Confidence intervals

Recap: confidence sets

Let $\theta \in \Theta$ be a parameter of interest, and $X_1, \ldots, X_n$ a sample. A set $C(X_1, \ldots, X_n) \subseteq \Theta$ is a $1 - \alpha$ **confidence set** for θ if

$$\inf_{\theta \in \Theta} P_\theta(\theta \in C(X_1, \ldots, X_n)) = 1 - \alpha$$

Θ: possible values of θ

$\forall \theta \in \Theta$, $P_\theta(\theta \in C(X_1, \ldots, X_n)) \geq 1 - \alpha$

$R(\theta_0) = \{ (x_1, \dots, x_n) : \text{reject } H_0 : \theta = \theta_0 \}$

$\left| \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \right| > z_{\alpha/2}$

(e.g., $H_0 : \mu = \mu_0$ rejection region:

$\Rightarrow \bar{x} > \mu + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}$ or $\bar{x} < \mu - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}$

Inverting a test

Theorem: Let $\theta \in \Theta$ be a parameter of interest.

For each value of $\theta_0 \in \Theta$, consider testing $H_0 : \theta = \theta_0$ vs. $H_A : \theta \neq \theta_0$,

and let $R(\theta_0)$ be the rejection region for a level α test of these hypotheses.

Let $C(X_1, \dots, X_n) = \{ \theta_0 : (X_1, \dots, X_n) \notin R(\theta_0) \}$

Then $C(X_1, \dots, X_n)$ is a $1-\alpha$ confidence set for θ

Pf: $\theta_0 \in C(X_1, \dots, X_n) \iff (X_1, \dots, X_n) \notin R(\theta_0)$

$$\Rightarrow P_{\theta_0}(\theta_0 \in C(X_1, \dots, X_n)) = P_{\theta_0}((X_1, \dots, X_n) \notin R(\theta_0))$$

$$= 1 - \underbrace{P_{\theta_0}((X_1, \dots, X_n) \in R(\theta_0))}_{\leq \alpha} \quad (\alpha\text{-level test})$$

$$\geq 1 - \alpha$$

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Example: Uniform

Suppose $X_1, \dots, X_n \overset{iid}{\sim} Uniform[0, \theta]$. We want to test

$$H_0 : \theta = \theta_0 \qquad H_A : \theta \neq \theta_0$$

reject when $X_{(n)}$ is big or small compared to θ_0

> Find the LRT statistic for this test.

Reject H_0 when $\dfrac{\sup_{\theta \neq \theta_0} L(\theta | X)}{\sup_{\theta = \theta_0} L(\theta | X)} = \dfrac{L(\hat{\theta}_{MLE} | X)}{L(\theta_0 | X)} > k$

$L(\theta | X) = \left(\frac{1}{\theta}\right)^n \mathbb{1}\{\theta \geq X_{(n)}\}$ $\qquad \hat{\theta}_{MLE} = X_{(n)}$

$\Rightarrow$ reject H_0 when $\dfrac{\left(\frac{1}{X_{(n)}}\right)^n \mathbb{1}\{X_{(n)} \geq X_{(n)}\}}{\left(\frac{1}{\theta_0}\right)^n \mathbb{1}\{\theta_0 \geq X_{(n)}\}} > k$

$\Rightarrow \dfrac{\theta_0^n}{X_{(n)}^n \mathbb{1}\{\theta_0 \geq X_{(n)}\}} > k$

if $\theta_0 < X_{(n)}$: reject

if $\dfrac{\theta_0}{X_{(n)}} > k^{\frac{1}{n}}$: reject

reject H_0 when $\theta_0 < X_{(n)}$ or $\dfrac{\theta_0}{X_{(n)}} > k^{\frac{1}{n}}$

$\Rightarrow$ fail to reject when $\underline{X_{(n)}} \le \theta_0 \le \underline{X_{(n)} k^{\frac{1}{n}}}$ $\leftarrow$

confidence set: $[X_{(n)}, X_{(n)} k^{\frac{1}{n}}] = [X_{(n)}, X_{(n)} k']$

$\Rightarrow$ need k' st $P_\theta(\theta \in [X_{(n)}, X_{(n)} k']) \ge 1-\alpha$

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\begin{aligned}
P_\theta(\theta \in [X_{(n)}, X_{(n)} k']) &= P_\theta(\theta \le X_{(n)} k') \\
&= 1 - P_\theta(\theta > X_{(n)} k') \\
&= 1 - \underbrace{P_\theta\left(X_{(n)} < \frac{\theta}{k'}\right)} \\
&= 1 - \left(\frac{\theta / k'}{\theta}\right)^n \\
&= 1 - \left(\frac{1}{k'}\right)^n
\end{aligned}
$$

$\leftarrow$ ($P_\theta(\theta < X_{(n)}) = 0$ b/c P_θ means "probability if θ is the true parameter")

