Lecture 20: Intro to Gaussian quadrature

Ciaran Evans

Recap: numerical integration rules (so far)

Divide interval $[a, b]$ into n subintervals of equal width $h = \dfrac{b-a}{n}$

- **Riemann rule:** (piecewise constant approximation)

$$\int_a^b f(x)dx \approx h \sum_{i=0}^{n-1} f(a+ih)$$

- **Trapezoid rule:** (piecewise linear approximation)

$$\int_a^b f(x)dx \approx h \sum_{i=1}^{n-1} f(a+ih) + \frac{h}{2}(f(a)+f(b))$$

- **Simpson's rule:** (piecewise quadratic approximation)

$$\int_a^b f(x)dx \approx \frac{h}{6} \sum_{i=0}^{n-1} \left[f(a+ih) + 4f\left(\frac{2a+2ih+h}{2}\right) + f(a+(i+1)h) \right]$$

Trapezoid rule

- Choose n. Interval widths are all the same: $h = (b-a)/n$

$$\int_a^b f(x)dx \;\approx\; h\sum_{i=1}^{n-1} f(a+ih) + \frac{h}{2}(f(a)+f(b))$$

$$= \sum_i w_i f(x_i)$$

Idea: What if we try another weighted sum,

$$\int_a^b f(x)dx \approx \sum_i w_i f(x_i)$$

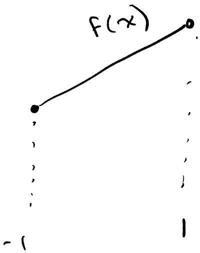
but this time we try to be more clever with the points x_i?

Motivation: using fewer points

- For now, restrict attention to functions on $[-1, 1]$ (can generalize to other intervals with a change of variables later)

- Suppose $f(x) = c_0 + c_1 x$ on $[-1, 1]$ (linear function)

Question: Will trapezoid rule do a good job?



Trapezoid rule will be exact

Motivation: using fewer points

- For now, restrict attention to functions on $[-1, 1]$ (can generalize to other intervals with a change of variables later)
- Suppose $f(x) = c_0 + c_1 x$ on $[-1, 1]$ (linear function) $\Rightarrow$ trapezoid rule is **exact**
- Trapezoid rule requires evaluation f at two points: $\int_{-1}^{1} f(x)dx = f(-1) + f(1)$

Claim: If $f(x) = c_0 + c_1 x$, there exist w_1, x_1 such that

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1)$$

(only need a single point to exactly integrate a linear function)

Motivation: using fewer points

Claim: If $f(x) = c_0 + c_1 x$, there exist w_1, x_1 such that

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1)$$

$$\int_{-1}^{1} (c_0 + c_1 x)\, dx = \left[c_0 x + c_1 \frac{x^2}{2} \right]_{-1}^{1}$$

$$= \left(c_0 + \frac{c_1}{2} \right) - \left(c_0(-1) + c_1 \frac{(-1)^2}{2} \right)$$

$$= 2c_0$$

$$= 2(c_0 + c_1(0))$$

$$= 2f(0)$$

$$\Rightarrow \quad w_1 = 2 \qquad x_1 = 0$$

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"Best" single point approximation

The "best" approximation with a single point x_1 is

$$\int_{-1}^{1} f(x)dx \approx w_1 f(x_1) = 2f(0)$$

- exact for linear functions
- close for functions which are approximately linear

What can we do with two points?

Question: A single point can integrate a linear function (1st order polynomial) exactly. What order of polynomial do you think we could integrate with *two* points?

What can we do with two points?

Claim: If $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$, then there exist $\underbrace{x_1, x_2, w_1, w_2}_{4 \text{ parameters}}$ such that

4 parameters

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2)$$

$$\int_{-1}^{1} (c_0 + c_1 x + c_2 x^2 + c_3 x^3)\, dx = \left[c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + c_3 \frac{x^4}{4} \right]_{-1}^{1}$$

$$= 2c_0 + \frac{2}{3} c_2$$

$$\Rightarrow \quad 2c_0 + \frac{2}{3}c_2 \overset{\text{set}}{=} w_1 f(x_1) + w_2 f(x_2)$$

$$= w_1(c_0 + c_1 x_1 + c_2 x_1^2 + c_3 x_1^3) + w_2(c_0 + c_1 x_2 + c_2 x_2^2 + c_3 x_2^3)$$

$$= c_0(w_1 + w_2) + c_1(w_1 x_1 + w_2 x_2) + c_2(w_1 x_1^2 + w_2 x_2^2) + c_3(w_1 x_1^3 + w_2 x_2^3)$$

$$\rightarrow \quad w_1 + w_2 = 2$$

$$w_1 x_1 + w_2 x_2 = 0$$

$$w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3}$$

$$w_1 x_1^3 + w_2 x_2^3 = 0$$

What can we do with two points?

Claim: If $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$, then there exist x_1, x_2, w_1, w_2 such that

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2)$$

<u>Solution</u> : $w_1 = w_2 = 1$ $\quad x_1 = \frac{-1}{\sqrt{3}}$ $\quad x_2 = \frac{1}{\sqrt{3}}$

$$\Rightarrow \int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \quad //$$

Example

$$f(x) = x^3 - 2x^2 + 3$$

$$\int_{-1}^{1} (x^3 - 2x^2 + 3)\,dx = \left[\frac{x^4}{4} - \frac{2}{3}x^3 + 3x \right]_{-1}^{1} = \frac{14}{3}$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = -\frac{1}{3\sqrt{3}} - \frac{2}{3} + 3$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3\sqrt{3}} - \frac{2}{3} + 3$$

$$\Rightarrow \quad f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 6 - \frac{4}{3} = \frac{14}{3} \quad \checkmark$$

Example

$$f(x) = x^3 - 2x^2 + 3$$

$$\int\limits_{-1}^{1} f(x)dx = \left(\frac{x^4}{4} - \frac{2x^3}{3} + 3x\right)\Bigg|_{-1}^{1} = \frac{14}{3}$$

$$f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = -\frac{1}{3\sqrt{3}} - \frac{2}{3} + 3 + \frac{1}{3\sqrt{3}} - \frac{2}{3} + 3 = \frac{14}{3}$$

Gaussian quadrature

General result: If $f(x) = \overbrace{c_0 + c_1 x + c_2 x^2 + \cdots + c_{2n-1} x^{2n-1}}^{2n \text{ parameters } c_0, c_1, \ldots, c_{2n-1}}$, then there exist **nodes** $\underbrace{x_1, \ldots, x_n}$ and **weights** $\underbrace{w_1, \ldots, \overset{w_n}{\cancel{x_n}}}$ such that

2n parameters

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

n-node Gaussian quadrature rule: For general function f,

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

Your turn

Practice questions on the course website:

https://sta379-s25.github.io/practice_questions/pq_20.html

- Try Gaussian quadrature with 3 nodes
- Start in class. You are welcome to work with others
- Practice questions are to help you practice. They are not submitted and not graded
- Solutions are posted on the course website