(x)dx$$

So far: $\sum_i w_i f(x_i) = \int_{-1}^{1} r(x)dx$

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (g(x)p_n(x) + r(x))dx$$

$$= \underbrace{\int_{-1}^{1} g(x)\,p_n(x)\,dx}_{=0 \text{ b/c } \deg g \le n-1 < n = \deg p_n} + \int_{-1}^{1} r(x)dx$$

$$= \int_{-1}^{1} r(x)dx = \sum_i w_i f(x_i) \qquad //$$

Your turn

Practice questions with roots of Legendre polynomials and Gaussian quadrature:

https://sta379-s25.github.io/practice_questions/pq_22.html

- Start in class
- You are welcome and encouraged to work with your neighbors
- Solutions posted on course website