STA 711 Homework 8

Due: Tuesday, April 23, 11am on Canvas.

Instructions: Submit your work as a single PDF. For this assignment, you may include written work by scanning it and incorporating it into the PDF. Include all R code needed to reproduce your results in your submission.

Confidence intervals

1. In class, we have worked with Wald confidence intervals for a binomial proportion. Now let's try inverting the test. Suppose we have data $X_1, ..., X_n \overset{iid}{\sim} Bernoulli(p)$. Derive a $1 - \alpha$ confidence interval for p by inverting the LRT of $H_0 : p = p_0$ vs. $H_A : p \neq p_0$. (It may be difficult to completely simplify the interval).

2. Suppose $X_1, ..., X_n \overset{iid}{\sim} N(\theta, \theta)$, where $\theta > 0$. Find a pivotal quantity $Q(X_1, ..., X_n, \theta)$, and use the quantity to create a $1 - \alpha$ confidence interval for θ.

3. Suppose $X_1 \sim Uniform[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$. Find a $1 - \alpha$ confidence interval for θ.

4. Suppose that $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

   (a) If σ^2 is known, the interval for μ is $\overline{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$, and the *width* of the interval is $2z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$. Find the minimum value of n so that a 95% confidence interval for μ will have a length of at most $\sigma/4$.

   (b) If σ^2 is unknown, the interval for μ is $\overline{X} \pm t_{n-1,\alpha/2} \frac{s}{\sqrt{n}}$, where $s^2 = \frac{1}{n-1} \sum\limits_{i=1}^{n} (X_i - \overline{X})^2$. Find the minimum value of n such that, with probability 0.9, a 95% confidence interval for μ will have a length of at most $\sigma/4$.

Delta method

Let $\theta \in \mathbb{R}^d$ be a parameter of interest, and $\widehat{\theta}$ be an estimate (e.g., the MLE). Our Wald tests and intervals depend on convergence in distribution to a normal:

$$\sqrt{n}(\widehat{\theta} - \theta) \xrightarrow{d} N(0, \Sigma).$$

We also know that if $\boldsymbol{a} \in \mathbb{R}^d$, then $\boldsymbol{a}^T \widehat{\theta} \approx N(\boldsymbol{a}^T \theta, \frac{1}{n} \boldsymbol{a}^T \Sigma \boldsymbol{a})$.

But what if we are interested in a *nonlinear* function $g(\theta)$, for some $g : \mathbb{R}^d \to \mathbb{R}$? It turns out that, under certain conditions, $g(\widehat{\theta})$ is actually (approximately) normal too! Formally, if g is a continuously differentiable function, then

$$\sqrt{n}(g(\widehat{\theta}) - g(\theta)) \xrightarrow{d} N\left(0, \left(\frac{\partial g}{\partial \theta}\right)^T \Sigma \left(\frac{\partial g}{\partial \theta}\right)\right),$$

where $\frac{\partial g}{\partial \theta}$ is the gradient of g evaluated at θ. This is called the *(multivariate) delta method.*

The purpose of this problem is to derive the delta method in the univariate case (the same intuition applies to the multivariate case). In the univariate case, $d = 1$ and $\theta \in \mathbb{R}$.

5. The univariate delta method is the following: if $\sqrt{n}(\widehat{\theta}-\theta) \xrightarrow{d} N(0,\sigma^2)$, and g is a continuously differentiable function with $g'(\theta) \neq 0$, then

$$\sqrt{n}(g(\widehat{\theta}) - g(\theta)) \xrightarrow{d} N(0, \sigma^2[g'(\theta)]^2).$$

(a) Using a first-order Taylor expansion, show that

$$\sqrt{n}(g(\widehat{\theta}) - g(\theta)) \approx \sqrt{n}g'(\theta)(\widehat{\theta} - \theta)$$

(b) Using Slutsky's theorem, argue that $\sqrt{n}g'(\theta)(\widehat{\theta} - \theta) \xrightarrow{d} N(0, \sigma^2[g'(\theta)]^2)$ and therefore $\sqrt{n}(g(\widehat{\theta}) - g(\theta)) \xrightarrow{d} N(0, \sigma^2[g'(\theta)]^2)$.

6. Suppose that $X_1, ..., X_n \overset{iid}{\sim} Poisson(\lambda)$. A $1-\alpha$ Wald interval for λ is $\widehat{\lambda} \pm z_{\alpha/2}\sqrt{\frac{\widehat{\lambda}}{n}}$, where $\widehat{\lambda} = \overline{X}$. Clearly, the variance of $\widehat{\lambda}$ depends on λ.

(a) Using the univariate delta method, find a transformation g such that the variance of $g(\widehat{\lambda})$ does not depend on λ (this is called a *variance stabilizing transformation*).

(b) Use (a) to find a $1-\alpha$ confidence interval for λ.