Confidence intervals

Recap: Pivotal quantities

Let $X_1, \ldots, X_n$ be a sample and θ be an unknown parameter. A function $Q(X_1, \ldots, X_n, \theta)$ is called a pivot if the distribution of $Q(X_1, \ldots, X_n, \theta)$ does not depend on θ

To construct a $1-\alpha$ confidence set:

Find a, b st $P_\theta(a \le Q(X_1, \ldots, X_n, \theta) \le b) = 1-\alpha$

The $1-\alpha$ confidence set for θ is

$$\{\theta : a \le Q(X_1, \ldots, X_n, \theta) \le b\}$$

$$P_\theta\Big(\underbrace{\alpha^{\frac{1}{n}} \le \frac{X_{(n)}}{\theta} \le 1}_{1 \le \frac{\theta}{X_{(n)}} \le \frac{1}{\alpha^{\frac{1}{n}}}}\Big) = 1-\alpha$$

$$\Rightarrow CI = \left[X_{(n)},\ \frac{X_{(n)}}{\alpha^{\frac{1}{n}}}\right]$$

Example

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} Uniform[0, \theta]$. We want to construct a $1-\alpha$ confidence set for θ using a pivot.

Last time:

$$\underbrace{Q(X_1, \ldots, X_n, \theta)}_{\text{pivot!}} = \underbrace{\frac{X_{(n)}}{\theta}}_{\in [0,1]} \Rightarrow P\left(\frac{X_{(n)}}{\theta} \le t\right) = \left(\frac{t \cdot \theta}{\theta}\right)^n = t^n$$

does not depend on θ!

Next: Choose a, b st $P_\theta\left(a \le \frac{X_{(n)}}{\theta} \le b\right) = 1-\alpha$

e.g. $b = 1$, $P_\theta\left(a \le \frac{X_{(n)}}{\theta} \le 1\right) = P_\theta\left(a \le \frac{X_{(n)}}{\theta}\right)$

$$= 1 - P_\theta\left(\frac{X_{(n)}}{\theta} \le a\right)$$

$$= 1 - a^n \overset{set}{=} 1-\alpha$$

$$\Rightarrow a = \alpha^{\frac{1}{n}}$$

Example

Let $X_1, \ldots, X_n \overset{iid}{\sim} Exponential(\theta)$, with density $f(x|\theta) = \theta e^{-\theta x}$.

Find a pivotal quantity $Q(X_1, \ldots, X_n, \theta)$ and construct a $1 - \alpha$ confidence interval for θ using the pivotal quantity.

Hints:

- Begin with the maximum likelihood estimate of θ, which is $\hat{\theta} = \dfrac{n}{\displaystyle\sum_{i=1}^{n} X_i}$
- If $X \sim Exponential(\theta)$, then $cX \sim Exponential\left(\frac{\theta}{c}\right)$
- $Exponential\left(\frac{1}{2}\right) = \chi^2_2$

$X_1, \ldots, X_n \overset{iid}{\sim} \text{Exp}(\theta)$

$$Q(X_1, \ldots, X_n, \theta) = 2\theta \textstyle\sum_i X_i$$

$$= \textstyle\sum_i 2\theta X_i \qquad\qquad 2\theta X_i \sim \chi^2_2$$

$$\sim \chi^2_{2n} \qquad \text{pivotal } \checkmark$$

want a, b st $\quad P_\theta\left(a \le 2\theta \sum_i X_i \le b\right) = 1 - \alpha$

$a = \chi^2_{2n, 1-\frac{\alpha}{2}}$ $\qquad$ or $\qquad a = 0$ $\qquad$ etc...

$b = \chi^2_{2n, \frac{\alpha}{2}}$ $\qquad\qquad\quad b = \chi^2_{2n, \alpha}$

$$\Rightarrow \left[\frac{a}{2\sum_i X_i}, \frac{b}{2\sum_i X_i}\right]$$

$$\mathcal{I}(\theta) \cdot \mathcal{I}^{-1}(\theta) = 1$$

Wald CI

Let $X_1, \ldots, X_n \overset{iid}{\sim} Exponential(\theta)$, with density $f(x|\theta) = \theta e^{-\theta x}$.

MLE: $\hat{\theta} = \frac{1}{\bar{x}} = \frac{n}{\sum_i X_i}$

$$\hat{\theta} \approx N(\theta, \mathcal{I}^{-1}(\theta))$$

$$\approx N\left(\theta, \frac{\theta^2}{n}\right)$$

$$\mathcal{I}(\theta) = \frac{n}{\theta^2}$$

$$\to \mathcal{I}^{-1}(\theta) = \frac{\theta^2}{n}$$

Wald CI for θ: $\hat{\theta} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\theta^2}{n}}$ ← don't know θ

variance depends on θ

$$\hat{\theta} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{\theta}^2}{n}}$$

Can we find a transformation $g(\hat{\theta})$ such that the variance $Var(g(\hat{\theta}))$ does <u>not</u> depend on θ?

Exponential: $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \theta^2)$

$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \xrightarrow{d} N(0, \theta^2 \cdot [g'(\theta)]^2)$

Delta method

Suppose $\hat{\theta}$ is an estimate of $\theta \in \mathbb{R}$, such that

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \sigma^2)$$

for some σ^2, and g is a continuously differentiable function with $g'(\theta) \neq 0$. Then

$$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \xrightarrow{d} N(0, \sigma^2[g'(\theta)]^2)$$

Proof sketch:

- First-order Taylor expansion of $g(\hat{\theta})$ around θ
- Slutsky's theorem

want $g'(\theta) = \frac{1}{\theta}$ $\qquad g(\theta) = \log(\theta)$

$\sqrt{n}(\log(\hat{\theta}) - \log(\theta)) \xrightarrow{d} N(0, 1)$

Variance stabilizing transformations