Lecture 8: Acceptance-rejection sampling continued

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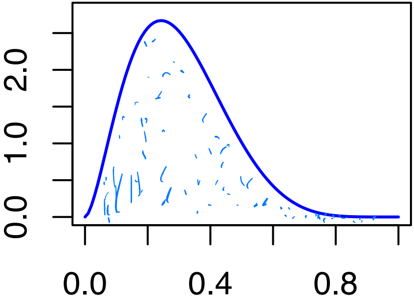
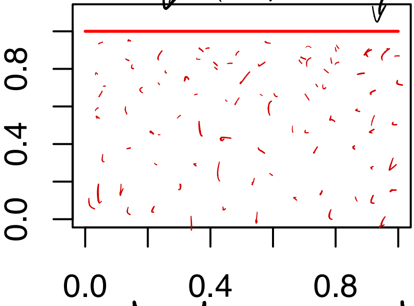
Acceptance-rejection sampling

Suppose we want to simulate from this Beta distribution:



We can *get* samples from this uniform distribution:



accept most of these points

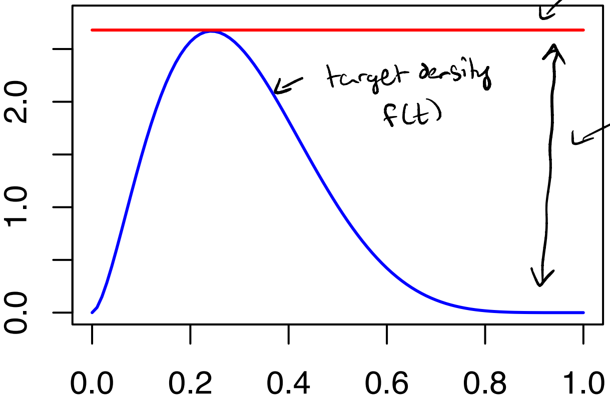
reject most of these points

this region is more likely for our Beta

more samples over here than if we had this Beta

Acceptance-rejection sampling

$c \cdot g(t)$



look at ratio between

$$\frac{f(t)}{cg(t)}$$

if ratio is small, reject most of the points

if ratio is large, keep most of the points

Acceptance-rejection sampling

target density f

candidate density

- Want to sample continuous r.v. $X \sim f$
- Can easily sample from a different density: $Y \sim g$, such that

$$\frac{f(t)}{g(t)} \leq c \qquad \text{for all } t \text{ where } f(t) > 0$$

(f and g are "similar enough")

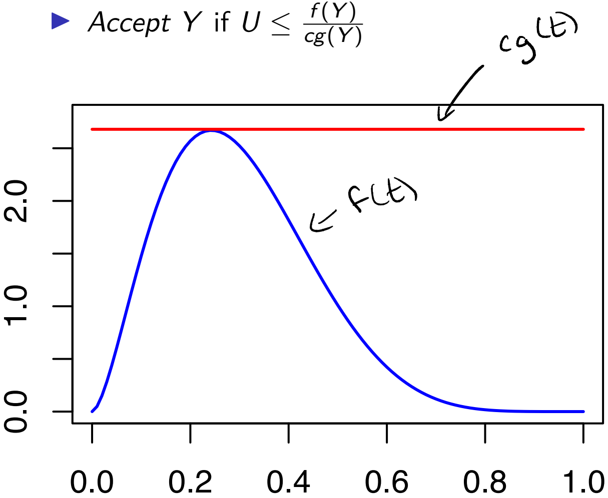
ideally: g is as close as possible to f, so we reject as few points as possible

Do the following:

1. Sample $Y \sim g$
2. Sample $U \sim Uniform(0, 1)$
3. If $U \leq \dfrac{f(Y)}{cg(Y)}$, set $\underbrace{X = Y}_{\text{accept sample}}$. Otherwise, return to step 1. (reject sample)

Illustration

- $Y \sim g$ and $U \sim Uniform(0, 1)$
- $Accept\ Y$ if $U \leq \frac{f(Y)}{cg(Y)}$



· need to choose g (here we chose uniform)

· need to find value of c

Finding c

Acceptance-rejection sampling requires that

$$\frac{f(t)}{g(t)} \leq c \qquad \text{for all } t \text{ where } f(t) > 0$$

So,

$$c = \max_{t:f(t)>0} \frac{f(t)}{g(t)}$$

If we have $g(t)$, we can find $\frac{f(t)}{g(t)}$

and then we maximize to find c

Finding c: example

$$\frac{\partial}{\partial t} u(t)v(t) = u'(t)v(t) + v'(t)u(t)$$

Example from last time:

- $f(t) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1}$
- $g(t) = 1$

$$c = \max_{t:f(t)>0} \frac{f(t)}{g(t)}$$

$$\frac{f(t)}{g(t)} = \frac{\overbrace{\Gamma(\alpha+\beta)}^{\text{doesn't depend on } t}}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1} \leftarrow \text{want to } \underline{\text{maximize}}$$

