Lecture 23: Changing the range of integration

Ciaran Evans

Gauss-Legendre quadrature

- Let $x_1, ..., x_n \in (-1, 1)$ be the n roots of the nth Legendre polynomial p_n
- Use $x_1, ..., x_n$ as quadrature nodes to approximate integrals:

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i) \qquad w_i = \int_{-1}^{1} L_{n,i}(x)dx$$

- If $f(x)$ is a polynomial of degree $\leq 2n - 1$, approximation is **exact**:

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

Changing the range of integration

Gauss-Legendre quadrature allows us to approximate $\int_{-1}^{1} f(x)dx$.

Question: What should I do if I want to approximate

$$\int_a^b f(x)dx$$

for a finite interval $[a, b]$? $x = h(u)$ $u \in (-1,1)$

$$x = \left(\frac{b-a}{2}\right)u + \left(\frac{a+b}{2}\right) \qquad u \in (-1,1)$$

$$\partial x = \left(\frac{b-a}{2}\right)\partial u$$

$$\Rightarrow \int_a^b f(x)\partial x = \int_{-1}^{1} f\left(\left(\frac{b-a}{2}\right)u + \left(\frac{a+b}{2}\right)\right)\left(\frac{b-a}{2}\right)\partial u$$

$$\approx \left(\frac{b-a}{2}\right)\sum_i w_i f\left(\left(\frac{b-a}{2}\right)x_i + \left(\frac{a+b}{2}\right)\right)$$

Integrating over an infinite range

Integrals in statistics often involve an infinite range. For example, standard normal cdf:

$$\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

Question: How could we use Gauss-Legendre quadrature to approximate the integral?

Integrating over an infinite range: truncation

The standard normal density is mostly concentrated around 0, so for many values of t

$$\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \approx \int_{-5}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

```r
pnorm(-1)
```

```
## [1] 0.1586553
```

```r
pnorm(-1) - pnorm(-5)
```

```
## [1] 0.158655
```

Integrating over an infinite range: truncation

```
library(rootSolve)
p4_roots <- uniroot.all(function(x){
  (1/8) * (35*x^4 - 30*x^2 + 3)},
                              c(-1, 1), tol=1e-12)
weights <- c((18 - sqrt(30))/36,
             (18 + sqrt(30))/36,
             (18 + sqrt(30))/36,
             (18 - sqrt(30))/36)
```

$$\left(\frac{b-a}{2}\right) \sum_i w_i f\left(\left(\frac{b-a}{2}\right)x_i + \left(\frac{a+b}{2}\right)\right)$$

```
a <- -5; b <- -1
(b - a)/2*sum(weights*dnorm((b-a)/2*p4_roots + (a+b)/2))
```

```
## [1] 0.1585709
```

```
pnorm(-1)
```

```
## [1] 0.1586553
```

Integrating over an infinite range: transformation

Find a transformation $x = h(u)$ such that

possibly infinite

$$\int_a^b f(x)dx = \int_{-1}^{1} f(h(u))h'(u)du$$

Question: For

$$\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

what transformations could I consider?

$u \in (-1, 1)$ $\quad$ $x = t + \log\left(\frac{u+1}{2}\right)$

Integrating over an infinite range: transformation

$$\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

Let $x = t + \log\left(\frac{u+1}{2}\right)$

$$dx = \frac{1}{u+1} du$$

$$\Rightarrow \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(t + \log\left(\frac{u+1}{2}\right)\right)^2\right\}\left(\frac{1}{u+1}\right) du$$

Now apply Gauss-Legendre quadrature

Integrating over an infinite range: transformation

```r
f <- function(u){
  exp(-0.5*(-1 + log(0.5*(u+1)))^2)/(u+1)
}

sum(weights * f(p4_roots))/sqrt(2*pi)
```

```
## [1] 0.1586723
```

```r
pnorm(-1)
```

```
## [1] 0.1586553
```

Gauss-Hermite quadrature

Lots of integrals in statistics involve the normal distribution, and so look like

$$\int_{-\infty}^{\infty} f(x) e^{-\frac{1}{2}x^2} dx$$

Gauss-Hermite quadrature is a quadrature rule that is good at these types of integrals:

$$\int_{-\infty}^{\infty} f(x) e^{-\frac{1}{2}x^2} dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

- Need to choose the x_i and w_i differently to Gauss-Legendre quadrature

Gauss-Hermite quadrature

$L_{n,i}(x) = \prod_{u: u \neq i} \frac{(x - x_u)}{(x_i - x_u)}$

Gauss-Legendre quadrature: $\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$

