

Lecture 27: Antithetic variables

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Course logistics

- ▶ Project 2 due April 18
- ▶ No more HW until after project 2
- ▶ Next week:
 - ▶ Monday: project work day
 - ▶ Wednesday: begin EM algorithm

Warmup: variance reduction

Work with your neighbor on the questions on the handout / course website:

https://sta379-s25.github.io/practice_questions/pq_27_warmup.html

Then we will discuss as a class

- ▶ Use Monte Carlo integration to approximate another integral
- ▶ Explore variability of two different estimators

Warmup

$$\theta = \int_{-\infty}^{\infty} \frac{x}{2^x - 1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \mathbb{E}[g(X)]$$

Basic Monte Carlo estimator:

- ▶ Sample $X_1, \dots, X_n \stackrel{iid}{\sim} N(0, 1)$
- ▶ $\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^n g(X_i)$

Question: From your Monte Carlo integration, what is the approximate value of θ ?

Warmup

```
g <- function(x){  
  x/(2^x - 1)  
}  
n <- 1000; nsim <- 1000  
theta_hat1 <- rep(NA, nsim)  
  
for(i in 1:nsim){  
  x <- rnorm(n)  
  theta_hat1[i] <- mean(g(x))  
}  
  
mean(theta_hat1)  
  
## [1] 1.498544  
  
var(theta_hat1)  
  
## [1] 0.0002586099
```

Warmup

$$\hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^{n/2} (g(X_i) + g(-X_i))$$

Questions:

- ▶ How does $\mathbb{E}[\hat{\theta}_2]$ compare to $\mathbb{E}[\hat{\theta}_1]$?
- ▶ How does $\text{Var}(\hat{\theta}_2)$ compare to $\text{Var}(\hat{\theta}_1)$?

Warmup

```
theta_hat2 <- rep(NA, nsim)
```

```
for(i in 1:nsim){  
  x <- rnorm(n/2)  
  theta_hat2[i] <- sum(g(x) + g(-x))/n  
}
```

```
mean(theta_hat2)
```

```
## [1] 1.499174
```

```
var(theta_hat2)
```

```
## [1] 1.273848e-05
```

```
(var(theta_hat1) - var(theta_hat2))/var(theta_hat1) * 100
```

```
## [1] 95.07425
```

Warmup

$$\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^n g(X_i) \quad \hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^{n/2} (g(X_i) + g(-X_i))$$

where $X_1, \dots, X_n \stackrel{iid}{\sim} N(0, 1)$

► $\mathbb{E}[\hat{\theta}_1] = \theta \quad \text{Var}(\hat{\theta}_1) = \frac{\text{Var}(g(X))}{n}$

► $\mathbb{E}[\hat{\theta}_2] =$

► $\text{Var}(\hat{\theta}_2) =$

Reducing variation

Let Y, Y^* be two random variables with the same distribution, with mean μ and variance σ^2 .

► $\mathbb{E}\left(\frac{Y + Y^*}{2}\right) = \mu$

► If Y, Y^* are independent, then

$$\text{Var}\left(\frac{Y + Y^*}{2}\right) = \frac{1}{4}(\text{Var}(Y) + \text{Var}(Y^*)) = \frac{\sigma^2}{2}$$

► If Y, Y^* are correlated, with correlation ρ , then

$$\text{Var}\left(\frac{Y + Y^*}{2}\right) = \frac{1}{4}(\text{Var}(Y) + \text{Var}(Y^*) + 2\text{Cov}(Y, Y^*)) = \frac{\sigma^2}{2} + \frac{\rho\sigma^2}{2}$$

Reducing variation: antithetic variables

Suppose that we want to estimate

$$\theta = \mathbb{E}[g(X)] \quad X \sim N(0, 1)$$

- ▶ Sample $X_1, \dots, X_{n/2} \stackrel{iid}{\sim} N(0, 1)$
- ▶ Antithetic Monte Carlo estimate:

$$\hat{\theta}_{AS} = \frac{1}{n} \sum_{i=1}^{n/2} (g(X_i) + g(-X_i))$$

- ▶ $\mathbb{E}[\hat{\theta}_{AS}] = \theta \quad \text{Var}(\hat{\theta}_{AS}) = \frac{(1 + \rho) \text{Var}(g(X))}{n}$ where
 $\rho = \text{Cor}(g(X), g(-X))$

- ▶ **Theorem:** If g is a *monotone* function, then $\rho \leq 0$ (and so we reduce the variance)

Another example

Suppose we want to estimate

$$\theta = \int_0^1 \log(x+1)e^x dx$$

Question: How would we do this with the basic Monte Carlo approach we learned previously?

Antithetic variables

$$\theta = \int_0^1 \log(x+1)e^x dx = \mathbb{E}[g(U)] \quad U \sim \text{Uniform}(0,1)$$

Antithetic sampling:

- ▶ **Theorem:** If g is *monotone*, then $\text{Cor}(g(U), g(1-U)) \leq 0$
- ▶ Sample $U_1, \dots, U_{n/2} \stackrel{iid}{\sim} \text{Uniform}(0,1)$
- ▶ $\hat{\theta}_{AS} = \frac{1}{n} \sum_{i=1}^{n/2} (g(U_i) + g(1-U_i))$
- ▶ $\text{Var}(\hat{\theta}_{AS}) = \frac{(1+\rho)\text{Var}(g(X))}{n}$ where $\rho = \text{Cor}(g(U), g(1-U))$