Confidence intervals

Recap: Pivotal quantities

Example

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} Uniform[0, \theta]$. We want to construct a $1 - \alpha$ confidence set for θ using a pivot.

Example

Let $X_1, \ldots, X_n \overset{iid}{\sim} Exponential(\theta)$, with density $f(x|\theta) = \theta e^{-\theta x}$.

Find a pivotal quantity $Q(X_1, \ldots, X_n, \theta)$ and construct a $1 - \alpha$ confidence interval for θ using the pivotal quantity.

Hints:

- Begin with the maximum likelihood estimate of θ, which is $\hat{\theta} = \dfrac{n}{\displaystyle\sum_{i=1}^{n} X_i}$
- If $X \sim Exponential(\theta)$, then $cX \sim Exponential\left(\frac{\theta}{c}\right)$
- $Exponential\left(\frac{1}{2}\right) = \chi^2_2$

Wald CI

Let $X_1, \ldots, X_n \overset{iid}{\sim} Exponential(\theta)$, with density $f(x|\theta) = \theta e^{-\theta x}$.

Delta method

Suppose $\hat{\theta}$ is an estimate of $\theta \in \mathbb{R}$, such that

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \sigma^2)$$

for some σ^2, and g is a continuously differentiable function with $g'(\theta) \neq 0$. Then

$$\sqrt{n}(g(\hat{\theta} - g(\theta)) \xrightarrow{d} N(0, \sigma^2[g'(\theta)]^2)$$

Proof sketch:

- First-order Taylor expansion of $g(\hat{\theta})$ around θ
- Slutsky's theorem

Variance stabilizing transformations