t-tests

Next steps

> So far, we have discussed the Wald test in detail. What other hypothesis tests have you seen in statistics courses?

Recap: power function

$$H_0 : \theta \in \Theta_0 \qquad H_A : \theta \in \Theta_1$$

Suppose we reject H_0 when $(X_1, \dots, X_n) \in R$. The **power function** $\beta(\theta)$ is

$$\beta(\theta) = P_\theta((X_1, \dots, X_n) \in R)$$

Example: $X_1, \dots, X_n$ iid from a population with mean μ and variance σ^2.

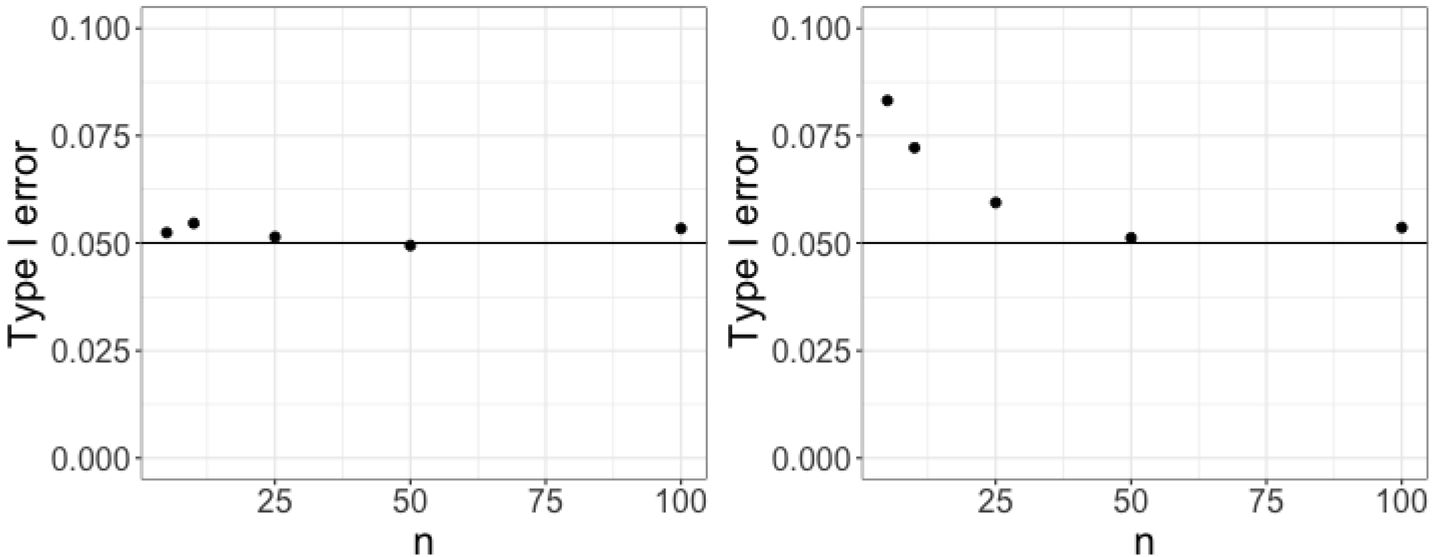
$$H_0 : \mu = \mu_0 \qquad H_A : \mu > \mu_0$$

$$\beta(\mu) \approx 1 - \Phi\left(z_\alpha - \frac{(\mu - \mu_0)}{\sigma/\sqrt{n}}\right)$$

Class activity, Part I

https://sta711-s23.github.io/class_activities/ca_lecture_21.html

Class activity



If we reject $H_0 : \mu = \mu_0$ when $\frac{\sqrt{n}(\overline{X}_n - \mu_0)}{s} > z_\alpha$, why does type I error increase as n decreases?

Issue: Wald tests with small n

The Wald test for a population mean μ relies on

$$Z_n = \frac{\sqrt{n}(\overline{X}_n - \mu)}{s} \approx N(0,1)$$

- $Z_n \xrightarrow{d} N(0,1)$ as $n \to \infty$
- But for small n, Z_n is not normal, even if $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

> Suppose $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. What is the exact distribution of $\frac{\sqrt{n}(\overline{X}_n - \mu)}{s}$?

