Cramer-Rao lower bound

(CRLB)

Cramer-Rao lower bound

<u>Thm</u>: Let $X_1, \dots, X_n$ be a sample from a distribution with probability function $f(x \mid \theta)$, and $\theta \in \mathbb{R}$. And let $\hat{\theta}$ be an unbiased estimator of θ.

Under regularity conditions,

$$Var(\hat{\theta}) \geq \frac{1}{\mathcal{I}(\theta)}$$

<u>Ex</u>: $X_1, \dots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$

MLE: $\hat{\lambda} = \bar{X}$ $\qquad Var(\hat{\lambda}) = \frac{\lambda}{n}$

$$\mathcal{I}(\lambda) = \frac{n}{\lambda} \implies CRLB = \frac{\lambda}{n}$$

$$\mathbb{E}[\hat{\lambda}] = \lambda \implies Bias(\hat{\lambda}) = 0, \; Var(\hat{\lambda}) = CRLB$$

$\implies \hat{\lambda} = \bar{X}$ is a <u>best unbiased estimator</u> of λ

(a.k.a <u>minimum-variance unbiased estimator</u>)

$s^2 = \frac{1}{n-1} \sum_i (X_i - \bar{X})^2$ unbiased ✓

$var(s^2) = \frac{2\sigma^4}{n-1}$

question: does s^2 attain CRLB?

Example

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Want to estimate σ^2

Need to find $\mathcal{I}(\sigma^2)$ to get CRLB

$$\ell(\mu, \sigma^2) = \log\left(\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{ -\frac{1}{2\sigma^2} \sum_i (X_i - \mu)^2 \right\} \right)$$

$$= -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\sigma^2) - \frac{1}{2\sigma^2} \sum_i (X_i - \mu)^2$$

$$\frac{\partial \ell}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_i (X_i - \mu)^2$$

$$\frac{\partial^2 \ell}{\partial \sigma^4} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6} \sum_i (X_i - \mu)^2$$

$$-\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \sigma^4} \right] = -\left(\frac{n}{2\sigma^4} - \frac{n}{\sigma^6} \cdot \sigma^2 \right) = \frac{n}{2\sigma^4} = \mathcal{I}(\sigma^2)$$

$$CRLB = \frac{2\sigma^4}{n} < Var(s^2)$$

- If μ is known:

  Consider $\hat{\sigma}^2 = \frac{1}{n} \sum_i (X_i - \mu)^2$

  $$\mathrm{var}(\hat{\sigma}^2) = \frac{2\sigma^4}{n}$$

  attains CRLB ✓

  unbiased ✓

- If μ is <u>unknown</u>: CRLB cannot be attained

<u>Thm</u>: $\hat{\theta}$ attains the CRLB <u>if and only if</u>

$$U(\theta) = a(\theta)\,[\hat{\theta} - \theta] \qquad \text{(for some } a(\theta)\text{)}$$

<u>Ex</u>: $N(\mu, \sigma^2)$: $U(\sigma^2) = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_i (X_i - \mu)^2$

$$= \frac{n}{2\sigma^4}\left(\frac{1}{n} \sum_i (X_i - \mu)^2 - \sigma^2 \right)$$

$\Rightarrow$ if μ is unknown, CRLB <u>cannot</u> be attained for $N(\mu, \sigma^2)$

<u>EDM</u>: $$f(y;\ \theta, \phi) = a(y, \phi) \exp\left\{ \frac{y\theta - \kappa(\theta)}{\phi} \right\}$$

$$\ell(\theta) = \sum_i \log(a(Y_i, \phi)) + \frac{1}{\phi} \sum_i (Y_i \theta - \kappa(\theta))$$

$$\frac{\partial \ell}{\partial \theta} = \frac{1}{\phi} \sum_i (Y_i - \mu) \qquad \frac{\partial \kappa}{\partial \theta} = \mu$$

$$= \frac{n}{\phi} \cdot \Big(\underbrace{\tfrac{1}{n} \textstyle\sum_i Y_i}_{\hat{\mu}} - \mu \Big)$$

<u>If</u> ϕ is known

MLE for mean μ

$g(\mu_i) = \beta^T X_i$

$$\text{Score} = \frac{n}{\phi} (\hat{\mu} - \mu)$$

$\hat{\mu}$ attain CRLB

Sufficient statistics

> Given an unbiased estimator, can I improve its variance?

<u>Def</u>: Let $X_1, \dots, X_n$ be a sample from a distribution $f(x|\theta)$. Let $T \equiv T(X_1, \dots, X_n)$ be a statistic. If the conditional distribution of $X_1, \dots, X_n \mid T$ does not depend on θ, then T is a <u>sufficient statistic</u> for θ.

<u>Ex</u>: Suppose $X_1, \dots, X_n \overset{iid}{\sim}$ Poisson(λ). Let $T = \sum_i X_i$

$T \sim \text{Poisson}(n\lambda)$

$$f(X_1, \dots, X_n \mid T) = \frac{f(X_1, \dots, X_n, T)}{f(T)} = \frac{f(X_1, \dots, X_n)}{f(T)}$$

$$= \frac{\cancel{e^{-n\lambda}}\, \cancel{\lambda^{\sum_i X_i}} / \prod_i X_i!}{\cancel{e^{-n\lambda}}\, (n\cancel{\lambda})^T / T!} = \frac{T!\, n^{-T}}{\prod_i X_i!} \Leftarrow \text{does not depend on } \lambda$$

Rao-Blackwell

Let θ be a parameter of interest, and $\tau(\theta)$ be some function of θ.

Let $\hat{\tau}$ be some unbiased estimator of $\tau(\theta)$, and T be a sufficient statistic for θ.

Let $\tau^* = \mathbb{E}[\hat{\tau} \mid T]$. Then:

1. $\mathbb{E}[\tau^*] = \tau$

2. $\mathrm{Var}(\tau^*) \leq \mathrm{Var}(\hat{\tau})$