Method of moments estimators

Course so far

- Maximum likelihood estimation
- Logistic regression
- Asymptotics
- Asymptotic properties of MLEs
- Hypothesis testing
- Confidence intervals

Common theme: Likelihoods and MLEs

Question: Why maximum likelihood estimation?

Example

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} Uniform[0, \theta]$. How could I estimate θ?

1. $\hat{\theta}_{MLE} = X_{(n)}$

2. $\mathbb{E}[X] = \frac{\theta}{2}$ $\qquad 2\bar{X} = \hat{\theta}$

3. $\mathbb{E}[X^2] = \frac{\theta^2}{12} + \frac{\theta^2}{4} = \frac{\theta^2}{3}$

   $\hat{\theta} = \sqrt{\frac{3}{n} \sum_i X_i^2}$

4. median $U[0, \theta] = \frac{\theta}{2}$

   $\hat{\theta} =$ sample median $\times 2$

5. $\hat{\theta} = 5$ (probably a terrible estimate)

$$\Rightarrow \quad \hat{a} = \hat{\mu}_1 - \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)}$$

Example

$$\hat{b} = \hat{\mu}_1 + \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)}$$

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} Uniform[a, b]$. How could I estimate a and b?

① MLE: $\hat{a}_{MLE} = X_{(1)}$ $\quad \hat{b}_{MLE} = X_{(n)}$

② Moments: $\mathbb{E}[X] = \frac{a+b}{2} = \mu_1 \quad \hat{\mu}_1 = \bar{X}$

$$\mathbb{E}[X^2] = \frac{1}{3}(a^2 + ab + b^2) = \mu_2 \quad \hat{\mu}_2 = \frac{1}{n}\sum_i X_i^2$$

$$b = 2\mu_1 - a$$

$$\mu_2 = \frac{1}{3}(a^2 + ab + b^2)$$

$$= \frac{1}{3}(a^2 + a(2\mu_1 - a) + (2\mu_1 - a)^2)$$

$$= \frac{1}{3}(a^2 - 2a\mu_1 + 4\mu_1^2)$$

$$\Rightarrow 3\mu_2 - 3\mu_1^2 = (a - \mu_1)^2$$

$$\Rightarrow a = \mu_1 - \sqrt{3(\mu_2 - \mu_1^2)} \qquad b = \mu_1 + \sqrt{3(\mu_2 - \mu_1^2)}$$

Method of moments

Let $X_1, \ldots, X_n$ be a sample from a distribution with probability function $f(x|\theta_1, \ldots, \theta_k)$, with k parameters $\theta_1, \ldots, \theta_k$.

Let

$$
\begin{aligned}
\mu_1 &= \mathbb{E}[X] = g_1(\theta_1, \ldots, \theta_k) & \hat{\mu}_1 &= \tfrac{1}{n}\textstyle\sum_i X_i \\
\mu_2 &= \mathbb{E}[X^2] = g_2(\theta_1, \ldots, \theta_k) & \hat{\mu}_2 &= \tfrac{1}{n}\textstyle\sum_i X_i^2 \\
&\vdots & &\vdots \\
\mu_k &= \mathbb{E}[X^k] = g_k(\theta_1, \ldots, \theta_k) & \hat{\mu}_k &= \tfrac{1}{n}\textstyle\sum_i X_i^k
\end{aligned}
$$

The method of moments (MoM) approach estimates $\theta_1, \ldots, \theta_k$ by the solutions to

$$
\begin{aligned}
\hat{\mu}_1 &= g_1(\hat{\theta}_1, \ldots, \hat{\theta}_k) \\
\hat{\mu}_2 &= g_2(\hat{\theta}_1, \ldots, \hat{\theta}_k) \\
&\vdots \\
\hat{\mu}_k &= g_k(\hat{\theta}_1, \ldots, \hat{\theta}_k)
\end{aligned}
$$

Example

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

Find the method of moments estimates $\widehat{\mu}$ and $\widehat{\sigma}^2$.

$\mu_1 = \mu$ $\qquad \hat{\mu}_1 = \bar{X}$ $\qquad \hat{\mu} = \bar{X}$

$$\mu_2 = \mathbb{E}[X^2]$$
$$= \text{Var}(X) + (\mathbb{E}[X])^2$$
$$= \sigma^2 + \mu^2$$

$\hat{\mu}_2 = \frac{1}{n} \sum_i X_i^2$

$\sigma^2 = \mu_2 - \mu_1^2 \qquad \Rightarrow \quad \hat{\sigma}^2 = \frac{1}{n} \sum_i X_i^2 - (\bar{X})^2$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_i (X_i - \bar{X})^2$$

Example

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} Gamma(\alpha, \beta)$, i.e. $f(x|\alpha, \beta) = \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$. Then

$$\mu_1 = \mathbb{E}[X] = \frac{\alpha}{\beta} \qquad \mu_2 = \mathbb{E}[X^2] = \left(\frac{\alpha}{\beta}\right)^2 + \frac{\alpha}{\beta^2}$$

> Use the method of moments to estimate α and β.

$$\hat{\alpha} = \frac{\hat{\mu}_1^2}{\hat{\mu}_2 - \hat{\mu}_1^2} \qquad \hat{\beta} = \frac{\hat{\mu}_1}{\hat{\mu}_2 - \hat{\mu}_1^2} \qquad (\hat{\mu}_2 > \hat{\mu}_1^2)$$

$$\mu_1 = \frac{\alpha}{\beta} \qquad \mu_2 = \left(\frac{\alpha}{\beta}\right)^2 + \frac{\alpha}{\beta^2}$$

$$\Rightarrow \quad \alpha = \beta\mu_1 \qquad \mu_2 = \left(\frac{\beta\mu_1}{\beta}\right)^2 + \frac{\beta\mu_1}{\beta^2}$$

$$= \mu_1^2 + \frac{\mu_1}{\beta}$$

$$\Rightarrow \quad \beta\mu_2 = \beta\mu_1^2 + \mu_1$$

$$\Rightarrow \quad \mu_1 = \beta(\mu_2 - \mu_1^2) \quad \Rightarrow \quad \beta = \frac{\mu_1}{\mu_2 - \mu_1^2} \qquad \alpha = \mu_1\beta$$

$$= \frac{\mu_1^2}{\mu_2 - \mu_1^2}$$