t distribution

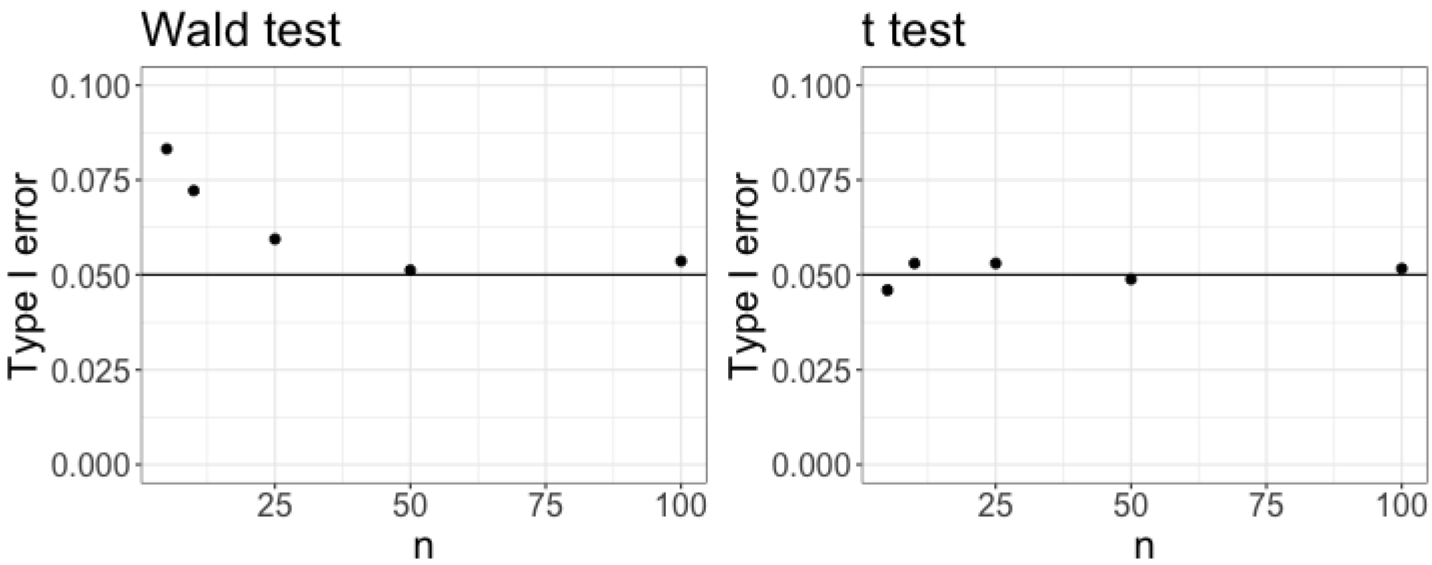
If $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{s} \sim t_{n-1}$$

Class activity, Part II

https://sta711-s23.github.io/class_activities/ca_lecture_21.html

Class activity



Example: two-sample t-test for a difference in means

Suppose that $X_1, \ldots, X_{n_1} \overset{iid}{\sim} N(\mu_1, \sigma^2)$ and $Y_1, \ldots, Y_{n_2} \overset{iid}{\sim} N(\mu_2, \sigma^2)$ are independent samples. We want to test

$$H_0 : \mu_1 = \mu_2 \qquad H_A : \mu_1 \neq \mu_2$$

Example: test for a population mean

Suppose $Y_1, \dots, Y_n \overset{iid}{\sim} Bernoulli(p)$. We want to test

$$H_0 : p = p_0 \qquad H_A : p \neq p_0$$

Wald test:

> Why is a t-test not appropriate?

Example: logistic regression

$$Y_i \sim Bernoulli(p_i) \qquad \log\left(\frac{p_i}{1-p_i}\right) = \beta^T X_i$$

We want to test

$$H_0 : C\beta = \gamma_0 \qquad H_A : C\beta \neq \gamma_0$$

> Why is a t-test not appropriate?

Philosophical question

- If $X_1, \ldots, X_n$ are iid from a population with mean μ and variance σ^2, then $\frac{\sqrt{n}(\overline{X}_n - \mu)}{s} \xrightarrow{d} N(0,1)$
- If $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then $\frac{\sqrt{n}(\overline{X}_n - \mu)}{s} \sim t_{n-1}$
- **Position 1:** For any reasonable sample size, the test statistic is approximately normal. And we never really have data from a normal distribution, so the t distribution is an approximation anyway. So always use the normal distribution
- **Position 2:** We always have a finite sample size, so our test statistic is never truly normal. And the t distribution is more conservative than the normal (heavier tails). So always use the t distribution

> With which position do you agree?