Lecture 27: Antithetic variables

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Course logistics

- Project 2 due April 18
- No more HW until after project 2
- Next week:
  - Monday: project work day
  - Wednesday: begin EM algorithm

Warmup: variance reduction

Work with your neighbor on the questions on the handout / course website:

https://sta379-s25.github.io/practice_questions/pq_27_warmup.html

Then we will discuss as a class

- Use Monte Carlo integration to approximate another integral
- Explore variability of two different estimators

Warmup

$$\theta = \int_{-\infty}^{\infty} \frac{x}{2^x - 1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \mathbb{E}[g(X)]$$

Basic Monte Carlo estimator:

- Sample $X_1, ..., X_n \overset{iid}{\sim} N(0, 1)$
- $\widehat{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} g(X_i)$

Question: From your Monte Carlo integration, what is the approximate value of θ?

≈ 1.5

Warmup

```r
g <- function(x){
  x/(2^x - 1)
}
n <- 1000; nsim <- 1000
theta_hat1 <- rep(NA, nsim)

for(i in 1:nsim){
  x <- rnorm(n)
  theta_hat1[i] <- mean(g(x))
}

mean(theta_hat1)
```

$g(x) = \frac{x}{2^x - 1}$

$x_i \overset{iid}{\sim} N(0,1)$

$\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} g(x_i)$

```
## [1] 1.499676
```

```r
var(theta_hat1)
```

```
## [1] 0.0002591736
```

Warmup

$$\widehat{\theta}_1 = \frac{1}{n}\sum_{i=1}^{n} g(X_i) \qquad \widehat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n/2}(g(X_i) + g(-X_i))$$

Questions:

- How does $\mathbb{E}[\widehat{\theta}_2]$ compare to $\mathbb{E}[\widehat{\theta}_1]$? The same
- How does $Var(\widehat{\theta}_2)$ compare to $Var(\widehat{\theta}_1)$?

$Var(\hat{\theta}_2) \ll Var(\hat{\theta}_1)$

Warmup

```
theta_hat2 <- rep(NA, nsim)

for(i in 1:nsim){
  x <- rnorm(n/2)
  theta_hat2[i] <- sum(g(x) + g(-x))/n
}

mean(theta_hat2)
```

$$\hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^{n/2} (g(x_i) + g(-x_i))$$

```
## [1] 1.499159
```

```
var(theta_hat2)
```

```
## [1] 1.222002e-05
```

```
(var(theta_hat1) - var(theta_hat2))/var(theta_hat1) * 100
```

```
## [1] 95.28501
```

% reduction in variance

Warmup

$X \sim N(0,1) \Rightarrow -X \sim N(0,1)$ $X \overset{d}{=} -X$ ("equal in distribution")

$$\widehat{\theta}_1 = \frac{1}{n}\sum_{i=1}^{n} g(X_i) \qquad \widehat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n/2}(g(X_i) + g(-X_i))$$

where $X_1, ..., X_n \overset{iid}{\sim} N(0,1)$

- $\mathbb{E}[\widehat{\theta}_1] = \theta \qquad Var(\widehat{\theta}_1) = \dfrac{Var(g(X))}{n}$
- $\mathbb{E}[\widehat{\theta}_2] = \frac{1}{n}\sum_{i=1}^{n/2}(\mathbb{E}[g(X_i)] + \mathbb{E}[g(-X_i)])$

$\mathbb{E}[g(X)] = \theta \qquad \mathbb{E}[g(-X)] = \theta$ (b/c X & $-X$ have same distribution)

$\Rightarrow \mathbb{E}[\hat{\theta}_2] = \frac{1}{n}\cdot\frac{n}{2}\cdot(2\theta) = \theta$ ✓

- $Var(\widehat{\theta}_2) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n/2}(g(X_i) + g(-X_i))\right)$

$= \frac{1}{n^2}\sum_{i=1}^{n/2} Var(g(X_i) + g(-X_i))$ ($X_1, X_2, X_3, \dots$ independent)

$Var(g(X) + g(-X)) = Var(g(X)) + Var(g(-X)) + 2Cov(g(X), g(-X))$

Reducing variation

Let Y, Y^* be two random variables with the same distribution, with mean μ and variance σ^2.

- $\mathbb{E}\left(\frac{Y+Y^*}{2}\right) = \mu$
- If Y, Y^* are independent, then

$$Var\left(\frac{Y+Y^*}{2}\right) = \frac{1}{4}\left(Var(Y) + Var(Y^*)\right) = \frac{\sigma^2}{2}$$

- If Y, Y^* are correlated, with correlation ρ, then

$$Var\left(\frac{Y+Y^*}{2}\right) = \frac{1}{4}\left(Var(Y) + Var(Y^*) + 2Cov(Y, Y^*)\right) = \frac{\sigma^2}{2} + \frac{\rho\sigma^2}{2}$$

If $\rho < 0$, we <u>decrease</u> the variance!

Reducing variation: antithetic variables

Suppose that we want to estimate

$$\theta = \mathbb{E}[g(X)] \qquad X \sim N(0,1)$$

- Sample $X_1, \ldots, X_{n/2} \overset{iid}{\sim} N(0,1)$
- Antithetic Monte Carlo estimate:

$$\widehat{\theta}_{AS} = \frac{1}{n}\sum_{i=1}^{n/2}(g(X_i) + g(-X_i))$$

- $\mathbb{E}[\widehat{\theta}_{AS}] = \theta \qquad Var(\widehat{\theta}_{AS}) = \frac{(1+\rho)Var(g(X))}{n} \qquad$ where $\rho = Cor(g(X), g(-X))$
- **Theorem:** If g is a *monotone* function, then $\rho \leq 0$ (and so we reduce the variance)

Another example

$f(x) = 1$

$g(x) = \log(x+1)e^x$

Suppose we want to estimate

$$\theta = \int_0^1 \log(x+1)e^x dx \quad = \int_0^1 g(x) f(x) dx$$

Question: How would we do this with the basic Monte Carlo approach we learned previously?

- $\theta = \mathbb{E}[g(U)] \qquad U \sim Uniform(0,1)$
- sample $U_1, \dots, U_n \overset{iid}{\sim} Uniform(0,1)$
- $\hat{\theta}_{MC} = \frac{1}{n} \sum_{i=1}^{n} g(U_i)$

$$Var(\hat{\theta}_{MC}) = \frac{Var(g(U))}{n}$$

Antithetic Samples: $U, \quad 1-U$

Antithetic variables

$$\theta = \int_0^1 \log(x+1)e^x dx = \mathbb{E}[g(U)] \qquad U \sim Uniform(0,1)$$

Antithetic sampling:

- **Theorem:** If g is *monotone*, then $Cor(g(U),\ g(1-U)) \leq 0$
- Sample $U_1, ..., U_{n/2} \overset{iid}{\sim} Uniform(0,1)$
- $\widehat{\theta}_{AS} = \frac{1}{n} \sum_{i=1}^{n/2} (g(U_i) + g(1-U_i))$
- $Var(\widehat{\theta}_{AS}) = \frac{(1+\rho)Var(g(U))}{n}$ where $\rho = Cor(g(U), g(1-U))$

Compare to

$Var(\hat{\theta}_{MC}) = \frac{Var(g(U))}{n}$

(simple MC estimate)

Your turn

Try antithetic sampling with uniform random variables:

https://sta379-s25.github.io/practice_questions/pq_27.html

- Start in class
- Welcome to work with a neighbor
- Solutions are posted on the course website