

(1)

Ex 1. Derive Bayes' rule:

$$p(H_i | X) = \frac{p(H_i, X)}{p(X)} \quad \text{by P3}$$

$$= \frac{p(X|H_i)p(H_i)}{p(X)} \quad \text{by P3}$$

$$= \frac{p(X|H_i)p(H_i)}{\sum_k p(X|H_k)p(H_k)} \quad \text{by rule of marginal prob.}$$

Ex 2 : show $F \perp G | H \Rightarrow p(F|H, G) = p(F|H)$

(†) $p(F, G | H) = p(F | H)p(G | H)$ by definition
but also

(††) $p(F, G | H) = p(F | G, H)p(G | H)$ by P3

matching up (†) & (††) :

$$p(F | G, H)p(G | H) = p(F | H)p(G | H)$$

Support set of values a r.v. can take.

$$X \sim \text{binomial}(n, \theta)$$

$$X \in \{0, \dots, n\}$$

(2)

Exercise: identify the kernel

gamma kernel: $x^{\alpha-1} e^{-\beta x}$

Exercise:

$$\int_0^\infty x^{\alpha-1} e^{-\beta x} dx = ?$$

$$\Gamma(\alpha)$$

$$\beta^\alpha$$

Law of total expectation

$$\begin{aligned}
 & E E(X|\theta) \\
 &= \int \left[\int x p(x|\theta) dx \right] p(\theta) d\theta \\
 &= \int x \underbrace{\int p(x|\theta) p(\theta) d\theta}_{p(x)} dx \\
 &= \int x \underbrace{\int p(x|\theta) d\theta}_{p(x)} dx \\
 &= \int x p(x) dx \quad \text{by rule of marginal prob.} \\
 &= EX \quad \square
 \end{aligned}$$

(3)

Defin exchangeable (subscripts don't matter)

Let $p(y_1, \dots, y_n)$ be the joint density of Y_1, \dots, Y_n . If $p(y_1, \dots, y_n) = p(y_{\pi(1)}, \dots, y_{\pi(n)})$, for all permutations π of $\{1, \dots, n\}$ then Y_1, \dots, Y_n are exchangeable.

Ex 1: Urn with 2 red, 1 green

$$\begin{aligned} p(Y_1 = \text{red}, Y_2 = \text{green}) &= p(Y_1 = \text{red}) \cdot p(Y_2 = \text{green} | Y_1 = \text{red}) \\ &= \frac{2}{3} \cdot \frac{1}{2} \\ &= \frac{2}{6} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} p(Y_1 = \text{green}, Y_2 = \text{red}) &= p(Y_1 = \text{green}) \cdot p(Y_2 = \text{red} | Y_1 = \text{green}) \\ &= \frac{1}{3} \cdot 1 \\ &= \frac{1}{3} \end{aligned}$$

Y_1, Y_2 are exchangeable even though not independent.

Ex 2:

coin 1 is a fair coin

coin 2 is double sided (heads only)

$$Pr(Y_1 = H) = 0.5$$

$$Pr(Y_2 = H) = 1$$

$$p(0, 1) = 0.5$$

$$p(1, 0) = 0$$

Y_1, Y_2 are not exchangeable

(4)

Claim:

If $\theta \sim p(\theta)$ and Y_1, \dots, Y_n are conditionally iid given θ , then marginally (unconditional on θ) Y_1, \dots, Y_n are exchangeable.

Proof:

$$\begin{aligned}
 p(y_1, \dots, y_n) &= \int p(y_1, \dots, y_n | \theta) p(\theta) d\theta \quad \text{by rule of marginal prob.} \\
 &= \int \left\{ \prod_{i=1}^n p(y_i | \theta) \right\} p(\theta) d\theta \quad \text{by cond'l iid} \\
 &= \int \left\{ \prod_{i=1}^n p(y_{\pi(i)} | \theta) \right\} p(\theta) d\theta \quad \text{products commute} \\
 &= p(y_{\pi(1)}, \dots, y_{\pi(n)})
 \end{aligned}$$

de Finetti's thm:

exchangeable Y_1, \dots, Y_n \Leftrightarrow

$\Rightarrow Y_1, \dots, Y_n | \theta$ iid (for some parameter θ)
 and prior distribution $p(\theta)$.

• very cool because exchangeability is common!

$Y_1, \dots, Y_n \rightarrow$ from repeatable experiment
 → sample w/o replacement
 → ∞ population w/o replacement

(1)

$$X_i = \begin{cases} 1 & (\text{H}) \\ 0 & (\text{T}) \end{cases}$$

$X_1 \perp X_2 ?$

$$\begin{aligned} P(X_{100} = 1 \mid X_1 = 0, X_2 = 0, \dots, X_{99} = 0) &> .5 \\ &< .5 \\ &= .5 ? \end{aligned}$$

So $X_i \cancel{\perp} X_j$

But exchangeable seems plausible.

$\Rightarrow X_i \perp X_j \mid \theta$ & identically distributed.

$$\text{therefore } p(x_1, \dots, x_n \mid \theta) = p(x_1 \mid \theta) p(x_2 \mid \theta) \cdots p(x_n \mid \theta)$$

What could θ be here?

$$\text{Ex: } p(x_i = 1 \mid \theta) = \theta$$

$$p(x_i = 0 \mid \theta) = (1-\theta)$$

$$\text{Together, } p(x_i = x_i \mid \theta) = \theta^{x_i} (1-\theta)^{1-x_i}$$

Exercise: write the joint density:

$$p(x_1, \dots, x_n \mid \theta)$$

$$\text{Soln: } \theta^{\sum x_i} (1-\theta)^{n-\sum x_i}$$

The *likelihood* is the joint density of the data, viewed as a function of the parameters.

Notice that evaluation of the likelihood only requires us to know the following about the data:

- the sum total number of heads
- the total number of coin flips, n