Lecture 7: Generating random variables – acceptance-rejection sampling

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Recap: Inverse transform method

Suppose X is a random variable with cdf F. Let

$$F^{-1}(u) = \inf\{t : F(t) \geq u\}$$

(If F invertible, this is the usual inverse)

1. Generate $U \sim \mathit{Uniform}(0, 1)$
2. Let $X = F^{-1}(U)$

Then, $X \sim F$

Today: How else can we generate random variables?

Example

Suppose we would like to generate $X \sim Beta(\alpha, \beta)$

$$f_X(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad x \in (0,1)$$

- Inverse transform method: $F_X(t) = ?$ $\int_0^t f_X(x)\,dx$

No closed form solution

except in special cases

Acceptance-rejection sampling: motivation

- **Want:** $X \sim Beta(\alpha, \beta)$, $f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

However, it may be difficult to directly simulate from this distribution. Can you think of another distribution on $(0,1)$ which is *easier* to simulate?

Uniform $(0,1)$!

Want: $X \sim Beta(\alpha, \beta)$

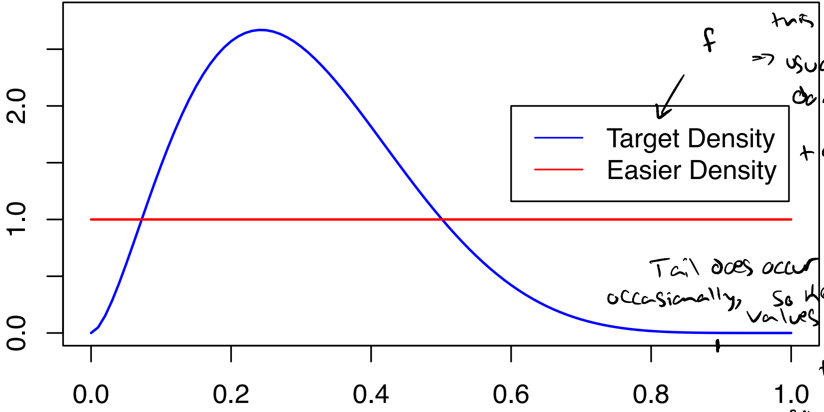
can easily get: $Y \sim U(0,1)$

Idea:
- Generate $Y \sim U(0,1)$
- If Y looks like what we might expect from Beta, keep and set $X = Y$
- If Y does not match $Beta(\alpha, \beta)$ try again

Acceptance-rejection sampling: motivation

Intuition: 0.9 is in the tail ⟹ unlikely to occur in this Beta ⟹ usually don't want to keep this sample

- **Want:** $X \sim Beta(\alpha, \beta)$, $f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$
- **Can get:** $Y \sim Uniform(0, 1)$

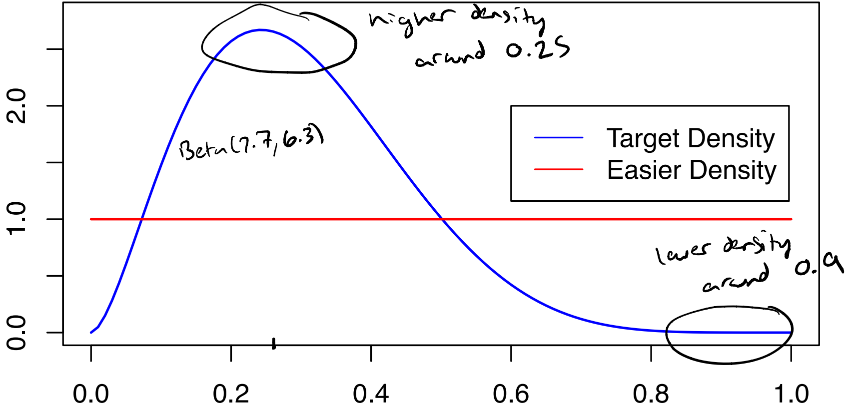


Tail does occur occasionally, so keep values in the tail sometimes

Suppose we sample $Y \sim Uniform(0, 1)$ and observe $y = 0.9$. Is it likely we would observe that draw from the Beta distribution shown here? No! Probably don't keep $Y = 0.9$, try again

