Lecture 21: Gaussian quadrature

Ciaran Evans

Gaussian quadrature

General result: If $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{2n-1} x^{2n-1}$, then there exist **nodes** $x_1, \ldots, x_n$ and **weights** $w_1, \ldots, \overset{w_n}{\cancel{x_n}}$ such that

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

n-node Gaussian quadrature rule: For general function f,

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

Activity, Part 1

Class activity to help motivate Gaussian quadrature:

https://sta379-s25.github.io/practice_questions/pq_21.html

- Work with your neighbors on Part 1 of the activity
- In a bit, we will discuss key points as a class

Activity, Part 1

Lagrange polynomial

$$L_{n,i}(x) = \prod_{k:k\neq i} \frac{(x - x_k)}{(x_i - x_k)}$$

$$L_{n,i}(x_i) = \prod_{k:k\neq i} \frac{(x_i - x_k)}{(x_i - x_k)} = \prod_{k:k\neq i} 1 = 1$$

$$L_{n,i}(x_j) = \prod_{k:k\neq i} \frac{(x_j - x_k)}{(x_i - x_k)} = \frac{(x_j - x_1)}{(x_i - x_1)} \cdots \frac{(x_j - x_j)}{(x_i - x_j)} \cdots \frac{(x_j - x_n)}{(x_i - x_n)} = 0$$

$\leftarrow 0$

$$q(x) = \sum_{i=1}^{n} y_i L_{n,i}(x)$$

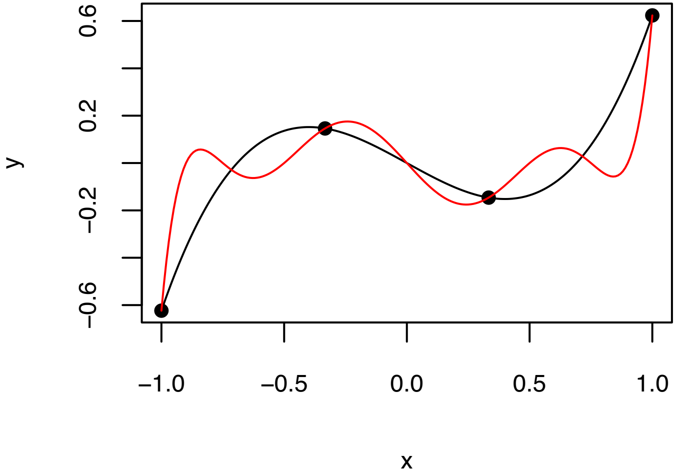
$$q(x_i) = y_i$$

q goes through all n points (x_i, y_i)

Activity, Part 1

Original: $f(x) = 10(x^7 - 1.6225x^5 + 0.79875x^3 - 0.113906x)$

Polynomial interpolation: $q(x)$, with $n = 4$ points



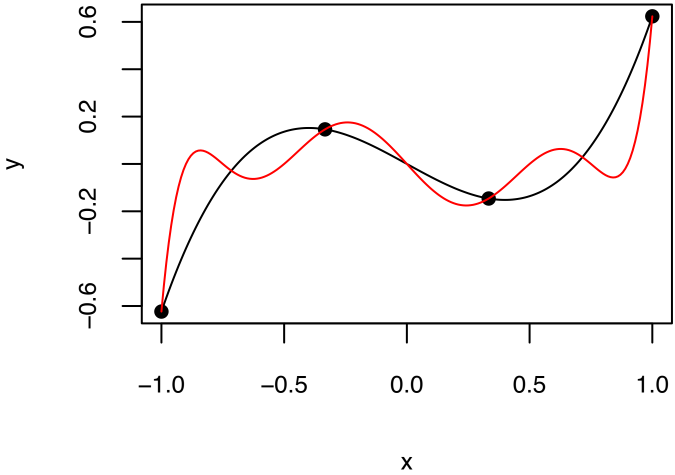
q is an

interpolating

polynomial

Activity, Part 1

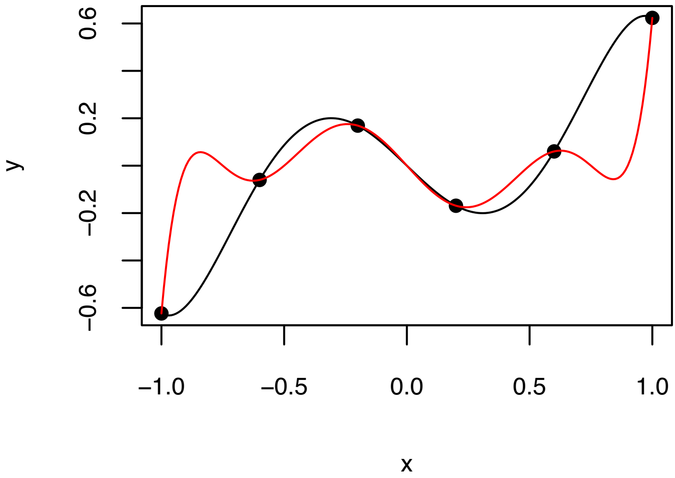
Polynomial interpolation: $q(x)$, with $n = 4$ points



Question: What happens as I change the number of points n used for $q(x)$?

