STA 711 Homework 5

Due: Friday, February 21, 12:00pm (noon) on Canvas.

Instructions: Submit your work as a single PDF. For this assignment, you may include written work by scanning it and incorporating it into the PDF. Include all R code needed to reproduce your results in your submission.

Convergence of random variables

1. For each of the following sequences $\{Y_n\}$, show that $Y_n \xrightarrow{p} 1$.
   (a) $Y_n = 1 + nX_n$, where $X_n \sim Bernoulli(1/n)$
   (b) $Y_n = \frac{1}{n}\sum\limits_{i=1}^{n} X_i^2$, where $X_i \overset{iid}{\sim} N(0,1)$
2. Suppose that $Y_1, Y_2, ... \overset{iid}{\sim} Beta(1, \beta)$. Find a value of ν such that $n^{\nu}(1 - Y_{(n)})$ converges in distribution.
3. Suppose that $Y_1, Y_2, ... \overset{iid}{\sim} Exponential(1)$. Find a sequence a_n such that $Y_{(n)} - a_n$ converges in distribution.
4. In this problem, we will prove part of the continuous mapping theorem. Let $\{Y_n\}$ be a sequence of random variables such that $Y_n \xrightarrow{p} Y$ for some random variable Y. Let g be a continuous function; recall that g is continuous if for all $\varepsilon > 0$, there exists some $\delta > 0$ such that $|g(x) - g(y)| < \varepsilon$ whenever $|x - y| < \delta$. Prove that $g(Y_n) \xrightarrow{p} g(Y)$.

Normal distributions and the Wald test

Suppose that $\widehat{\theta}$ is some estimator of a parameter of interest $\theta \in \mathbb{R}^d$. We want to test the hypotheses

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_A : \theta \neq \theta_0.$$

If $\widehat{\theta}$ is approximately normal, then we can use the Wald test (often $\widehat{\theta}$ will be the MLE, but the Wald test can be applied to any asymptotically normal estimator, not just to the MLE). Formally, suppose that

$$\sqrt{n}(\widehat{\theta} - \theta) \xrightarrow{d} N(0, V),$$

and let $\widehat{V}$ be some estimator of the covariance matrix V, such that $\widehat{V} \xrightarrow{p} V$. Then the Wald test statistic is

$$W = n(\widehat{\theta} - \theta_0)^T \widehat{V}^{-1}(\widehat{\theta} - \theta_0).$$

The goal of this section is to verify that $W \xrightarrow{d} \chi^2_d$ if H_0 is true. Our derivation will rely on the following properties of multivariate normal distributions, and positive semi-definite matrices:

- Recall from HW 4 that if $X \sim N(\mu, \Sigma)$, then

$$\boldsymbol{a} + \boldsymbol{B}X \sim N(\boldsymbol{a} + \boldsymbol{B}\mu, \boldsymbol{B}\Sigma\boldsymbol{B}^T)$$

- For any random vector X, the covariance matrix $\Sigma = Var(X)$ is positive semi-definite (you may use this without proof)

- If Σ is a positive semi-definite matrix, then there exists a unique positive semi-definite matrix $\Sigma^{\frac{1}{2}}$ such that $\Sigma = \Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}$ (you may use this without proof)
- $Z \sim N(0, \mathbf{I})$ if and only if $Z = (Z_1, ..., Z_q)^T \overset{iid}{\sim} N(0, 1)$ (you may use this without proof).
- Suppose that $X = (X_1, ..., X_q)^T \sim N(\mu, \Sigma)$. The entries X_i and X_j are independent *if and only* if $\Sigma_{ij} = Cov(X_i, X_j) = 0$. This is a special property of multivariate normal distributions, which we will prove below.
- If $Z \sim N(0, \mathbf{I})$ is a q-dimensional multivariate normal variable, where $\mathbf{I}$ is the identity matrix, then $Z^T Z \sim \chi^2_q$ (we will prove this below).

5. Let us begin by proving some results for the multivariate normal.

   (a) Show that if $X \sim N(\mu, \Sigma)$, then $\Sigma^{-\frac{1}{2}}(X - \mu) \sim N(0, \mathbf{I})$, where $\mathbf{I}$ is the identity matrix.

   (b) Show that $X \sim N(\mu, \Sigma)$ if and only if $X = \mu + \Sigma^{\frac{1}{2}} Z$ where $Z \sim N(0, \mathbf{I})$.

   (c) Let $X \sim N(\mu, \Sigma)$, where $X \in \mathbb{R}^q$. Suppose that for some $1 \leq p < q$, Σ can be partitioned as

   $$\Sigma = \begin{pmatrix} \Sigma_{11} & 0_{p\times(q-p)} \\ 0_{(q-p)\times p} & \Sigma_{22} \end{pmatrix},$$

   where Σ_{11} is $p \times p$, Σ_{22} is $(q-p) \times (q-p)$, and $0_{m\times n}$ denotes the matrix of zeros of the specified dimensions. Similarly partition

   $$X = \begin{pmatrix} X_{(1)} \\ X_{(2)} \end{pmatrix} \qquad \mu = \begin{pmatrix} \mu_{(1)} \\ \mu_{(2)} \end{pmatrix},$$

   into vectors of length p and $q - p$. Prove that

   $$X_{(1)} \sim N(\mu_{(1)}, \Sigma_{11}), \qquad X_{(2)} \sim N(\mu_{(2)}, \Sigma_{22}),$$

   and $X_{(1)}$ and $X_{(2)}$ are independent.

   (d) Using (c), conclude that if $X = (X_1, ..., X_q)^T \sim N(\mu, \Sigma)$, then the entries X_i and X_j are independent *if and only* if $\Sigma_{ij} = Cov(X_i, X_j) = 0$.

6. Now let's derive the relationship between the normal distribution and the χ^2 distribution.

   (a) Let $Z \sim N(0, 1)$ be a standard normal variable. Show that $Z^2 \sim \chi^2_1$ (a χ^2 distribution with 1 degree of freedom), by proving that the pdf of $Y = Z^2$ is

   $$f_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-y/2}.$$

   (b) Suppose that $Z_1, Z_2, ..., Z_q \overset{iid}{\sim} N(0, 1)$. Show that $\sum_{i=1}^{q} Z_i^2 \sim \chi^2_q$ (a χ^2 distribution with q degrees of freedom).

   (c) Let $\theta \in \mathbb{R}$ be a parameter of interest, and $\widehat{\theta}_n$ the maximum likelihood from a sample of size n. Let

   $$Z_n = \sqrt{n\mathcal{I}_1(\theta)}(\widehat{\theta}_n - \theta).$$

   Asymptotic normality of the MLE tells us that $Z_n \overset{d}{\to} N(0, 1)$. Show that $Z_n^2 \overset{d}{\to} \chi^2_1$.

7. Finally, let's connect the multivariate normal with the χ^2.

(a) Show that if $X \sim N(\mu, \Sigma)$, then $(X-\mu)^T\Sigma^{-1}(X-\mu) \sim \chi^2_q$.

(b) Suppose that $\widehat{\theta}$ is some estimator of $\theta \in \mathbb{R}^d$, and $\sqrt{n}(\widehat{\theta}-\theta) \xrightarrow{d} N(0, V)$. Let $\widehat{V}$ be an estimator of V such that $\widehat{V} \xrightarrow{p} V$, and let $W = n(\widehat{\theta}-\theta_0)^T\widehat{V}^{-1}(\widehat{\theta}-\theta_0)$. Prove that $W \xrightarrow{d} \chi^2_d$ if the null hypothesis $H_0 : \theta = \theta_0$ is true.