Lecture 19: Numerical Integration

Ciaran Evans

Numerical integration

Numerical integration: Numerical approximations to definite integrals that are hard to solve in closed form.

Question: In statistics, when do integrals without a closed form solution arise?

Numerical integration: motivation

- CDFs with intractable integrals: e.g.,

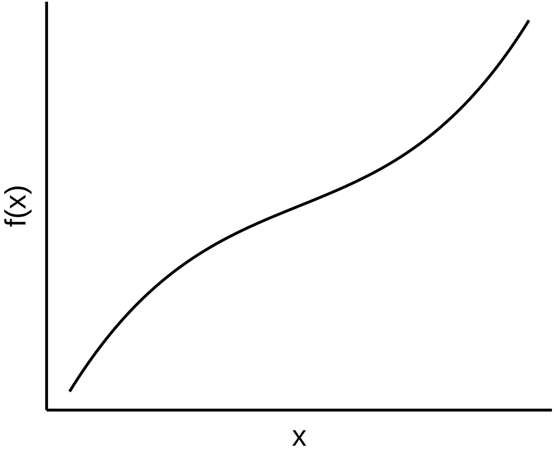
$$\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \qquad \int_{0}^{t} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx$$

- Optimization problems involving integrals: e.g., mixed effects models,

$$\prod_i \int \left[\phi(\gamma_i; 0, \sigma^2) \prod_j f(y_{ij}|\theta_{ij}) \right] d\gamma_i$$

Numerical integration

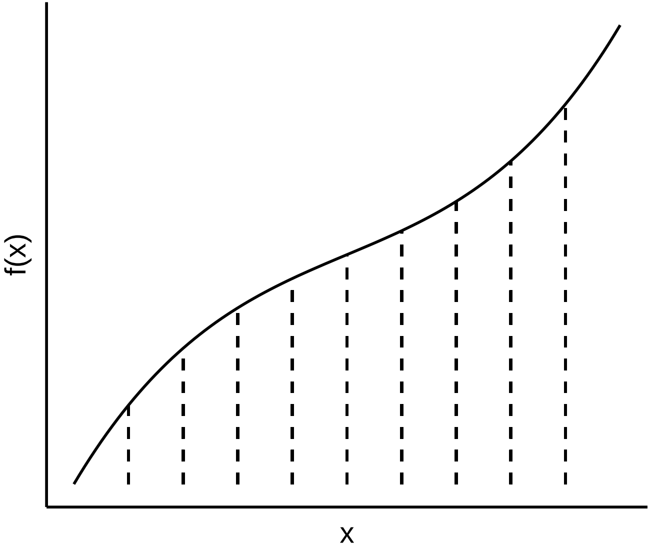
Suppose I want to calculate $\int_a^b f(x)dx$, but I can't get a closed form for the anti-derivative.



Question: How can we approximate the integral without a closed form?

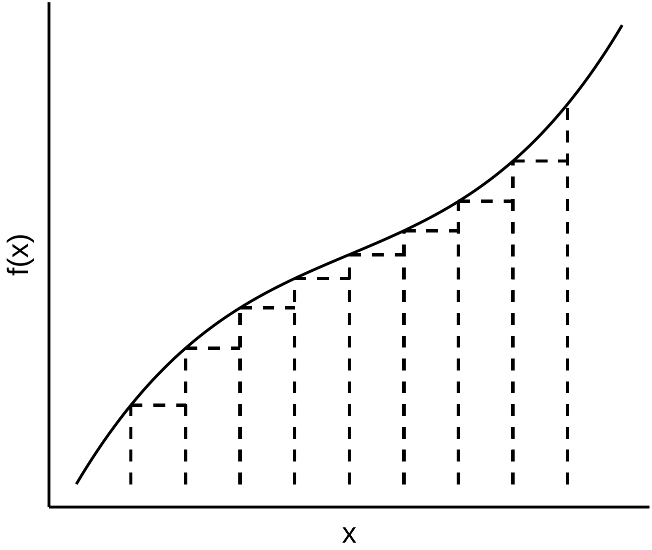
Numerical integration: Riemann sum

Divide $[a, b]$ into n subintervals:



