Lecture 28: Antithetic variables

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Warmup: antithetic sampling

Work with your neighbor on the questions on the handout / course website:

https://sta379-s25.github.io/practice_questions/pq_28_warmup.html

Then we will discuss as a class

- Use Monte Carlo integration to approximate another integral
- Explore variability of two different estimators

Warmup: antithetic sampling

$$\hat{\theta}_{AS} = \frac{1}{n} \sum_{i=1}^{n/2} (g(u_i) + g(1-u_i))$$

```
theta_hat_as <- rep(NA, nsim)
for(i in 1:nsim){
  u <- runif(n/2)
  theta_hat_as[i] <- sum(g(u) + g(1-u))/n
}

(var(theta_hat_mc) - var(theta_hat_as))/var(theta_hat_mc)
```

```
## [1] 0.9649457
```

```
u <- runif(10000)
cor(g(u), g(1-u))
```

```
## [1] -0.9678501
```

• if g is monotone then reduction in variance is given by $\text{cor}(g(u), g(1-u))$

Antithetic variables

$$\theta = \mathbb{E}[g(U)] \qquad U \sim \mathit{Uniform}(0,1)$$

Antithetic sampling:

- **Theorem:** If g is *monotone*, then $\mathit{Cor}(g(U),\ g(1-U)) \leq 0$
- Sample $U_1, ..., U_{n/2} \overset{iid}{\sim} \mathit{Uniform}(0,1)$
- $\widehat{\theta}_{AS} = \frac{1}{n}\sum_{i=1}^{n/2}(g(U_i) + g(1-U_i))$
- $\mathit{Var}(\widehat{\theta}_{AS}) = \dfrac{(1+\rho)\mathit{Var}(g(U))}{n}$ where $\rho = \mathit{Cor}(g(U), g(1-U))$

Another example

Suppose we want to approximate the integral

$$\theta = \int_1^\infty e^{-x} \frac{4}{x^5} dx = \mathbb{E}[e^{-X}]$$

(Pareto)

where X has pdf $f(x) = \frac{4}{x^5}$, $x > 1$.

Simple Monte Carlo:

- $X_1, ..., X_n \overset{iid}{\sim} f$
- $\widehat{\theta}_{MC} = \frac{1}{n} \sum_{i=1}^{n} e^{-X_i}$

Question: How do we sample $X_i \sim f$? inverse transform!

$F(x) = 1 - x^{-4} \implies F^{-1}(u) = (1-u)^{-1/4}$

sample $u \sim \text{Uniform}$

$X = (1-u)^{-1/4}$

Another example

Suppose we want to approximate the integral

$$\theta = \int_1^\infty e^{-x} \frac{4}{x^5} dx = \mathbb{E}[e^{-X}]$$

where X has pdf $f(x) = \dfrac{4}{x^5}$, $x > 1$.

Simple Monte Carlo:

- $U_1, ..., U_n \overset{iid}{\sim} Uniform(0, 1)$
- $X_i = F^{-1}(U_i)$
- $\widehat{\theta}_{MC} = \frac{1}{n} \sum_{i=1}^{n} e^{-X_i} = \frac{1}{n} \sum_{i=1}^{n} e^{-F^{-1}(U_i)}$

Question: Can we use antithetic sampling here?

Another example

Suppose we want to approximate the integral

$F^{-1}(u) = (1-u)^{-1/4}$

$$\theta = \int_1^\infty e^{-x} \frac{4}{x^5} dx = \mathbb{E}[e^{-X}]$$

$g(x) = e^{-x}$

where X has pdf $f(x) = \frac{4}{x^5}$, $x > 1$.

To sample X: $u \sim$ Uniform $(0,1)$, $X = F^{-1}(u)$

$\hat{\theta}_{MC} = \frac{1}{n} \sum_i g(X_i) = \frac{1}{n} \sum_i g(F^{-1}(u_i))$

- g is monotone
- F^{-1} is monotone (b/c cdf is always monotone)

$\hat{\theta}_{AS} = \frac{1}{n} \sum_{i=1}^{n/2} \left(g(F^{-1}(u_i)) + g(F^{-1}(1-u_i)) \right)$

Antithetic sampling with inverse transform

Suppose we want to approximate

$$\theta = \mathbb{E}[g(X)] \qquad X \sim f$$

If we can generate X with the inverse transform method, then
$X = F^{-1}(u)$

$$\theta = \mathbb{E}[g(F^{-1}(U))] \qquad U \sim Uniform(0,1)$$

Antithetic sampling:

- **Theorem:** If g is *monotone*, then $Cor(g(F^{-1}(U)),\ g(F^{-1}(1-U))) \leq 0$
- Sample $U_1, ..., U_{n/2} \overset{iid}{\sim} Uniform(0,1)$
- $\widehat{\theta}_{AS} = \frac{1}{n}\sum_{i=1}^{n/2}(g(F^{-1}(U_i)) + g(F^{-1}(1-U_i)))$

Your turn

Try antithetic sampling with the inverse transform method:

https://sta379-s25.github.io/practice_questions/pq_28.html

- Start in class
- Welcome to work with a neighbor
- Solutions are posted on the course website