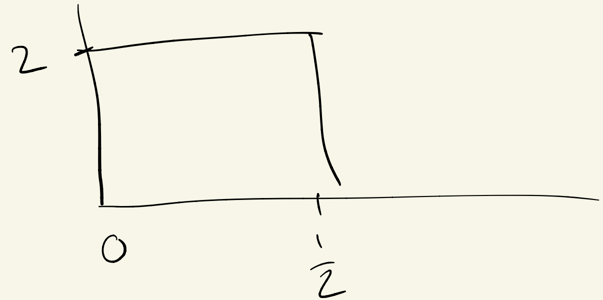
Acceptance-rejection sampling: motivation

- **Want:** $X \sim Beta(\alpha, \beta)$, $f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$
- **Can get:** $Y \sim Uniform(0, 1)$



Suppose we sample $Y \sim Uniform(0, 1)$ and observe $y = 0.25$. Is it likely we would observe that draw from the Beta distribution shown here? Yes! Probably keep $Y = 0.25$, $X = Y$

<u>Sidebar</u>: $V \sim U(0, \frac{1}{2})$



Fine for pdf > 1, just has to integrate to 1

Acceptance-rejection sampling

Support of f:

$\{t : f(t) > 0\}$

i.e. support of Beta $\to (0,1)$

Suppose we would like to generate a continuous random variable X with pdf f. Let Y be another continuous r.v., with density g, such that

1) Y is easy to simulate

2) $\exists\ c > 0$ (constant) st $\dfrac{f(t)}{g(t)} \le c$

for all t st $f(t) > 0$

Then: to generate $X \sim f$, do the following:

1) Sample $Y \sim g$

2) Sample $U \sim U(0,1)$

3) If $U \le \dfrac{f(Y)}{c\,g(Y)}$, set $X = Y$ (accept)

otherwise, go back to 1 (reject)

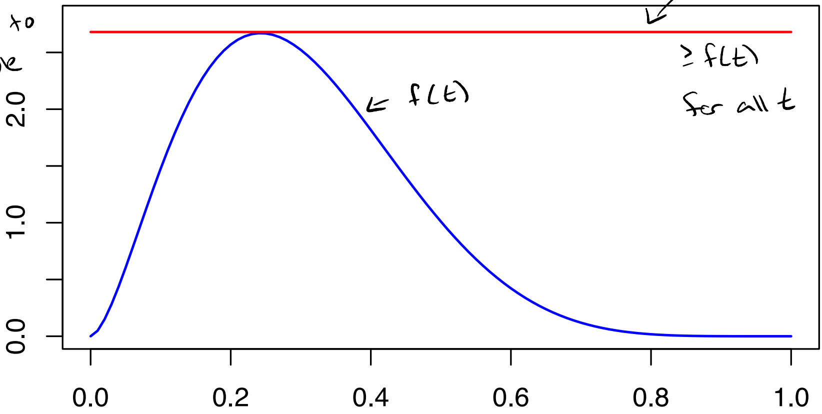
Illustration

want to sample from

Target density: $f(t) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1},\ t \in (0,1)$

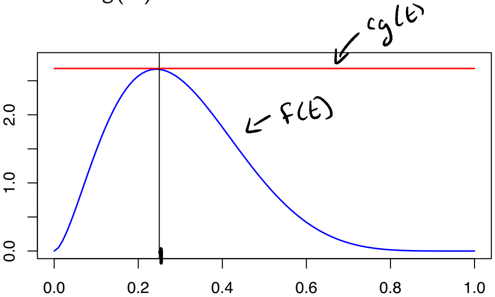
Candidate density: $g(t) = 1,\ t \in (0,1)$

