# Lecture 11: Probability inequalities

# What we need to do

$$\hat{\beta} \approx N(\beta, \tilde{I}^{-1}(\beta))$$

Need $\hat{\beta} \to \beta$ as $n \to \infty$

(need a definition of convergence)

Need $\hat{\beta} \approx$ Normal

Need $Var(\hat{\beta}) \approx \tilde{I}^{-1}(\beta)$

1. preliminary machinery
   - probability inequalities
   - types of convergence
   - theorems about convergence
2. Properties of maximum likelihood estimators
   - consistency $(\hat{\theta} \to \theta)$
   - asymptotic normality $(\hat{\theta} \approx \text{normal})$
3. Start hypothesis testing theorem

# Markov's inequality

**Theorem:** Let $Y$ be a non-negative random variable, and suppose that $\mathbb{E}[Y]$ exists. Then for any $t > 0$,

$$P(Y \geq t) \leq \frac{\mathbb{E}[Y]}{t}$$

(Do the case where Y is continuous, discrete case is similar)

Pf:

$$\mathbb{E}(Y) = \int_0^\infty y f(y)\,dy$$

$$= \int_0^t y f(y)\,dy + \int_t^\infty y f(y)\,dy$$

$$\geq \int_t^\infty y f(y)\,dy \geq t \int_t^\infty f(y)\,dy$$

$$= t\,P(Y \geq t)$$

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# Chebyshev's inequality

**Theorem:** Let $Y$ be a random variable, and let $\mu = \mathbb{E}[Y]$ and $\sigma^2 = Var(Y)$. Then

$(t > 0)$

$$P(|Y - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

With your neighbor, apply Markov's inequality to prove Chebyshev's inequality.

Pf: $P(|Y-\mu| \geq t) = P(|Y-\mu|^2 \geq t^2)$

$\leq \frac{\mathbb{E}(|Y-\mu|^2)}{t^2}$ (Markov's)

$= \frac{\sigma^2}{t^2}$

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# Cauchy-Schwarz inequality

**Theorem:** For any two random variables $X$ and $Y$, (Casella & Berger, Theorem 4.7.3)

$$|\mathbb{E}[XY]| \leq \mathbb{E}|XY| \leq (\mathbb{E}[X^2])^{1/2}(\mathbb{E}[Y^2])^{1/2}$$

**Example:** The *correlation* between $X$ and $Y$ is defined by

$\mu_X = \mathbb{E}[X]$

$\mu_Y = \mathbb{E}[Y]$

$$\underbrace{\rho(X,Y)}^{\rho(X,Y)} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$$

Using the Cauchy-Schwarz inequality, we can show that $-1 \leq \rho(X,Y) \leq 1$. $\iff$ $|\mathrm{Cov}(X,Y)| \leq \sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}$

$$|\mathrm{Cov}(X,Y)| = \left|\mathbb{E}\left[(X-\mu_X)(Y-\mu_Y)\right]\right| \leq \mathbb{E}\left[\,|(X-\mu_X)(Y-\mu_Y)|\,\right]$$

(Cauchy-Schwarz)

$$\leq \mathbb{E}\left[(X-\mu_X)^2\right]^{\frac{1}{2}}\mathbb{E}\left[(Y-\mu_Y)^2\right]^{\frac{1}{2}} = \sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}$$

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