

# Warmup: Variance reduction in Monte Carlo integration

Group members:

## Monte Carlo integration

Suppose we wish to approximate the integral

$$\theta = \int_{-\infty}^{\infty} \frac{x}{2^x - 1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \mathbb{E}[g(X)]$$

where  $X \sim N(0, 1)$  and  $g(x) = \frac{x}{2^x - 1}$ .

The basic Monte Carlo approach is to do the following:

- Sample  $X_1, \dots, X_n \stackrel{iid}{\sim} N(0, 1)$
- $\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^n g(X_i)$

## Questions

1. Use Monte Carlo integration with  $n = 1000$  to estimate  $\theta$ . Repeat the process many times to approximate  $\mathbb{E}[\hat{\theta}_1]$  and  $Var(\hat{\theta}_1)$ . Write down the approximate values of  $\mathbb{E}[\hat{\theta}_1]$  and  $Var(\hat{\theta}_1)$ .

## A different Monte Carlo estimator

Now consider the estimator

$$\hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^{n/2} (g(X_i) + g(-X_i))$$

where  $X_1, \dots, X_{n/2} \sim N(0, 1)$  as before.

2. Use  $n = 1000$  (so  $n/2 = 500$ ) to compute this second estimator  $\hat{\theta}_2$ . Repeat the process many times to approximate  $\mathbb{E}[\hat{\theta}_2]$  and  $Var(\hat{\theta}_2)$ .

3. What is the percent reduction in variance of  $\hat{\theta}_2$  compared to  $\hat{\theta}_1$ ? Remember that the percent reduction in variance is calculated by

$$100 \cdot \frac{Var(\hat{\theta}_1) - Var(\hat{\theta}_2)}{Var(\hat{\theta}_1)}$$

4. Using R, estimate the correlation  $cor(g(X_i), g(-X_i))$  with  $X_i \sim N(0, 1)$  and  $g$  given as above (the `cor` function will be helpful). How is the correlation related to the percent reduction in variance?