Lecture 24: Neyman-Pearson and Likelihood ratio tests

Recap: Neyman-Pearson test

Let $X_1, \ldots, X_n$ be a sample from a distribution with probability function f, and parameter θ. To test

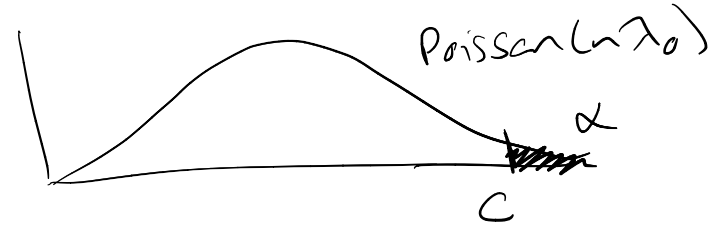
$$H_0 : \theta = \theta_0 \qquad H_A : \theta = \theta_1,$$

the Neyman-Pearson test rejects H_0 when

$$\frac{L(\theta_1 | \mathbf{X})}{L(\theta_0 | \mathbf{X})} > k,$$

where k is chosen so that $\beta(\theta_0) = \alpha$.

Warm-up



Let $X_1, \ldots, X_n \overset{iid}{\sim} Poisson(\lambda)$. We wish to test $H_0 : \lambda = \lambda_0$ vs. $H_A : \lambda = \lambda_1$, where $\lambda_1 > \lambda_0$. Find a most powerful test for these hypotheses (that is, determine a rejection region such that the test has a desired type I error rate α).

$$L(\lambda|X) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda} \lambda^{\sum_i X_i}}{\prod_i X_i!}$$

$$\frac{L(\lambda_1|X)}{L(\lambda_0|X)} = \frac{e^{-n\lambda_1} \lambda_1^{\sum_i X_i}}{e^{-n\lambda_0} \lambda_0^{\sum_i X_i}} = \left(\frac{\lambda_1}{\lambda_0}\right)^{\sum_i X_i} e^{n(\lambda_0 - \lambda_1)} > k$$

$$\left(\sum_i X_i\right)\left(\log(\lambda_1) - \log(\lambda_0)\right) + n(\lambda_0 - \lambda_1) > \log(k)$$

$$\underbrace{\sum_i X_i}_{\sim Poisson(n\lambda)} > \frac{\log(k) + n(\lambda_1 - \lambda_0)}{\log(\lambda_1) - \log(\lambda_0)} = c$$

c = upper α quantile Poisson$(n\lambda_0)$ //

Recap: Neyman-Pearson lemma

The Neyman-Pearson test is a *uniformly most power* level α test of $H_0 : \theta = \theta_0$ vs. $H_A : \theta = \theta_1$.

Proof: N-P rejects when $\frac{f(x|\theta_1)}{f(x|\theta_0)} > k$

Let β_{NP} denote power function of NP test. choose k s.t. $\beta_{NP}(\theta_0) = \alpha$

Let β^* denote power of another level α test of these hypotheses
$\beta^*(\theta_0) \leq \alpha$. WTS $\beta_{NP}(\theta_1) \geq \beta^*(\theta_1)$

Let ϕ_{NP} denote N-P rejection function: $\phi_{NP}(X) = \begin{cases} 1 & \text{N-P test rejects} \\ 0 & \text{N-P test fails to reject} \end{cases}$

Similarly, let ϕ^* denote rejection function for the other test

$$\beta_{NP}(\theta) = P_\theta(\text{reject } H_0) = P_\theta(\phi_{NP}(X) = 1) = \mathbb{E}_\theta[\phi_{NP}(X)]$$

$$= \int_{\mathcal{X}} \phi_{NP}(x) f(x|\theta)\,dx$$

$$\beta^*(\theta) = \int_{\mathcal{X}} \phi^*(x) f(x|\theta)\,dx$$

know: • N-P rejects when $f(x|\theta_1) > k f(x|\theta_0)$

$\iff f(x|\theta_1) - k f(x|\theta_0) > 0$

• $\beta_{NP}(\theta_0) = \alpha$, $\beta^*(\theta_0) \leq \alpha$ $\Rightarrow$ $\beta_{NP}(\theta_0) - \beta^*(\theta_0) \geq 0$

$$\Rightarrow \quad \beta_{NP}(\theta_0) - \beta^*(\theta_0) = \int_x (\phi_{NP}(x) - \phi^*(x)) f(x|\theta_0)\, dx \geq 0$$

Now lets look at $\int_x \underbrace{(\phi_{NP}(x) - \phi^*(x))}_{} \underbrace{(f(x|\theta_1) - k f(x|\theta_0))}_{} dx$

= 0 if tests agree; = 1 N-P test rejects, other fails; = −1 N-P fails, other test rejects

> 0 if $\phi_{NP}(x) = 1$; ≤ 0 if $\phi_{NP}(x) = 0$

so: $(\phi_{NP}(x) - \phi^*(x))(f(x|\theta_1) - k f(x|\theta_0))$ = 0 if tests agree; > 0 if NP test rejects, other fails; ≥ 0 if NP fails, other test rejects

$$\Rightarrow \int_x (\phi_{NP}(x) - \phi^*(x))(f(x|\theta_1) - k f(x|\theta_0)) \geq 0$$

$$\Rightarrow \underbrace{\int_x (\phi_{NP}(x) - \phi^*(x)) f(x|\theta_1)}_{\beta_{NP}(\theta_1) - \beta^*(\theta_1)} \geq \underbrace{k \int_x (\phi_{NP}(x) - \phi^*(x)) f(x|\theta_0)}_{= k(\beta_{NP}(\theta_0) - \beta^*(\theta_0)) \geq 0}$$

Another question

Let $X_1, \ldots, X_n \overset{iid}{\sim} Poisson(\lambda)$. We wish to test $H_0 : \lambda = \lambda_0$ vs. $H_A : \lambda \neq \lambda_0$.

Likelihood ratio tests

Back to the Poisson example

Let $X_1, \ldots, X_n \overset{iid}{\sim} Poisson(\lambda)$. We wish to test $H_0 : \lambda = \lambda_0$ vs. $H_A : \lambda \neq \lambda_0$.

Asymptotics of the LRT