Assume $\alpha, \beta > 1$

$$\frac{\partial}{\partial t} t^{\alpha-1}(1-t)^{\beta-1} = (\alpha-1)t^{\alpha-2}(1-t)^{\beta-1} + (\beta-1)(1-t)^{\beta-2}(-1)t^{\alpha-1}$$

$$= (\alpha-1)t^{\alpha-2}(1-t)^{\beta-1} - (\beta-1)(1-t)^{\beta-2}t^{\alpha-1} \overset{\text{set}}{=} 0$$

$$\Rightarrow (\alpha-1)t^{\alpha-2}(1-t)^{\beta-1} = (\beta-1)t^{\alpha-1}(1-t)^{\beta-2}$$

$$\Rightarrow (\alpha-1)(1-t) = (\beta-1)t$$

$$\Rightarrow (\alpha-1) - (\alpha-1)t = (\beta-1)t$$

$$\Rightarrow t = \frac{\alpha-1}{\alpha+\beta-2}$$

So: $\max \frac{f(t)}{g(t)}$ occurs at $t_{max} = \frac{\alpha-1}{\alpha+\beta-2}$

$$\Rightarrow \quad c = \max \frac{f(t)}{g(t)} = \frac{f\left(\frac{\alpha-1}{\alpha+\beta-2}\right)}{g\left(\frac{\alpha-1}{\alpha+\beta-2}\right)}$$

evaluate this in R!

Why does acceptance-rejection sampling work?

Suppose we want to generate X with pdf f and cdf F, using acceptance-rejection sampling

Know: target pdf is f, candidate g, generate $Y \sim g$

want: f to be pdf of X

1) $Y \sim g$
2) $U \sim U(0,1)$
3) set $X = Y$ if $U \le \frac{f(Y)}{cg(Y)}$, else go back to step 1

Prove: $P(X \le t) = F(t)$

① Note: $X = Y$ when Y accepted, i.e. $U \le \frac{f(Y)}{cg(Y)}$

$$P(X \le t) = P\left(Y \le t \,\middle|\, U \le \frac{f(Y)}{cg(Y)}\right)$$

② Recall: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$$P(X \le t) = P\left(Y \le t \,\middle|\, U \le \frac{f(Y)}{cg(Y)}\right)$$

Recall: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$$P\left(Y \le t \,\middle|\, U \le \frac{f(Y)}{cg(Y)}\right) = \frac{P\left(Y \le t \underline{\text{ and }} U \le \frac{f(Y)}{cg(Y)}\right)}{P\left(U \le \frac{f(Y)}{cg(Y)}\right)}$$

3) Recall: Law of total probability

Discrete case: if $B_1, \dots, B_n$ partition space

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

Continuous case: $P(A) = \displaystyle\int_{-\infty}^{\infty} P(A|V=v) f_V(v)\,dv$

any random variable V

$$P\left(U \le \frac{f(Y)}{cg(Y)}\right) = \int_{-\infty}^{\infty} P\left(U \le \frac{f(y)}{cg(y)} \,\middle|\, Y=y\right) g(y)\,dy$$

$$P\left(U \le \frac{f(Y)}{cg(Y)}\right) = \int_{-\infty}^{\infty} \underbrace{P\left(U \le \frac{f(y)}{cg(y)} \,\middle|\, Y=y\right)}_{U \text{ is uniform } (0,1)} g(y)\,dy$$

$$P(U \le u) = u \qquad\qquad P\left(U \le \frac{f(y)}{cg(y)}\right) = \frac{f(y)}{cg(y)}$$

$$= \int_{-\infty}^{\infty} \frac{f(y)}{c\cancel{g(y)}} \cdot \cancel{g(y)}\,dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{c} f(y)\,dy = \frac{1}{c} \underbrace{\int_{-\infty}^{\infty} f(y)\,dy}_{=1}$$

$$= \frac{1}{c}$$

$$P\left(Y \le t \text{ and } U \le \frac{f(Y)}{cg(Y)}\right) = \int_{-\infty}^{t} P\left(U \le \frac{f(y)}{cg(y)}\right) g(y)\,dy$$

$$= \int_{-\infty}^{t} \frac{f(y)}{c\cancel{g(y)}} \cancel{g(y)}\,dy = \underbrace{\frac{1}{c}\int_{-\infty}^{t} f(y)\,dy}_{F(t)}$$

$$= \frac{1}{c} F(t)$$

$$\frac{P\left(Y \le t \text{ and } U \le \frac{f(Y)}{cg(Y)}\right)}{P\left(U \le \frac{f(Y)}{cg(Y)}\right)} = \frac{\frac{1}{c}F(t)}{\frac{1}{c}} = F(t)$$

$$\Rightarrow \quad P(X \le t) = F(t)$$

i.e. X really does have cdf F! //

Your turn

Practice questions on the course website:

https://sta379-s25.github.io/practice_questions/pq_8.html

- Implement acceptance-rejection sampling for the beta example
- Start in class. You are welcome to work with others
- Practice questions are to help you practice. They are not submitted and not graded
- Solutions are posted on the course website
- If done early, start on HW 3

Next time:

- Challenges with acceptance-rejection sampling
- Other transformation methods to generate random variables