

①

Ex 1. Derive Bayes' rule:

$$p(H_i | X) = \frac{p(H_i, X)}{p(X)} \quad \text{by P3}$$

$$= \frac{p(X | H_i) p(H_i)}{p(X)} \quad \text{by P3}$$

$$= \frac{p(X | H_i) p(H_i)}{\sum_k p(X | H_k) p(H_k)} \quad \text{by rule of marginal prob.}$$

Ex 2 : show $F \perp G | H \Rightarrow p(F | H, G) = p(F | H)$

(i) $p(F, G | H) = p(F | H) p(G | H)$ by definition
but also

(ii) $p(F, G | H) = p(F | G, H) p(G | H)$ by P3

matching up (i) & (ii) :

$$p(F | G, H) \cancel{p(G | H)} = p(F | H) \cancel{p(G | H)}$$

Support set of values a r.v. can take.

$$X \sim \text{binomial}(n, \theta)$$

$$X \in \{0, \dots, n\}$$

(2)

Exercise: identify the kernel

gamma kernel: $x^{\alpha-1} e^{-\beta x}$

Exercise:

$$\int_0^{\infty} x^{\alpha-1} e^{-\beta x} dx = ?$$

$$\frac{\Gamma(\alpha)}{\beta^{\alpha}}$$

$$\beta^{\alpha}$$

Law of total expectation

$$E E(X|\theta) \quad p(\theta)$$

$$= \int \left[\int x p(x|\theta) dx \right] p(\theta) d\theta$$

$$= \int x \int p(x|\theta) p(\theta) d\theta dx$$

$$= \int x \int p(x, \theta) d\theta dx$$

$$= \int x p(x) dx$$

by rule of marginal prob.

$$= EX \quad \square$$

3

Defin exchangeable (subscripts don't matter)

Let $p(y_1, \dots, y_n)$ be the joint density of Y_1, \dots, Y_n . If $p(y_1, \dots, y_n) = p(y_{\pi_1}, \dots, y_{\pi_n})$ for all permutations π of $\{1, \dots, n\}$ then Y_1, \dots, Y_n are exchangeable.

Ex 1: Urn with 2 red, 1 green

$$\begin{aligned} p(Y_1 = \text{red}, Y_2 = \text{green}) &= p(Y_1 = \text{red}) \cdot P(Y_2 = \text{green} | Y_1 = \text{red}) \\ &= \frac{2}{3} \cdot \frac{1}{2} \\ &= \frac{2}{6} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} p(Y_1 = \text{green}, Y_2 = \text{red}) &= p(Y_1 = \text{green}) \cdot P(Y_2 = \text{red} | Y_1 = \text{green}) \\ &= \frac{1}{3} \cdot 1 \\ &= \frac{1}{3} \end{aligned}$$

Y_1, Y_2 are exchangeable even though ~~not~~ independent.

Ex 2:

coin 1 is a fair coin

coin 2 is double sided (heads only)

$$\Pr(Y_1 = H) = 0.5$$

$$\Pr(Y_2 = H) = 1$$

$$p(0, 1) = 0.5$$

$$p(1, 0) = 0$$

Y_1, Y_2 are not exchangeable.

(4)

Claim:

If $\theta \sim p(\theta)$ and Y_1, \dots, Y_n are conditionally iid given θ , then marginally (unconditional on θ) Y_1, \dots, Y_n are exchangeable.

Proof:

$$\begin{aligned}
 p(y_1, \dots, y_n) &= \int p(y_1, \dots, y_n | \theta) p(\theta) d\theta \quad \text{by rule of marginal prob.} \\
 &= \int \left\{ \prod_{i=1}^n p(y_i | \theta) \right\} p(\theta) d\theta \quad \text{by cond'l iid} \\
 &= \int \left\{ \prod_{i=1}^n p(y_{\pi_i} | \theta) \right\} p(\theta) d\theta \quad \text{products commute} \\
 &= p(y_{\pi_1}, \dots, y_{\pi_n})
 \end{aligned}$$

de Finetti's thm:

exchangeable $Y_1, \dots, Y_n \quad \forall n$

$\Rightarrow Y_1, \dots, Y_n | \theta$ iid (for some parameter θ)
and prior distribution $p(\theta)$.

• very cool because exchangeability is common!

$Y_1, \dots, Y_n \rightarrow$ from repeatable experiment
 \rightarrow sampled w/ replacement
 $\rightarrow \infty$ pop'n w/o replacement

①

$$X_i = \begin{cases} 1 & (H) \\ 0 & (T) \end{cases}$$

$X_1 \perp X_2$?

$$P(X_{100} = 1 \mid X_1 = 0, X_2 = 0, \dots, X_{99} = 0) \begin{matrix} > .5 \\ < .5 \\ = .5 ? \end{matrix}$$

So $X_i \not\perp X_j$

But exchangeable seems plausible.

$\Rightarrow X_i \perp X_j \mid \theta$ & identically distributed.

therefore $p(X_1, \dots, X_n \mid \theta) = p(X_1 \mid \theta) p(X_2 \mid \theta) \dots p(X_n \mid \theta)$.

What could θ be here?

Ex:
$$\begin{aligned} p(X_i = 1 \mid \theta) &= \theta \\ p(X_i = 0 \mid \theta) &= (1 - \theta) \end{aligned}$$

Together,
$$p(X_i = x_i \mid \theta) = \theta^{x_i} (1 - \theta)^{1 - x_i}$$

Exercise: write the joint density:

$$p(X_1, \dots, X_n \mid \theta)$$

Sol'n:
$$\theta^{\sum x_i} (1 - \theta)^{n - \sum x_i}$$

The *likelihood* is the joint density of the data, viewed as a function of the parameters.

Notice that evaluation of the likelihood only requires us to know the following about the data:

- the sum total number of heads
- the total number of coin flips, n