Lecture 23: Neyman-Pearson lemma

Wald test for normal mean

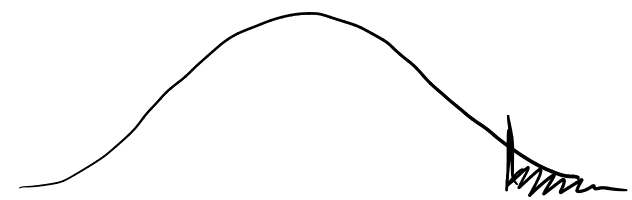
Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \quad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$.

Wald test: reject H_0 if

$$\frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} > z_\alpha$$



z_α

i.e., reject H_0 if $\bar{X}_n > \underbrace{\mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha}_{c_0}$

(c_0 satisfies $\beta(\mu_0) = \alpha$)

Wald test for normal mean

Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \quad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$. The Wald test rejects if

$$\overline{X}_n > \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$$

We know that $\beta(\mu_0) = \alpha$ for this test.

Does there exist a different test, with power function $\beta^*(\mu)$, such that $\beta^*(\mu_0) \leq \alpha$ and $\beta^*(\mu_1) > \beta(\mu_1)$?

Rearranging

$(H_0: \mu = \mu_0$ vs. $H_A: \mu = \mu_1$ $\mu_1 > \mu_0)$

rejects H_0 when $\bar{X}_n > c_0$

$$\iff 2n\bar{X}_n(\mu_0 - \mu_1) < 2nc_0(\mu_0 - \mu_1) \qquad (\mu_0 - \mu_1 < 0)$$

$$\iff 2n\bar{X}_n(\mu_0 - \mu_1) - n\mu_0^2 + n\mu_1^2 < 2nc_0(\mu_0 - \mu_1) - n\mu_0^2 + n\mu_1^2$$

$$\iff \sum_{i=1}^n X_i^2 - \sum_{i=1}^n X_i^2 + 2n\bar{X}_n(\mu_0 - \mu_1) - n\mu_0^2 + n\mu_1^2$$

$$< 2nc_0(\mu_0 - \mu_1) - n\mu_0^2 + n\mu_1^2$$

$$\iff \sum_{i=1}^n (X_i - \mu_1)^2 - \sum_{i=1}^n (X_i - \mu_0)^2 < 2nc_0(\mu_0 - \mu_1) - n\mu_0^2 + n\mu_1^2$$

$$\iff -\frac{1}{2\sigma^2}\sum_{i=1}^n (X_i - \mu_1)^2 + \frac{1}{2\sigma^2}\sum_{i=1}^n (X_i - \mu_0)^2 > -\frac{1}{2\sigma^2}\left(2nc_0(\mu_0 - \mu_1) - n\mu_0^2 + n\mu_1^2\right)$$

$$\iff \frac{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_i (X_i - \mu_1)^2\right\}}{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_i (X_i - \mu_0)^2\right\}} > \underbrace{\exp\left\{\frac{-(2nc_0(\mu_0 - \mu_1) - n\mu_0^2 + n\mu_1^2)}{2\sigma^2}\right\}}_{= K_0}$$

$$\iff \frac{f(X_1, \dots, X_n \mid \mu_1)}{f(X_1, \dots, X_n \mid \mu_0)} > K_0$$

Rearranging

Let $\mathbf{X} = X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \quad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$.

chosen so that $\beta(\mu_0) = \alpha$

The Wald test rejects if $\overline{X}_n > c_0$, which is equivalent to rejecting when

chosen so that $\beta(\mu_0) = \alpha$

$$\frac{L(\mu_1|\mathbf{X})}{L(\mu_0|\mathbf{X})} = \frac{f(X_1, \ldots, X_n|\mu_1)}{f(X_1, \ldots, X_n|\mu_0)} > k_0$$

Intuition: reject H_0 if the likelihood of μ_1 is sufficiently greater than the likelihood of μ_0

Neyman-Pearson test

Let $X_1, \dots, X_n$ be a sample from some distribution with probability function f, and parameter θ

