Convergence of random variables

Where we're heading

Convergence in probability

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in probability* to a random variable X if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

We write $X_n \xrightarrow{p} X$.

Example: (Weak law of large numbers)

WLLN

Theorem: Let $X_1, X_2, \ldots$ be iid random variables with $\mathbb{E}[X_i] = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Then

$$\overline{X}_n \xrightarrow{p} \mu$$

> Working with your neighbor, apply Chebyshev's inequality to prove the WLLN.

Another example

Let $U \sim Uniform(0,1)$, and let $X_n = \sqrt{n}\,\mathbb{I}\{U \leq 1/n\}$.

Show that $X_n \xrightarrow{p} 0$.

Almost sure convergence

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges almost surely* to a random variable X if, for every $\varepsilon > 0$,

$$P(\lim_{n\to\infty} |X_n - X| < \varepsilon) = 1$$

We write $X_n \overset{a.s.}{\to} X$.

Example: (Strong law of large numbers)

Convergence in distribution

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in distribution* to a random variable X if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

at all points where $F_X(x)$ is continuous. We write $X_n \xrightarrow{d} X$.

Example: (Central limit theorem)

Another example

Let $X \sim N(0, 1)$, and let $X_n = -X$ for $n = 1, 2, 3, \ldots$.

> Show that $X_n \xrightarrow{d} X$, but X_n does *not* converge to X in probability.

Relationships between types of convergence