STA 711 Homework 5

Due: Friday, February 21, 12:00pm (noon) on Canvas.

Instructions: Submit your work as a single PDF. For this assignment, you may include written work by scanning it and incorporating it into the PDF. Include all R code needed to reproduce your results in your submission.

Central limit theorem with estimated variance

The central limit theorem tells us that if $Y_1, Y_2, ...$ is a sequence of iid random variables, then

$$\frac{\sqrt{n}(\overline{Y}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1),$$

where $\overline{Y}_n = \frac{1}{n}\sum\limits_{i=1}^{n} Y_i$, $\mu = \mathbb{E}[Y_i]$, and $\sigma^2 = Var(Y_i)$. This limiting distribution is useful when we want to construct confidence intervals and tests for μ, but it requires us to know σ^2. When σ^2 is unknown, we replace it with an estimate. Two possible estimators of σ^2 are:

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y}_n)^2$$
$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y}_n)^2$$

5. Our goal is to show that using $\widehat{\sigma}^2$ or s^2 in place of σ^2 does not change our limiting normal distribution. For the purposes of this problem, suppose that $Y_1, Y_2, ...$ is a sequence of iid random variables, and that the moment generating function of Y_i exists in a neighborhood of 0.

   (a) Show that $\widehat{\sigma}^2 \xrightarrow{p} \sigma^2$ and $s^2 \xrightarrow{p} \sigma^2$.

   (b) Show that

   $$\frac{\sqrt{n}(\overline{Y}_n - \mu)}{\widehat{\sigma}} \xrightarrow{d} N(0,1)$$

   and

   $$\frac{\sqrt{n}(\overline{Y}_n - \mu)}{s} \xrightarrow{d} N(0,1).$$

   (c) If both $\widehat{\sigma}^2$ and s^2 can be used as estimates of the population variance σ^2, why do we have two estimates? The reason is that $\widehat{\sigma}^2$ is a *biased* estimator of σ^2 (that is, $\mathbb{E}[\widehat{\sigma}^2] \neq \sigma^2$), whereas s^2 is *unbiased* (that is, $\mathbb{E}[s^2] = \sigma^2$). Later in the course we will discuss the bias of estimators in more detail.

   Calculate $\mathbb{E}[\widehat{\sigma}^2]$ and $\mathbb{E}[s^2]$.