# MLE with mis-specified model

no $\theta$, b/c we have the wrong model

# Maximum likelihood with mis-specified models

$Y_1, Y_2, \ldots, Y_n \sim G$ w/ probability function $g$

Assume (incorrectly!) that $Y_i \sim F_\theta$, and we estimate $\theta$

Still write down $\ell(\theta) = \sum_{i=1} \log f(Y_i; \theta)$

Estimate: $\hat{\theta}$ solves $U(\theta) = 0$

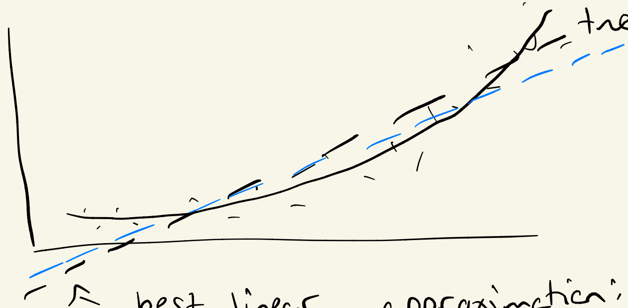
$\ell'(\theta)$

Q: What is $\hat{\theta}$ actually estimating??

Let $\theta^*$ be the value of $\theta$ that solves

expectation w/ respect to the distribution $\rightarrow$ $\mathbb{E}_g[U(\theta)] = 0$

Now: $\theta^*$ is the parameter which best approximates the true model $g(\cdot)$ in the space of all models considered $f(\cdot;\theta)$




best linear approximation: $\beta_0 + \beta_1 X$ $\quad \theta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$

Sample data $\rightarrow$ estimated line $\quad \hat{\beta}_0 + \hat{\beta}_1 X \quad \hat{\theta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}$

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$\hat{\theta} \xrightarrow{P} \theta^*$ (under weak assumptions) as $n \to \infty$

$$\sqrt{n}(\hat{\theta} - \theta^*) \rightarrow N(0, S_1)$$

(and regularity)

<u>If</u> model is correctly specified:

$$Var(\ell'(\theta)) = -\mathbb{E}[\ell''(\theta)] = \mathcal{I}(\theta)$$

and asymptotic variance $= \mathcal{I}_1^{-1}(\theta)$

If model is incorrectly specified: $Var(\ell'(\theta)) \neq -\mathbb{E}[\ell''(\theta)]$

$$\hat{\theta} \xrightarrow{P} \theta^*$$

$$\sqrt{n}(\hat{\theta} - \theta^*) \xrightarrow{d} N(0, S_1)$$

Let $V_n(\theta) = \mathrm{Var}_g(\ell'(\theta))$

$J_n(\theta) = -\mathbb{E}_g[\ell''(\theta)]$

Asymptotics: $-\frac{1}{n}\ell''(\theta) \xrightarrow{P} J_1(\theta^*)$

$\frac{1}{\sqrt{n}}\ell'(\theta) \xrightarrow{d} N(0, V_1(\theta^*))$

$$\sqrt{n}(\hat{\theta} - \theta^*) \xrightarrow{d} N(0, \underbrace{J_1^{-1}(\theta^*)\, V_1(\theta^*)\, J_1^{-1}(\theta^*)}_{S_1})$$

Sandwich variance

$$\hat{\theta} \approx N(\theta^*, \hat{J}_n^{-1}(\hat{\theta})\hat{V}_n(\hat{\theta})\hat{J}_n^{-1}(\hat{\theta}))$$

$\exp\{\beta^T X_i\}$

Poisson regression: $U(\beta) = X^T(Y - \mu) = \sum_i (Y_i - \mu_i) X_i$

$\hat{J}_n(\hat{\beta}) = \sum_i \hat{\mu}_i X_i X_i^T = X^T \operatorname{diag}(\hat{\mu}_i) X$

$\hat{V}_n(\hat{\beta}) = \sum_i (Y_i - \hat{\mu}_i)^2 X_i X_i^T = X^T \operatorname{diag}((Y_i - \hat{\mu}_i)^2) X$