- $w_i = \int_{-1}^{1} L_{n,i}(x)dx$
- $x_1, ..., x_n$ are the roots of the nth Legendre polynomial p_n. Legendre polynomials satisfy $\int_{-1}^{1} (c_0 + c_1 x + \cdots + c_{n-1}x^{n-1})p_n(x)dx \quad = 0$

Gauss-Hermite quadrature: $\int_{-\infty}^{\infty} f(x)e^{-\frac{1}{2}x^2}dx \approx \sum_{i=1}^{n} w_i f(x_i)$

- $w_i = \int_{-\infty}^{\infty} L_{n,i}(x)\, e^{-\frac{1}{2}x^2}\, \partial x$

Gauss-Hermite quadrature

Gauss-Legendre quadrature: $\int\limits_{-1}^{1} f(x)dx \approx \sum\limits_{i=1}^{n} w_i f(x_i)$

- $w_i = \int\limits_{-1}^{1} L_{n,i}(x)dx$
- $x_1, ..., x_n$ are the roots of the nth Legendre polynomial p_n. Legendre polynomials satisfy
  $\int\limits_{-1}^{1} (c_0 + c_1 x + \cdots + c_{n-1}x^{n-1})p_n(x)dx \quad = 0$

Gauss-Hermite quadrature: $\int\limits_{-\infty}^{\infty} f(x)e^{-\frac{1}{2}x^2}dx \approx \sum\limits_{i=1}^{n} w_i f(x_i)$

- $w_i = \int\limits_{-\infty}^{\infty} L_{n,i}(x)e^{-\frac{1}{2}x^2}dx$
- $x_1, ..., x_n$ are the roots of the nth **Hermite** polynomial h_n. Hermite polynomials satisfy
  $\int\limits_{-\infty}^{\infty} (c_0 + c_1 x + \cdots + c_{n-1}x^{n-1})h_n(x)e^{-\frac{1}{2}x^2}dx = 0$

Example

Hermite polynomial for $n = 2$: $h_2(x) = x^2 - 1$

- Roots of h_2: $x_1 = -1$, $x_2 = 1$
- Weights: $w_i = \int\limits_{-\infty}^{\infty} L_{n,i}(x) e^{-\frac{1}{2}x^2} dx$

$$L_{2,1}(x) = \frac{x - x_2}{x_1 - x_2}$$

$$w_1 = \int_{-\infty}^{\infty} \left(\frac{x - x_2}{x_1 - x_2}\right) e^{-\frac{1}{2}x^2} dx = \frac{1}{x_1 - x_2}\left[\int_{-\infty}^{\infty} x e^{-\frac{1}{2}x^2} dx - x_2 \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx\right]$$

$$\int_{-\infty}^{\infty} x e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi} \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}\, \mathbb{E}[X] \quad X \sim N(0,1)$$

$$= 0$$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$$

$$\Rightarrow \quad w_1 = \frac{-x_2\sqrt{2\pi}}{x_1 - x_2} = \frac{-\sqrt{2\pi}}{-2} = \sqrt{\frac{\pi}{2}} \qquad w_2 = \sqrt{\frac{\pi}{2}}$$

Example

$X \sim N(0,1)$

$f(x)$

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi} \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx =$$

$$= \sqrt{2\pi}\, \mathbb{E}[X^2] = \sqrt{2\pi}$$

Gauss-Hermite quadrature with $n = 2$: $w_1 f(x_1) + w_2 f(x_2) =$

$$\sqrt{\frac{\pi}{2}}\,(-1)^2 + \sqrt{\frac{\pi}{2}}\,(1)^2 = \sqrt{2\pi} \quad \checkmark$$

Example

Gauss- Hermite quadrature: $\int_{-\infty}^{\infty} f(x) e^{-\frac{1}{2}x^2} dx$ is exact for polynomials f of deg $\leq 2n-1$

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$$

Gauss-Hermite quadrature with $n = 2$:

$$w_1 f(x_1) + w_2 f(x_2) = \sqrt{\frac{\pi}{2}} \cdot (-1)^2 + \sqrt{\frac{\pi}{2}} \cdot (1)^2 = \sqrt{2\pi}$$

```
nodes <- c(-1, 1)
weights <- c(sqrt(pi/2), sqrt(pi/2))
sum(weights * nodes^2)
```

```
## [1] 2.506628
```

```
sqrt(2*pi)
```

```
## [1] 2.506628
```

Your turn

Try Gauss-Hermite quadrature for calculating expectations of functions of normal distributions:

https://sta379-s25.github.io/practice_questions/pq_23.html

- Start in class
- You are welcome and encouraged to work with your neighbors
- Solutions posted on course website