(a) If $X_n \xrightarrow{a.s.} X$, then $X_n \xrightarrow{P} X$ (Casella & Berger)

(b) If $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{d} X$

(c) If $X_n \xrightarrow{d} c$, where c is a constant, then $X_n \xrightarrow{P} c$

(we will prove (b) & (c) later)

<u>Counterexample</u>: $X_n \xrightarrow{P} X$ does not imply $X_n \xrightarrow{a.s.} X$

Let $U \sim \text{Uniform}(0,1)$ and construct the following sequence:

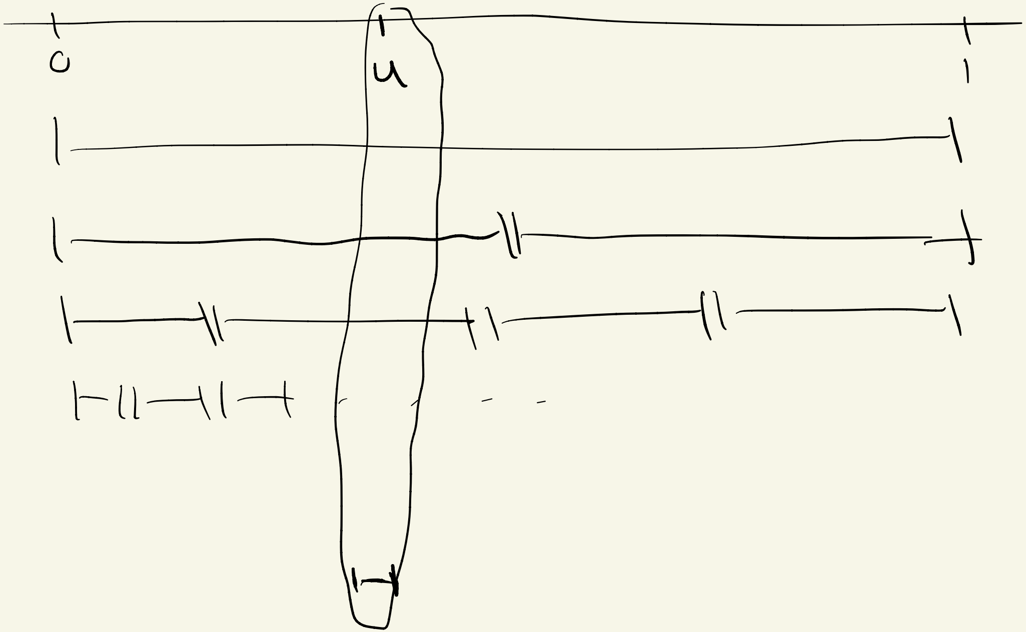
$X_1 = \mathbb{1}\{U \in [0,1]\}$, $X_2 = \mathbb{1}\{U \in [0,\frac{1}{2}]\}$, $X_3 = \mathbb{1}\{U \in [\frac{1}{2},1]\}$, ...

$X_n = \mathbb{1}\{U \in [\frac{j}{2^k}, \frac{j+1}{2^k}]\}$, $k = \lfloor \log_2(n) \rfloor$, $j = 2^k - n$

① $X_n \xrightarrow{P} 0$: <u>Pf</u>: $P(|X_n - 0| > \varepsilon) = P(U \in [\frac{j}{2^k}, \frac{j+1}{2^k}]) = \frac{1}{2^k} \to 0$ as $n \to \infty$

② $X_n \not\xrightarrow{a.s.} 0$: <u>Pf</u>: $X_n \xrightarrow{a.s.} 0$ if $\lim_{n\to\infty} |X_n - 0| = 0$ (w/prob. 1)

But $\lim_{n\to\infty} |X_n - 0|$ does not exist. For any U, there are infinitely many n st $X_n = 1$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A.S. convergence:

$X_n \xrightarrow{a.s.} X$ if for all $\varepsilon > 0$,

$$P\left(\lim_{n\to\infty} |X_n - X| < \varepsilon\right) = 1$$

$$\lim_{n\to\infty} |X_n - X| < \varepsilon \quad \forall \quad \varepsilon > 0$$

$$\Rightarrow \lim_{n\to\infty} |X_n - X| = 0$$

($X_n \xrightarrow{a.s.} X$ if $\lim_{n\to\infty} X_n = X$

or $\lim_{n\to\infty} |X_n - X| = 0$

w/ probability 1)