To test $H_0: \theta = \theta_0$ vs. $H_A: \theta = \theta_1$, the Neyman-Pearson test rejects H_0 when

$$\frac{L(\theta_1 \mid x)}{L(\theta_0 \mid x)} = \frac{f(x \mid \theta_1)}{f(x \mid \theta_0)} > k$$

where k is chosen so that $\beta(\theta_0) = \alpha$

Neyman-Pearson lemma

Lemma: The Neyman-Pearson test is a uniformly most powerful level α test of $H_0: \theta = \theta_0$ vs. $H_A: \theta = \theta_1$

(i.e., $\beta_{NP}(\theta_0) \leq \alpha$ and $\beta_{NP}(\theta_1) \geq \beta^*(\theta_1)$ for any other test with power $\beta^*(\theta_0) \leq \alpha$)

Def: Consider testing $H_0: \theta \in \mathcal{H}_0$ vs. $H_A: \theta \in \mathcal{H}_1$,

Let $\mathcal{C}_\alpha$ be the set of level-α tests for these hypotheses. A test in $\mathcal{C}_\alpha$ is the uniformly most powerful level α test if it has power

$\beta(\theta) \leq \alpha \quad \forall \theta \in \mathcal{H}_0$

$\beta(\theta) \geq \beta^*(\theta) \quad \forall \theta \in \mathcal{H}_1$

where β^* is power for any other level α test

Example

Let $\mathbf{X} = X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \quad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$.

The Wald test rejects when

$$\frac{L(\mu_1|\mathbf{X})}{L(\mu_0|\mathbf{X})} > k,$$

an example of the Neyman-Pearson test

where k is chosen such that $\beta(\mu_0) = \alpha$.

$\Rightarrow$ Wald test for these hypotheses is a uniformly most powerful test

Example

Let $X_1, \ldots, X_n \overset{iid}{\sim} Exponential(\theta)$, with pdf $f(x|\theta) = \theta e^{-\theta x}$. We want to test

$$H_0 : \theta = \theta_0 \quad H_A : \theta = \theta_1,$$

where $\theta_1 < \theta_0$. The Neyman-Pearson test rejects when

$$\frac{L(\theta_1|\mathbf{X})}{L(\theta_0|\mathbf{X})} > k.$$

Find k such that the test has size α.

$\beta(\theta_0) = \alpha$

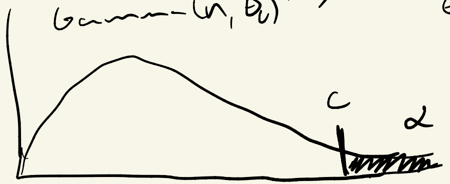
$$\frac{L(\theta_1 | X)}{L(\theta_0 | X)} = \frac{\theta_1^n e^{-\theta_1 \sum_i x_i}}{\theta_0^n e^{-\theta_0 \sum_i x_i}} > k \qquad f(x) = \theta e^{-\theta x}$$

$$L(\theta_1 | X) = f(X | \theta_1) = \prod_i \theta_1 e^{-\theta_1 x_i} = \theta_1^n e^{-\theta_1 \sum_i X_i}$$

$$\left(\frac{\theta_1}{\theta_0}\right)^n \exp\left\{ (\theta_0 - \theta_1) \sum_i X_i \right\} > k$$

$\sum_i X_i \sim$ Gamma(n, θ_0) $\Rightarrow$

$$\exp\left\{ (\theta_0 - \theta_1) \underbrace{\sum_i X_i}_{} \right\} > \left(\frac{\theta_0}{\theta_1}\right)^n k$$



$$X_i \sim \text{Exp}(\theta) = \text{Gamma}(1, \theta)$$

$$\sum_i X_i \sim \text{Gamma}(n, \theta)$$

$$\Rightarrow (\theta_0 - \theta_1) \sum_i X_i > \log k + n \log\left(\frac{\theta_0}{\theta_1}\right)$$

$$\Rightarrow \sum_i X_i > \frac{\log k + n \log\left(\frac{\theta_0}{\theta_1}\right)}{\theta_0 - \theta_1} = c$$

reject H_0 when $\sum_i X_i > c$

Under H_0, $\sum_i X_i \sim \text{Gamma}(n, \theta_0)$ $\quad \beta(\theta_0) = \alpha$ if c = upper α quantile of Gamma(n, θ_0)