Interval estimation

Motivation

Suppose we have data $(X_1, Y_1), \dots, (X_n, Y_n)$ with

$$Y_i \sim Bernoulli(p_i)$$

$$\log\left(\frac{p_i}{1 - p_i}\right) = \beta^T X_i$$

So far, we have discussed:

- Finding point estimates $\widehat{\beta}$
- Testing hypotheses about the true (but unknown) parameters β

> What are the limitations of point estimates and hypothesis tests for inference about β?

Confidence interval

```
...
##                Estimate Std. Error z value Pr(>|z|)
## (Intercept)  2.6415063  0.1213233   21.77   <2e-16 ***
## WBC         -0.2892904  0.0134349  -21.53   <2e-16 ***
## PLT         -0.0065615  0.0005932  -11.06   <2e-16 ***
## ---
...
```

> How would I calculate a 95% confidence interval for β_1 (the change in the log odds of dengue for a one-unit increase in WBC, holding PLT fixed)?

$\hat{\beta}_1 \pm z_{\frac{\alpha}{2}} SE(\hat{\beta}_1)$ <- $1-\alpha$ Wald CI

95% CI: $-0.289 \pm 1.96(0.0134)$

$(-0.315, -0.262)$

Confidence interval

$P(\beta_1 \in (-0.315, -0.262))$

$= \begin{cases} 1 & \text{(if interval contains } \beta_1) \\ 0 & \text{(if interval does not contain } \beta_1) \end{cases}$

```
...
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## ---
...
```

95% confidence interval for β_1: (-0.315, -0.262)

> How do I interpret this confidence interval?

95% confident : if we take many samples and we calculate an interval from each sample, 95% of those intervals should contain the true (unknown) parameter

Let L be the lower endpoint, U be the upper endpoint (random variables that are functions of the sample):

$P(\beta_1 \in (L, U)) = 0.95$ (for 95% interval)

$$\hat{\theta} \sim N(\theta, Var(\hat{\theta})) \Rightarrow \frac{\hat{\theta} - \theta}{SE(\hat{\theta})} \sim N(0,1)$$

Deriving the coverage probability

$$\underbrace{P\left(-z_{\frac{\alpha}{2}} \leq \frac{\hat{\theta} - \theta}{SE(\hat{\theta})} \leq z_{\frac{\alpha}{2}}\right)} = 1-\alpha$$

$$P\left(-z_{\frac{\alpha}{2}} SE(\hat{\theta}) \leq \hat{\theta} - \theta \leq z_{\frac{\alpha}{2}} SE(\hat{\theta})\right) = 1-\alpha$$

$$\Rightarrow P\left(\hat{\theta} - z_{\frac{\alpha}{2}} SE(\hat{\theta}) \leq \theta \leq \hat{\theta} + z_{\frac{\alpha}{2}} SE(\hat{\theta})\right) = 1-\alpha$$

endpoints of $1-\alpha$ Wald CI: $\hat{\theta} \pm z_{\frac{\alpha}{2}} SE(\hat{\theta})$

(Also true: $P\left(\theta - z_{\frac{\alpha}{2}} SE(\hat{\theta}) \leq \hat{\theta} \leq \theta + z_{\frac{\alpha}{2}} SE(\hat{\theta})\right) = 1-\alpha$

but we can't calculate these endpoints!)

Note: for any particular value θ_0 of θ

$\theta_0 \in (\hat{\theta} - z_{\alpha/2} SE(\hat{\theta}), \hat{\theta} + z_{\alpha/2} SE(\hat{\theta}))$ if and only if $\left|\frac{\hat{\theta} - \theta_0}{SE(\hat{\theta})}\right| \leq z_{\alpha/2}$

i.e. fail to reject $H_0: \theta = \theta_0$ $H_A: \theta \neq \theta_0$

Summerize : $\theta_0 \in \left(\hat{\theta} - z_{\frac{\alpha}{2}} SE(\hat{\theta}) , \hat{\theta} + z_{\frac{\alpha}{2}} SE(\hat{\theta})\right)$

if and only if fail to reject $H_0 : \theta = \theta_0$

(vs. $H_A : \theta \neq \theta_0$)

To test a hypothesis $H_0 : \theta = \theta_0$ vs. $H_A : \theta = \theta_0$ (at level α):

1) Construct a $1-\alpha$ CI for θ

2) Check if $\theta_0 \in$ CI

(But, we don't get a p-value)

To create a $1-\alpha$ CI for θ :

1) Test $H_0 : \theta = \theta_0$ vs. $H_A : \theta \neq \theta_0$ for all θ_0
(at level α)

2) $1-\alpha$ CI $= \{$all θ_0 for which we fail to reject$\}$

$X_1, \ldots, X_n \overset{iid}{\sim}$ with mean μ & variance σ^2

$$Var(\bar{X}_n) = \frac{\sigma^2}{n} = \frac{Var(X_i)}{n}$$

$$SE(\bar{X}_n) = \frac{\sigma}{\sqrt{n}}$$

Formal definition

Let $\theta \in \Theta$ be a parameter of interest, and $X_1, \dots, X_n$ a sample. Let $C(X_1, \dots, X_n) \subseteq \Theta$ be a set constructed from $X_1, \dots, X_n$ ($\Rightarrow$ $C(X_1, \dots, X_n)$ is a random set).

$C(X_1, \dots, X_n)$ is a $1-\alpha$ confidence set for θ

if

$$\inf_{\theta \in \Theta} P_\theta(\theta \in C(X_1, \dots, X_n)) = 1-\alpha$$

$$(\forall \theta, \quad P_\theta(\theta \in C(X_1, \dots, X_n)) \geq 1-\alpha)$$

Inverting a test

Theorem: Let $\theta \in \Theta$ be a parameter of interest.

For each value of $\theta_0 \in \Theta$, consider testing $H_0: \theta = \theta_0$ vs. $H_A: \theta \neq \theta_0$, and let $R(\theta_0)$ be the rejection region for a level α test of these hypotheses.

Let $C(X_1, \ldots, X_n) = \{ \theta_0 : (X_1, \ldots, X_n) \notin R(\theta_0) \}$

Then $C(X_1, \ldots, X_n)$ is a $1-\alpha$ confidence set for θ

Example

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} Uniform[0, \theta]$. We want to test

$$H_0 : \theta = \theta_0 \qquad H_A : \theta \neq \theta_0$$

Find the LRT statistic for this test.

Example

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} Uniform[0, \theta]$. Inverting the LRT gives us a confidence interval of the form

$$C(X_1, \ldots, X_n) = \left\{\theta : X_{(n)} \leq \theta \leq X_{(n)}k'\right\}$$

> Find a value k' such that the test is size α.