STA 711 HW 1 Solutions

Ciaran Evans

1. Suppose that $X \sim Poisson(\lambda)$.

   (a) We use the fact that $e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$.

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} \\
&= e^{-\lambda} \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = e^{-\lambda} \lambda \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \\
&= e^{-\lambda} \lambda e^{\lambda} \\
&= \lambda
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \lambda e^{-\lambda} \sum_{x=1}^{\infty} x^2 \frac{\lambda^{x-1}}{x!} \\
&= \lambda e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^{x-1}}{(x-1)!} \\
&= \lambda e^{-\lambda} \left(\sum_{x=1}^{\infty} (x-1) \frac{\lambda^{x-1}}{(x-1)!} + \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right) \\
&= \lambda e^{-\lambda} \left(\lambda e^{\lambda} + e^{\lambda} \right) \\
&= \lambda^2 + \lambda
\end{aligned}$$

   Thus, $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X] = \lambda$.

(b)

$$\mathbb{E}[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = \exp\{\lambda(e^t - 1)\}$$

(c) From (b), $M_X(t) = \exp\{\lambda(e^t - 1)\}$, so

$$M'_X(t) = \exp\{\lambda(e^t - 1)\}\lambda e^t$$

and

$$M''_X(t) = \exp\{\lambda(e^t - 1)\}\lambda^2 e^{2t} + \exp\{\lambda(e^t - 1)\}\lambda$$

At $t = 0$, we get

$$\mathbb{E}[X] = M'_X(0) = \lambda$$
$$\mathbb{E}[X^2] = M''_X(0) = \lambda^2 + \lambda$$

and thus $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X] = \lambda$.

2. Suppose that $X \sim Poisson(\lambda)$ and $Y \sim Poisson(\mu)$ are independent. Let $Z = X + Y$.

(a) We use the fact that the MGF for the sum of independent random variables is the product of their individual MGFs:

$$\begin{aligned} M_Z(t) &= M_X(t)M_Y(t) = \exp\{\lambda(e^t - 1)\}\exp\{\mu(e^t - 1)\} \\ &= \exp\{(\lambda + \mu)(e^t - 1)\} \end{aligned}$$

This is the MGF of a $Poisson(\lambda + \mu)$ distribution, and so by the uniqueness of MGFs we conclude $X + Y \sim Poisson(\lambda + \mu)$.

(b)

$$\begin{aligned} P(X = x|Z = n) &= \frac{P(X = x, Z = n)}{P(Z = n)} = \frac{P(X = x, Y = n - x)}{P(Z = n)} \\ &= \frac{\frac{\lambda^x e^{-\lambda}}{x!} \cdot \frac{\mu^{n-x} e^{-\mu}}{(n - x)!}}{\frac{(\lambda + \mu)^n e^{-(\lambda+\mu)}}{n!}} \\ &= \frac{n!}{x!(n - x)!} \left(\frac{\lambda}{\lambda + \mu}\right)^x \left(1 - \frac{\lambda}{\lambda + \mu}\right)^{n-x} \end{aligned}$$

This is the pmf of a $Binomial\left(n, \frac{\lambda}{\lambda+\mu}\right)$ distribution.

3. The joint pdf is

$$f(x,y) = \begin{cases} c(x+2y) & 0 < y < 1, 0 < x < 2 \\ 0 & \text{else} \end{cases}$$

(a) To find c, we note that $f(x,y)$ must integrate to 1.

$$\begin{aligned} c\int_0^1\int_0^2 (x+2y)dxdy &= c\int_0^1\int_0^2 xdxdy + c\int_0^1\int_0^2 2ydxdy \\ &= 2c + 2c \end{aligned}$$

so we conclude that $c = 1/4$.

(b)

$$f(x) = \int_0^1 f(x,y)dy = \frac{1}{4}\int_0^1 (x+2y)dy = \frac{x+1}{4}$$

(c)

$$\begin{aligned} P(Z < t) = P(9 \leq (1+X)^2 t) = P(\frac{3}{\sqrt{t}} - 1 \leq X) = \int_{3/\sqrt{t}-1}^1 (x+1)/4dx \\ = \frac{1}{2} - \frac{9}{8t} \end{aligned}$$

Then, differentiating,

$$f_Z(t) = \frac{9}{8t^2}$$

for $t \in [9/4, 9]$

4.

5.

6. (a) Since F_X is a monotonic, continuous cdf, then F_X and F_X^{-1} are both strictly increasing, and

$$P(X \le t) = P(F_X^{-1}(U) \le t) = P(U \le F_X(t)) = F_X(t),$$

where the final step follows from the cdf of a $Uniform(0,1)$ distribution.

(b) See the R part below.

7. (a)

$$\begin{aligned}
Cov\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) &= \mathbb{E}\left[\left(\sum_{i=1}^{m} a_i X_i - \sum_{i=1}^{m} a_i \mathbb{E}[X_i]\right)\left(\sum_{j=1}^{n} b_j Y_j - \sum_{j=1}^{n} b_j \mathbb{E}[Y_j]\right)\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{m} a_i (X_i - \mathbb{E}[X_i])\right)\left(\sum_{j=1}^{n} b_j (Y_j - \mathbb{E}[Y_j])\right)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j (X_i - \mathbb{E}[X_i])(Y_j - \mathbb{E}[Y_j])\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \mathbb{E}\left[(X_i - \mathbb{E}[X_i])(Y_j - \mathbb{E}[Y_j])\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j Cov(X_i, Y_j)
\end{aligned}$$

(b)

$$\begin{aligned}
Var\left(\sum_{i=1}^{m} X_i\right) &= Cov\left(\sum_{i=1}^{m} X_i, \sum_{i=1}^{m} X_i\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} Cov(X_i, X_j) \\
&= \sum_{i=1}^{m} Var(X_i) + \sum_{i=1}^{m}\sum_{j\neq i} Cov(X_i, X_j) \\
&= \sum_{i=1}^{m} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)
\end{aligned}$$

8. $X_1, ..., X_n \overset{iid}{\sim} Uniform(0, 1)$, and $Y = \max\{X_1, ..., X_n\}$

   (a) For $t \in [0, 1]$:

$$
\begin{aligned}
P(Y \leq t) &= P(X_1, ..., X_n \leq t) = \prod_{i=1}^{n} P(X_i \leq t) \text{ (independence)} \\
&= t^n
\end{aligned}
$$

   Then, the pdf is

$$f_Y(t) = nt^{n-1}$$