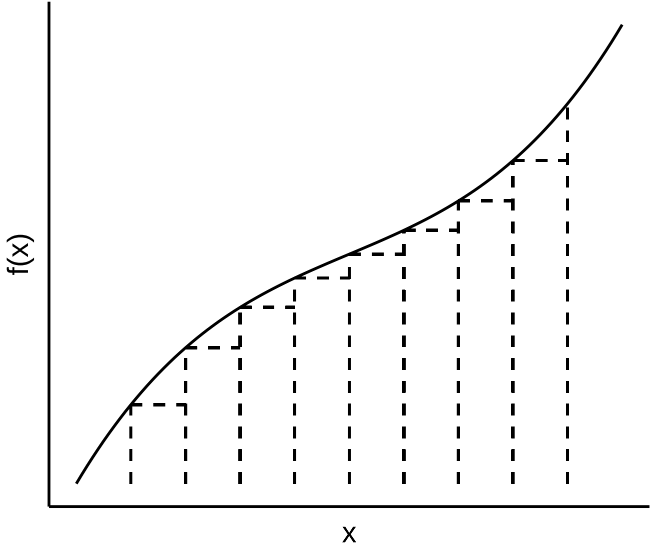
Numerical integration: Riemann sum

Approximate $f(x)$ in each interval:



Numerical integration: Riemann sum

Use the approximation in each interval to approximate the integral:



Numerical integration: Riemann sum

Want to approximate $\int_a^b f(x)dx$

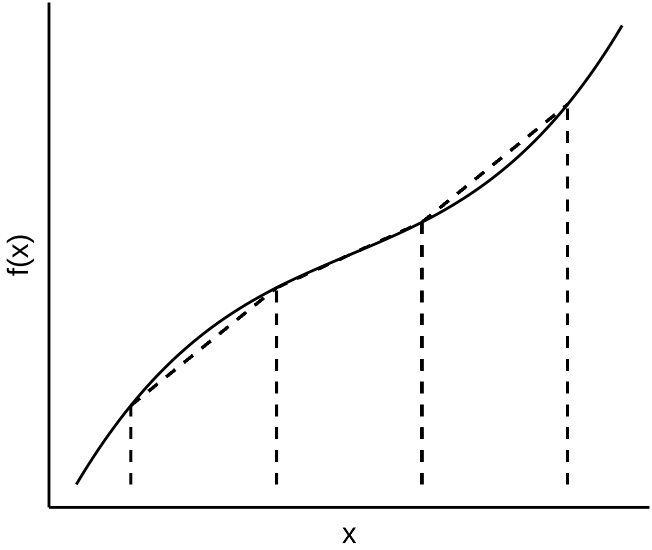
1. Divide $[a, b]$ into n subintervals of equal width $h = \dfrac{b-a}{n}$

2. Riemann sum approximation:

$$\int_a^b f(x)dx \approx h \sum_{i=0}^{n-1} f(a + ih)$$

Numerical integration: trapezoid rule

Instead of a constant value in each interval, use a line:



Numerical integration: trapezoid rule

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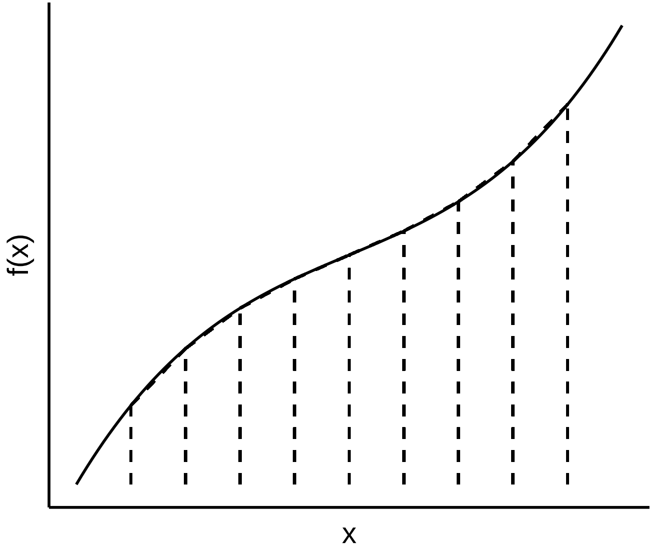

Numerical integration: trapezoid rule

Want to approximate $\int_a^b f(x)dx$

1. Divide $[a, b]$ into n subintervals of equal width $h = \dfrac{b-a}{n}$

2. Trapezoid rule approximation:

$$\int_a^b f(x)dx \;\approx\; \sum_{i=0}^{n-1} \frac{h}{2}\left[f(a+ih) + f(a+(i+1)h)\right]$$

$$= h\sum_{i=1}^{n-1} f(a+ih) + \frac{h}{2}(f(a) + f(b))$$