Lecture 12: Convergence

What we need to do

$$\hat{\beta} \approx N(\beta, \mathcal{I}^{-1}(\beta))$$

Need $\hat{\beta} \to \beta$ as $n \to \infty$

Convergence in probability

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in probability* to a random variable X if, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0$$

For any $\delta > 0$, $\exists$ N_0 st $\forall$ $n \geq N_0$

$P(|X_n - X| \geq \varepsilon) \leq \delta$

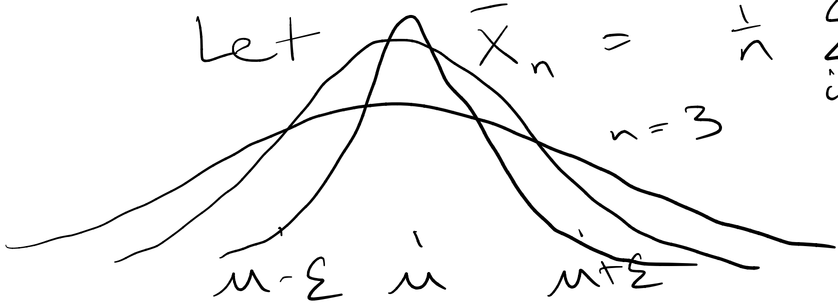
We write $X_n \xrightarrow{p} X$.

Example: (Weak law of large numbers)

Let $X_1, X_2, \ldots$ be iid with $E[X_i] = \mu$

and $Var(X_i) = \sigma^2 < \infty$

Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Then $\bar{X}_n \xrightarrow{p} \mu$



WLLN

$\mathbb{E}[\bar{X}_n] = \mu$

$Var(\bar{X}_n) = \frac{1}{n} \cdot Var(X_i) = \frac{\sigma^2}{n}$

Theorem: Let $X_1, X_2, \ldots$ be iid random variables with $\mathbb{E}[X_i] = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Then

$$\overline{X_n} \xrightarrow{p} \mu$$

Working with your neighbor, apply Chebyshev's inequality to prove the WLLN.

Pf: Let $\varepsilon > 0$, WTS $P(|\bar{X}_n - \mu| \geq \varepsilon) \to 0$ as $n \to \infty$

$0 \leq P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{Var(\bar{X}_n)}{\varepsilon^2}$ (Chebyshev)

$= \frac{\sigma^2}{n\varepsilon^2} \to 0$ as $n \to \infty$

$\Rightarrow P(|\bar{X}_n - \mu| \geq \varepsilon) \to 0$ as $n \to \infty$ //

Another example

$X_n = \begin{cases} 0 \\ \sqrt{n} \end{cases}$

Let $U \sim \mathrm{Uniform}(0, 1)$, and let $X_n = \sqrt{n}\, \mathbb{I}\{U \leq 1/n\}$.

Show that $X_n \xrightarrow{p} 0$.

Pf: Let $\varepsilon > 0$. WTS $P(|X_n - 0| \geq \varepsilon) \to 0$ as $n \to \infty$

$$P(|X_n - 0| \geq \varepsilon) = P(\sqrt{n}\, \mathbb{1}\{U \leq \tfrac{1}{n}\} \geq \varepsilon)$$

$$X_n = \begin{cases} 0 & U > \frac{1}{n} \\ \sqrt{n} & U \leq \frac{1}{n} \end{cases}$$

For sufficiently large n,
$\sqrt{n} \geq \varepsilon$
$\Rightarrow \sqrt{n}\, \mathbb{1}\{U \leq \frac{1}{n}\} \geq \varepsilon$
if and only if $\mathbb{1}\{U \leq \frac{1}{n}\} = 1$
(i.e., $U \leq \frac{1}{n}$)

$$\lim_{n \to \infty} P(|X_n - 0| \geq \varepsilon) = \lim_{n \to \infty} P(U \leq \tfrac{1}{n})$$

$$= \lim_{n \to \infty} \left(\tfrac{1}{n}\right) = 0$$

//

$\Rightarrow X_n \xrightarrow{p} 0$

Almost sure convergence

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges almost surely* to a random variable X if, for every $\varepsilon > 0$,

$$P(\lim_{n\to\infty} |X_n - X| < \varepsilon) = 1$$

We write $X_n \xrightarrow{a.s.} X$.

Example: (Strong law of large numbers)

Let $X_1, X_2, \ldots$ be iid with $\mathbb{E}[X_i] = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Then $\bar{X}_n \xrightarrow{a.s.} \mu$

i.e., $P(\underbrace{\bar{X}_n \to \mu}_{\text{usual "calculus" convergence}}) = 1$

(proof sketch in C & B)

Convergence in distribution

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in distribution* to a random variable X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

(pointwise convergence of the cdf)

at all points where $F_X(x)$ is continuous. We write $X_n \xrightarrow{d} X$.

Example: (Central limit theorem)

Let $X_1, X_2, \ldots$ be iid whose mgfs exist in a neighborhood of 0.

Let $\mu = \mathbb{E}[X_i]$ and $\sigma^2 = Var(X_i) < \infty$. Then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \qquad Z \sim N(0,1)$$

Another example

Let $X \sim N(0, 1)$, and let $X_n = -X$ for $n = 1, 2, 3, \ldots$.

Show that $X_n \xrightarrow{d} X$, but X_n does *not* converge to X in probability.

Relationships between types of convergence