$X_i \sim \text{Uniform}(0, \theta)$

$P(X_i \le t) = \dfrac{t}{\theta}$

$\Rightarrow P\left(X_i \le \dfrac{\theta}{k'}\right) = \dfrac{\theta / k'}{\theta}$

$\Rightarrow$ choose k' st $1 - \left(\frac{1}{k'}\right)^n = 1-\alpha$

$\Rightarrow \left(\frac{1}{k'}\right)^n = \alpha \Rightarrow k' = \dfrac{1}{\alpha^{1/n}}$

$\Rightarrow$ $1-\alpha$ CI for θ:

$$\left[X_{(n)}, \frac{X_{(n)}}{\alpha^{1/n}}\right]$$

Example: Inverting the t-test

Suppose that $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. We want to construct a $1-\alpha$ confidence interval for μ.

> Construct a $1-\alpha$ confidence interval for μ by inverting the t-test.

reject $H_0 : \mu = \mu_0$ when $\left| \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \right| > t_{n-1, \frac{\alpha}{2}}$

$\Rightarrow$ reject when $\bar{X} > \mu_0 + \frac{s}{\sqrt{n}} t_{n-1, \frac{\alpha}{2}}$ or

$\bar{X} < \mu_0 - \frac{s}{\sqrt{n}} t_{n-1, \frac{\alpha}{2}}$

$\Rightarrow$ $1-\alpha$ CI for μ:

$$\left[\bar{X} - t_{n-1, \frac{\alpha}{2}} \frac{s}{\sqrt{n}}, \quad \bar{X} + t_{n-1, \frac{\alpha}{2}} \frac{s}{\sqrt{n}} \right]$$

Another way to view this: $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

depends on μ, parameter we want to estimate

$$\underbrace{Q(X_1, \dots, X_n, \mu)}_{\text{pivotal quantity}} = \frac{\bar{X}_n - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

t_{n-1} does not depend on μ

pivotal quantity (distribution does not depend on μ)

To construct a $1-\alpha$ confidence set for μ:

Find a, b s.t. $P_\mu(a \le Q(X_1, \dots, X_n, \mu) \le b) = 1 - \alpha$

don't depend on μ b/c distribution of $Q(X_1, \dots, X_n, \mu)$ does not depend on μ

Ex: $Q(X_1, \dots, X_n, \mu) \sim t_{n-1}$ $\quad a = -t_{n-1, \frac{\alpha}{2}}$, $\quad b = t_{n-1, \frac{\alpha}{2}}$

$$P_\mu\left(-t_{n-1, \frac{\alpha}{2}} \le Q(X_1, \dots, \mu) \le t_{n-1, \frac{\alpha}{2}}\right) = 1 - \alpha$$

$1-\alpha$ confidence set: $\{\mu : a \le Q(X_1, \dots, X_n, \mu) \le b\}$

Ex: $\left\{\mu : -t_{n-1, \frac{\alpha}{2}} \le \frac{\bar{X}_n - \mu}{S/\sqrt{n}} \le t_{n-1, \frac{\alpha}{2}}\right\}$

$$\left[\bar{X}_n - \frac{S}{\sqrt{n}} t_{n-1, \frac{\alpha}{2}},\ \bar{X}_n + \frac{S}{\sqrt{n}} t_{n-1, \frac{\alpha}{2}}\right]$$

Pivotal quantities

Let $X_1, \dots, X_n$ be a sample and θ be an unknown parameter. A function $Q(X_1, \dots, X_n, \theta)$ is called a pivot if the distribution of $Q(X_1, \dots, X_n, \theta)$ does not depend on θ

To construct a $1-\alpha$ confidence set:

Find a, b st $P_\theta(a \le Q(X_1, \dots, X_n, \theta) \le b) = 1-\alpha$

The $1-\alpha$ confidence set for θ is

$$\{\theta : a \le Q(X_1, \dots, X_n, \theta) \le b\}$$

Ex: $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ $\quad \frac{X_1 - \mu}{S} \sim t_1$ pivotal (but X_1 is not sufficient)

(less information than $\frac{\bar{X}_n - \mu}{S/\sqrt{n}} \sim t_{n-1}$)

Example

Let $X_1, \dots, X_n \overset{iid}{\sim} \text{Uniform}(0, \theta)$

want a $1-\alpha$ confidence set for θ

Step 1: find a pivotal quantity $Q(X_1, \dots, X_n, \theta)$

intuition: $\hat{\theta} = X_{(n)}$, so maybe we can use $X_{(n)}$ to create a pivot

$$P(X_{(n)} \le t) = \left(\frac{t}{\theta}\right)^n \qquad (t \in [0, \theta])$$

$$\underbrace{Q(X_1, \dots, X_n, \theta)}_{\text{pivot!}} = \frac{X_{(n)}}{\theta} \Rightarrow P\left(\frac{X_{(n)}}{\theta} \le t\right) = \left(\frac{t \cdot \theta}{\theta}\right)^n = t^n$$

does not depend on θ!