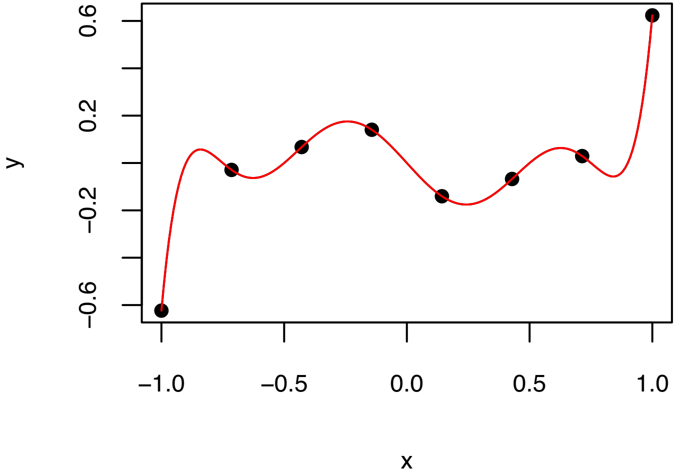
Activity, Part 1

Polynomial interpolation: $q(x)$, with $n = 6$ points



Activity, Part 1

Polynomial interpolation: $q(x)$, with $n = 8$ points



Polynomial interpolation

Let f be a function on $(-1, 1)$ we wish to approximate, and let $x_1, ..., x_n$ be n distinct points in $(-1, 1)$.

Interpolating polynomial: $q(x) = \sum_{i=1}^{n} f(x_i) L_{n,i}(x)$

- $q(x_i) = f(x_i)$ for $i = 1, ..., n$ (**interpolation**)
- If f is a polynomial of degree $\leq n - 1$, then $q(x) = f(x)$ for **all** x

Integration: $\int_{-1}^{1} f(x)dx \approx \int_{-1}^{1} q(x)dx$

$$= \int_{-1}^{1} \sum_i f(x_i) L_{n,i}(x)\, dx$$

$$= \sum_{i=1}^{n} f(x_i) \int_{-1}^{1} L_{n,i}(x)\, dx$$

$$= \sum_{i=1}^{n} w_i f(x_i) \qquad w_i = \int_{-1}^{1} L_{n,i}(x)\, dx$$

Summary so far

want to approximate $\int_{-1}^{1} f(x)dx$

1) Choose points $x_1, \dots, x_n$ in $(-1,1)$

2) Interpolating polynomial: $q(x) = \sum_{i=1}^{n} f(x_i) L_{n,i}(x)$

3) $\int_{-1}^{1} q(x)dx = \sum_{i=1}^{n} w_i f(x_i)$ $\quad w_i = \int_{-1}^{1} L_{n,i}(x)dx$

4) $\int_{-1}^{1} f(x)dx \approx \int_{-1}^{1} q(x)dx = \sum_{i=1}^{n} w_i f(x_i)$

<u>exact</u> if f is polynomial of deg $\le n-1$

Summary so far

To approximate $\int\limits_{-1}^{1} f(x)dx$:

1. Choose n points $x_1, ..., x_n$ in $(-1, 1)$
2. Construct the interpolating polynomial: $q(x) = \sum\limits_{i=1}^{n} f(x_i)L_{n,i}(x)$
3. Integrate q:

$$\int\limits_{-1}^{1} q(x)dx = \sum_{i=1}^{n} w_i f(x_i) \qquad w_i = \int\limits_{-1}^{1} L_{n,i}(x)dx$$

4. Approximate the integral of f:

$$\int\limits_{-1}^{1} f(x)dx \approx \int\limits_{-1}^{1} q(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

Next time: Which points $x_1, ..., x_n$ do we use??

Activity, Part 2

Verify calculation of the weights w_i:

https://sta379-s25.github.io/practice_questions/pq_21.html

- Work with your neighbors on Part 2 of the activity

$$\int_{-1}^{1} L_{2,1}(x)\,\partial x = \int_{-1}^{1} \left(\frac{x - x_2}{x_1 - x_2}\right) \partial x = \frac{1}{(x_1 - x_2)} \left(\frac{x^2}{2} - x \cdot x_2 \right)\Big|_{-1}^{1}$$

$$= \frac{-2x_2}{(x_1 - x_2)}$$

$x_1 = -\frac{1}{\sqrt{3}}$

$x_2 = \frac{1}{\sqrt{3}}$

$$\Rightarrow \quad \frac{-2x_2}{(x_1 - x_2)} = \frac{-2\left(\frac{1}{\sqrt{3}}\right)}{-2/\sqrt{3}} = 1 \quad \checkmark$$