Lecture 21: Gaussian quadrature

Ciaran Evans

Gaussian quadrature

General result: If $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{2n-1} x^{2n-1}$, then there exist **nodes** $x_1, ..., x_n$ and **weights** $w_1, ..., x_n$ such that

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

n-node Gaussian quadrature rule: For general function f,

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

Activity, Part 1

Class activity to help motivate Gaussian quadrature:

https://sta379-s25.github.io/practice_questions/pq_21.html

- Work with your neighbors on Part 1 of the activity
- In a bit, we will discuss key points as a class

Activity, Part 1

$$L_{n,i}(x) = \prod_{k:k\neq i} \frac{(x - x_k)}{(x_i - x_k)}$$

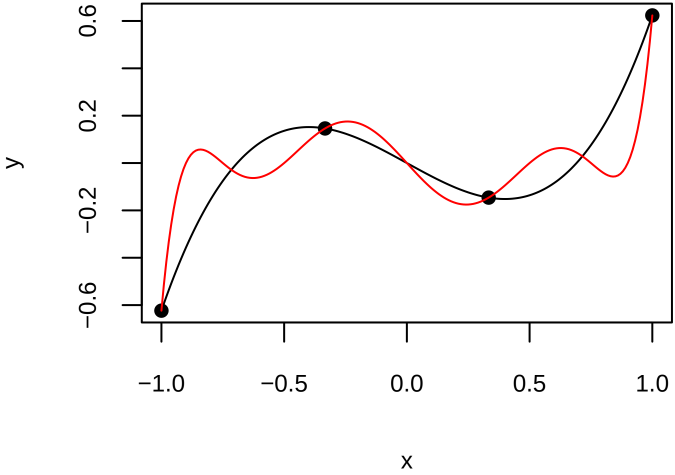
$$q(x) = \sum_{i=1}^{n} y_i L_{n,i}(x)$$

$$q(x_i) =$$

Activity, Part 1

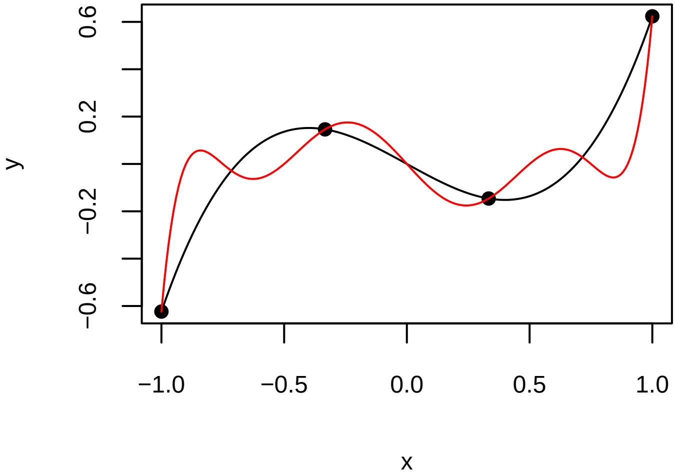
Original: $f(x) = 10(x^7 - 1.6225x^5 + 0.79875x^3 - 0.113906x)$

Polynomial interpolation: $q(x)$, with $n = 4$ points



Activity, Part 1

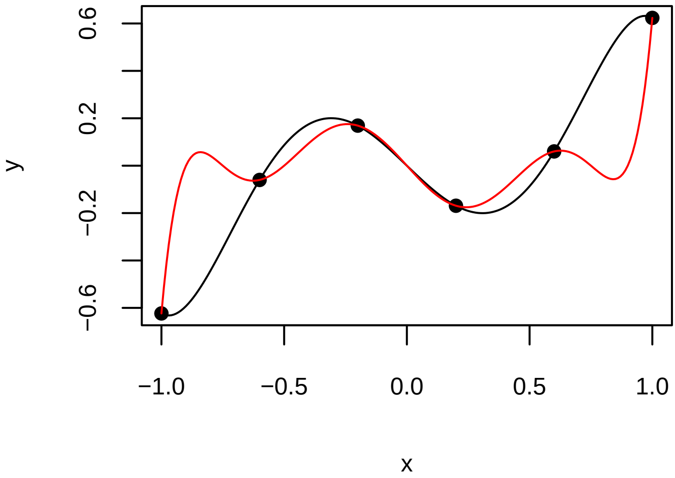
Polynomial interpolation: $q(x)$, with $n = 4$ points



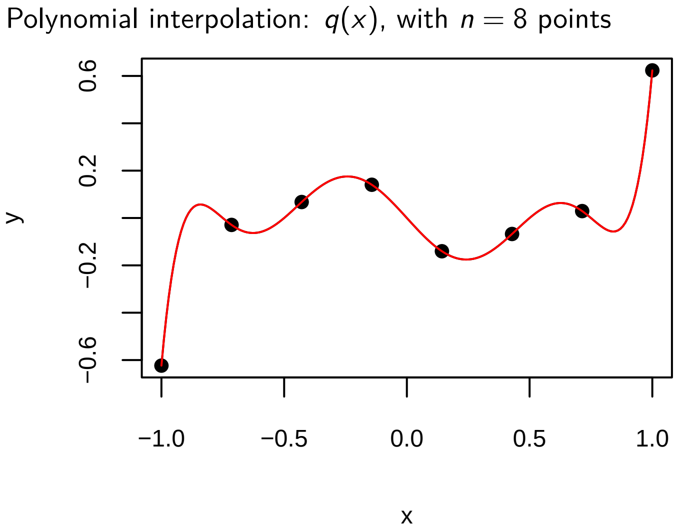
Question: What happens as I change the number of points n used for $q(x)$?

Activity, Part 1

Polynomial interpolation: $q(x)$, with $n = 6$ points



Activity, Part 1



Polynomial interpolation

Let f be a function on $(-1,1)$ we wish to approximate, and let $x_1, ..., x_n$ be n distinct points in $(-1,1)$.

Interpolating polynomial: $q(x) = \sum\limits_{i=1}^{n} f(x_i) L_{n,i}(x)$

- $q(x_i) = f(x_i)$ for $i = 1, ..., n$ (**interpolation**)
- If f is a polynomial of degree $\leq n-1$, then $q(x) = f(x)$ for **all** x

Integration: $\int\limits_{-1}^{1} f(x)dx \approx \int\limits_{-1}^{1} q(x)dx$

Summary so far

Summary so far

To approximate $\int_{-1}^{1} f(x)dx$:

1. Choose n points $x_1, ..., x_n$ in $(-1, 1)$
2. Construct the interpolating polynomial: $q(x) = \sum_{i=1}^{n} f(x_i)L_{n,i}(x)$
3. Integrate q:

$$\int_{-1}^{1} q(x)dx = \sum_{i=1}^{n} w_i f(x_i) \qquad w_i = \int_{-1}^{1} L_{n,i}(x)dx$$

4. Approximate the integral of f:

$$\int_{-1}^{1} f(x)dx \approx \int_{-1}^{1} q(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

Next time: Which points $x_1, ..., x_n$ do we use??

Activity, Part 2

Verify calculation of the weights w_i:

https://sta379-s25.github.io/practice_questions/pq_21.html

- Work with your neighbors on Part 2 of the activity