easy to sample from



Illustration

Now sample $Y \sim g$ and $U \sim Uniform(0,1)$. *Accept* Y if $U \leq \frac{f(Y)}{cg(Y)}$



$$\frac{f(0.25)}{cg(0.25)} \approx 1$$

$u \sim U(0,1)$

accept if $u \leq \frac{f(0.25)}{cg(0.25)} \approx 1$

$$P\left(u \leq \frac{f(0.25)}{cg(0.25)}\right) \approx 1$$

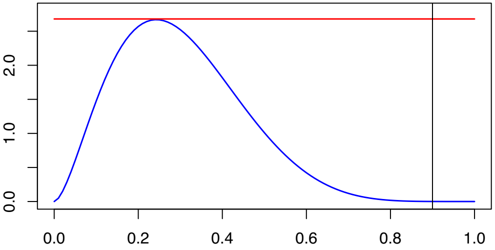
Suppose we observe $Y = 0.25$. Are we likely to *accept* or *reject* 0.25 as a sample from f?

$\Rightarrow P(\text{accept } Y \mid Y = 0.25) \approx 1$

probably keep 0.25!

Illustration

Now sample $Y \sim g$ and $U \sim \mathit{Uniform}(0,1)$. *Accept* Y if $U \leq \frac{f(Y)}{cg(Y)}$



$$\frac{f(0.9)}{cg(0.9)} \approx 0$$

$$P\left(U \leq \frac{f(0.9)}{cg(0.9)}\right) \approx 0$$

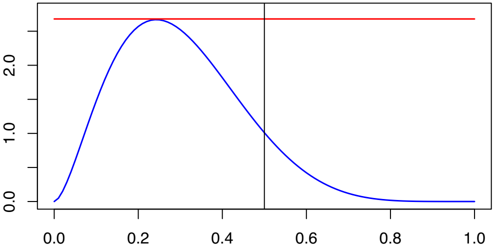
$$\Rightarrow P(\text{accept} \mid Y = 0.9) \approx 0$$

Suppose we observe $Y = 0.9$. Are we likely to *accept* or *reject* 0.9 as a sample from f?

probably reject 0.9!

Illustration

Now sample $Y \sim g$ and $U \sim \textit{Uniform}(0,1)$. *Accept* Y if $U \leq \frac{f(Y)}{cg(Y)}$



$\frac{f(0.5)}{cg(0.5)} \approx 0.38$

$P\left(U \leq \frac{f(0.5)}{cg(0.5)}\right) \approx 0.38$

often reject 0.5, but sometimes accept

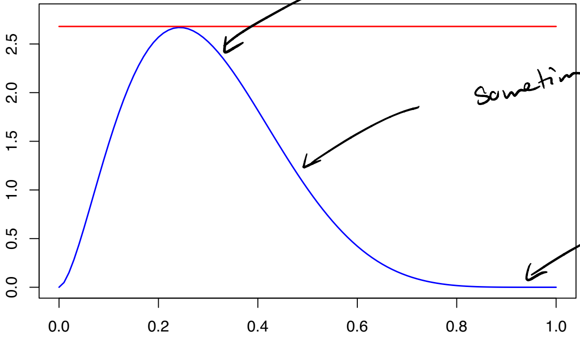
Suppose we observe $Y = 0.5$. Are we likely to *accept* or *reject* 0.5 as a sample from f?

(about $\frac{1}{3}$ of the time)

Illustration

- $Y \sim g$ and $U \sim Uniform(0, 1)$
- *Accept* Y if $U \leq \frac{f(Y)}{cg(Y)}$

mostly accept over here



Sometimes accept, Sometimes reject

mostly reject

General idea: Accept Y more often when Y better matches distribution f

Your turn

Practice questions on the course website:

https://sta379-s25.github.io/practice_questions/pq_7.html

- Implement acceptance-rejection sampling for the beta example
- Start in class. You are welcome to work with others
- Practice questions are to help you practice. They are not submitted and not graded
- Solutions are posted on the course website

Next time: formal proof that this sampling procedure works!