Warmup: Variance reduction in Monte Carlo integration

Group members:

Monte Carlo integration

Suppose we wish to approximate the integral

$$\theta = \int_{-\infty}^{\infty} \frac{x}{2^x - 1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \mathbb{E}[g(X)]$$

where $X \sim N(0,1)$ and $g(x) = \dfrac{x}{2^x - 1}$.

The basic Monte Carlo approach is to do the following:

- Sample $X_1, ..., X_n \overset{iid}{\sim} N(0,1)$
- $\widehat{\theta}_1 = \frac{1}{n} \sum\limits_{i=1}^{n} g(X_i)$

Questions

1. Use Monte Carlo integration with $n = 1000$ to estimate θ. Repeat the process many times to approximate $\mathbb{E}[\widehat{\theta}_1]$ and $Var(\widehat{\theta}_1)$. Write down the approximate values of $\mathbb{E}[\widehat{\theta}_1]$ and $Var(\widehat{\theta}_1)$.

A different Monte Carlo estimator

Now consider the estimator

$$\widehat{\theta}_2 = \frac{1}{n} \sum_{i=1}^{n/2} (g(X_i) + g(-X_i))$$

where $X_1, ..., X_{n/2} \sim N(0,1)$ as before.

2. Use $n = 1000$ (so $n/2 = 500$) to compute this second estimator $\widehat{\theta}_2$. Repeat the process many times to approximate $\mathbb{E}[\widehat{\theta}_2]$ and $Var(\widehat{\theta}_2)$.

3. What is the percent reduction in variance of $\widehat{\theta}_2$ compared to $\widehat{\theta}_1$? Remember that the percent reduction in variance is calculated by

$$100 \cdot \frac{Var(\widehat{\theta}_1) - Var(\widehat{\theta}_2)}{Var(\widehat{\theta}_1)}$$

4. Using R, estimate the correlation $cor(g(X_i), g(-X_i))$ with $X_i \sim N(0,1)$ and g given as above (the `cor` function will be helpful). How is the correlation related to the percent reduction in variance?