STA 711 Exam 1 Make-up

Due: Friday, March 3, 12:00pm (noon) on Canvas.

Instructions: Submit your work as a single PDF. For this assignment, you may include written work by scanning it and incorporating it into the PDF. Include all R code needed to reproduce your results in your submission.

Rules: This is a closed-book, closed-notes exam. You may:

- Email me, or come to office hours, with specific questions (I may be somewhat less helpful than for regular assignments)
- Use the internet for general R help and debugging
- Use R for question 7

You may *not*:

- Use any resources from the course (the textbook, the course website, class notes, previous assignments, etc.)
- Use the internet to look up any questions on the exam
- Discuss the exam with anyone else

Maximum likelihood and Fisher information

1. Let $Y_1, ..., Y_n$ be iid random variables with pdf

$$f(y|\mu, \sigma^2) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(\log(y) - \mu)^2\right\}$$

   where $\sigma^2 > 0$, $y > 0$, and $\mu \in \mathbb{R}$. Find the maximum likelihood estimators $\widehat{\mu}$ and $\widehat{\sigma}^2$.

2. Let $Y_1, ..., Y_n \overset{iid}{\sim} Gamma(\alpha, \beta)$. Recall that the gamma distribution has pdf

$$f(y|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}$$

   where $y > 0$, $\alpha > 0$, and $\beta > 0$. Find the Fisher information matrix $\mathcal{I}(\alpha, \beta)$. (You may assume that desired regularity conditions hold for the gamma distribution). Note: the derivative of the gamma function $\Gamma(x)$ has no nice closed form. You may use $\psi(x)$ to represent $\frac{d}{dx} \log \Gamma(x)$ (the digamma function), and $\psi_1(x)$ to represent $\frac{d}{dx}\psi(x)$ (the trigamma function).

Modeling an Exponential response

In this section, you will model a response variable $Y_i \sim Exponential(\lambda_i)$.

3. Recall that if $Y \sim Exponential(\lambda)$, then the pdf of Y is given by

$$f(Y;\lambda) = \frac{1}{\lambda} e^{-Y/\lambda}$$

Show that $\mathbb{E}[Y] = \lambda$ and $Var(Y) = \lambda^2$.

4. Suppose that Y_i is a response variable of interest, and we use the following model for Y_i:

$$\begin{aligned} Y_i &\sim Exponential(\lambda_i) \\ \frac{1}{\lambda_i} &= \beta^T X_i, \end{aligned}$$

where $X_i = (1, X_{i,1}, ..., X_{i,k})^T \in \mathbb{R}^{k+1}$, and $\beta = (\beta_0, ..., \beta_k) \in \mathbb{R}^{k+1}$ is the vector of regression coefficients. We observe n independent observations $(X_1, Y_1), ..., (X_n, Y_n)$.

   (a) Derive a matrix form for the score $U(\beta)$.

   (b) Derive a matrix form for the Fisher information $\mathcal{I}(\beta)$. You may assume that desired regularity conditions hold.

5. Now we apply the model from Question 4 to real data. A factory is interested in the relationship between the amount of stress applied to a piece of steel, and the time it takes until that steel breaks. We use the following model:

$$\begin{aligned} time_i &\sim Exponential(\lambda_i) \\ \frac{1}{\lambda_i} &= \beta_0 + \beta_1 stress_i \end{aligned}$$

The raw data contains $n = 40$ observations $(stress_1,\ time_1), ..., (stress_{40},\ time_{40})$.

You can load the data into R by

```
steel <- read.csv("https://sta711-s23.github.io/exams/steel.csv")
```

In this question, we want to begin Fisher scoring to estimate β_0 and β_1 using the observed data.

   (a) In R, write functions `U` and `I` to calculate $U(\beta)$ and $\mathcal{I}(\beta)$ using the observed data.

   (b) Use your functions to estimate β using Fisher scoring with the observed data. Iterate the Fisher scoring algorithm until convergence, beginning with $\beta^{(0)} = (1, 2)^T$. For the purpose of this question, stop when

$$\max\{|\beta_0^{(r+1)} - \beta_0^{(r)}|,\ |\beta_1^{(r+1)} - \beta_1^{(r)}|\} < 0.0001$$