Neyman-Pearson lemma

Wald test for normal mean

Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \qquad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$.

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The Wald test rejects if

$$\overline{X}_n > \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$$

We know that $\beta(\mu_0) = \alpha$ for this test.

> Does there exist a different test, with power function $\beta^*(\mu)$, such that $\beta^*(\mu_0) \leq \alpha$ and $\beta^*(\mu_1) > \beta(\mu_1)$?

Rearranging the Wald test for a population mean

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$$H_0 : \mu = \mu_0 \qquad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$.

The Wald test rejects if $\overline{X}_n > t_0$, which is equivalent to rejecting when

$$\frac{L(\mu_1|\mathbf{X})}{L(\mu_0|\mathbf{X})} = \frac{f(X_1, \ldots, X_n|\mu_1)}{f(X_1, \ldots, X_n|\mu_0)} > k_0$$

Intuition: Reject H_0 if the likelihood of μ_1 is sufficiently greater than the likelihood of μ_0.

Neyman-Pearson test

Neyman-Pearson lemma

Example

Let $\mathbf{X} = X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \qquad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$.

The Wald test rejects when

$$\frac{L(\mu_1|\mathbf{X})}{L(\mu_0|\mathbf{X})} > k,$$

where k is chosen such that $\beta(\mu_0) = \alpha$.

Example

Let $X_1, \ldots, X_n \overset{iid}{\sim} Exponential(\theta)$, with pdf $f(x|\theta) = \theta e^{-\theta x}$. We want to test

$$H_0 : \theta = \theta_0 \qquad H_A : \theta = \theta_1,$$

where $\theta_1 < \theta_0$. The Neyman-Pearson test rejects when

$$\frac{L(\theta_1|\mathbf{X})}{L(\theta_0|\mathbf{X})} > k.$$

Find k such that the test has size α.