Warmup: The importance of node choice in Gaussian quadrature

Group members:

Integral approximations

Given n nodes $x_1, ..., x_n$ in $(-1, 1)$, we approximated the integral of a function f by

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

where

$$w_i = \int_{-1}^{1} L_{n,i}(x) \qquad L_{n,i}(x) = \prod_{k:k\neq i} \frac{(x - x_k)}{(x_i - x_k)}$$

Our goal for today is to understand the importance of choosing the *right* nodes $x_1, ..., x_n$.

Questions

1. For the two-point rule ($n = 2$) we have $L_{2,1}(x) = \dfrac{x - x_2}{x_1 - x_2}$ and $L_{2,2}(x) = \dfrac{x - x_1}{x_2 - x_1}$.

   Suppose that we choose $n = 2$ nodes with $x_1 = -0.1$ and $x_2 = 0.5$. Show that $w_1 = 5/3$ and $w_2 = 1/3$.

2. Let $f(x) = x^3 - 2x^2 + 3$. Then, $\int_{-1}^{1} f(x)dx = 14/3$. Calculate $w_1 f(x_1) + w_2 f(x_2)$ using the two nodes and weights from question 1. Is our integral approximation exact with these two nodes?

3. Let $f(x) = x^3 - 2x^2 + 3$. Previously, we found that the "best" two-point rule uses $x_1 = -1\sqrt{3}$ and $x_2 = 1/\sqrt{3}$. Calculating the weights for these two nodes gives $w_1 = w_2 = 1$. Using these optimal nodes and weights, calculate $w_1 f(x_1) + w_2 f(x_2)$. Is our integral approximation exact with these two nodes?