Comparing estimators

MOM estimators: consistent & asymptotically normal under fairly weak assumptions

CLT: $\bar{X} \approx N$ for large n

Example

MLE: consistent, asymptotically normal under regularity conditions

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} Uniform[0, \theta]$. Some possible estimates:

MLE: $\hat{\theta} = X_{(n)}$

$$f_{X_{(n)}}(x) = \frac{n x^{n-1}}{\theta^n}$$

$$\mathbb{E}[X_{(n)}] = \frac{n}{n+1}\theta$$ (tends to underestimate)

$$Var(X_{(n)}) = \frac{\theta^2 n}{(n+1)^2(n+2)}$$

MOM: $\hat{\theta} = 2\bar{X}$

$$\mathbb{E}[\hat{\theta}] = 2\mathbb{E}[\bar{X}] = \frac{2\theta}{2} = \theta$$

$$Var(\hat{\theta}) = \frac{\theta^2}{3n}$$

> What properties might I want an estimator $\hat{\theta}$ to possess?

- robustness to outliers?
- $Var(\hat{\theta})$ is small ($Var(\hat{\theta}) \to 0$ as $n \to \infty$?)
- $\mathbb{E}[\hat{\theta}] = \theta$ (unbiased)
- $\hat{\theta} \xrightarrow{P} \theta$ (consistency)
- computational complexity?
- $\hat{\theta} \approx$ Normal?

Bias, Variance and MSE

Mean squared error (MSE): Let $\hat{\theta}$ be an estimator of θ

The MSE of $\hat{\theta}$ is: $\mathbb{E}_\theta[(\hat{\theta}-\theta)^2] = \mathrm{Var}(\hat{\theta}-\theta) + (\mathbb{E}_\theta[\hat{\theta}-\theta])^2$

$$\mathbb{E}_\theta[(\hat{\theta}-\theta)^2] = \mathbb{E}_\theta[(\hat{\theta} - \mathbb{E}_\theta[\hat{\theta}] + \mathbb{E}_\theta[\hat{\theta}] - \theta)^2]$$

$$= \underbrace{\mathbb{E}_\theta[(\hat{\theta} - \mathbb{E}_\theta[\hat{\theta}])^2]}_{\mathrm{Var}(\hat{\theta})} + \underbrace{\mathbb{E}_\theta[(\mathbb{E}_\theta[\hat{\theta}]-\theta)^2]}_{(\mathbb{E}_\theta[\hat{\theta}]-\theta)^2 = \mathrm{Bias}^2}$$

$$+ \underbrace{2\,\mathbb{E}_\theta[(\hat{\theta} - \mathbb{E}_\theta[\hat{\theta}])(\mathbb{E}_\theta[\hat{\theta}]-\theta)]}_{=0}$$

$$\mathrm{MSE} = \mathrm{Var}(\hat{\theta}) + \mathrm{Bias}^2(\hat{\theta})$$

One approach to choosing estimators: minimize MSE

MSE for Uniform $(0, \theta)$:

$$\text{Bias}(X_{(n)}) = \frac{n}{n+1}\theta - \theta = \frac{-\theta}{n+1}$$

$$\text{Var}(X_{(n)}) = \frac{\theta^2 n}{(n+1)^2(n+2)}$$

$$\text{MSE}(X_{(n)}) = \frac{\theta^2}{(n+1)^2} + \frac{\theta^2 n}{(n+1)^2(n+2)} = \frac{2\theta^2}{(n+1)(n+2)}$$

$$\text{MSE}(2\bar{X}) = \underset{\substack{\uparrow \\ \text{Bias}^2}}{0} + \frac{\theta^2}{3n} = \frac{\theta^2}{3n} > \frac{2\theta^2}{(n+1)(n+2)}$$

Try unbiasing $X_{(n)}$: $\hat{\theta} = \left(\frac{n+1}{n}\right) X_{(n)} \Rightarrow \mathbb{E}[\hat{\theta}] = \theta$

$$\text{Var}(\hat{\theta}) = \left(\frac{n+1}{n}\right)^2 \text{Var}(X_{(n)})$$

$$= \frac{\theta^2}{n(n+2)} < \frac{2\theta^2}{(n+1)(n+2)}$$

$V \sim \chi^2_\nu$ then $\mathbb{E}[V] = \nu$

$Var(V) = 2\nu$

Example

Suppose $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Previously, we considered

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 \qquad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

and we showed that $\mathbb{E}\widehat{\sigma}^2 = \frac{n-1}{n}\sigma^2$, $\mathbb{E}(s^2) = \sigma^2$, and $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$.

> Calculate the MSE of both $\widehat{\sigma}^2$ and s^2.

$\mathbb{E}[s^2] = \sigma^2$ $\quad$ $Bias(s^2) = 0$

$\mathbb{E}[\hat{\sigma}^2] = \frac{(n-1)}{n}\sigma^2$ $\quad$ $Bias(\hat{\sigma}^2) = -\frac{1}{n}\sigma^2$

$$Var(s^2) = Var\left(\frac{\sigma^2}{n-1} \cdot \frac{n-1}{\sigma^2} \cdot s^2\right) = \left(\frac{\sigma^2}{n-1}\right)^2 \overbrace{Var\left(\frac{n-1}{\sigma^2} s^2\right)}^{2(n-1)}$$

$$= \frac{2\sigma^4}{n-1}$$

$$MSE(s^2) = \frac{2\sigma^4}{n-1}$$

$$\mathbb{E}[\hat{\sigma}^2] = \frac{n-1}{n}\sigma^2 \qquad \text{Bias}(\hat{\sigma}^2) = \frac{-\sigma^2}{n}$$

$$\text{Var}(\hat{\sigma}^2) = \text{Var}\left(\frac{n-1}{n} S^2\right) = \left(\frac{n-1}{n}\right)^2 \text{Var}(S^2)$$

$$= \left(\frac{n-1}{n}\right)^2 \frac{2\sigma^4}{n-1}$$

$$= \frac{2\sigma^4(n-1)}{n^2}$$

$$\text{MSE}(\hat{\sigma}^2) = \left(\frac{\sigma^2}{n}\right)^2 + \frac{2\sigma^4(n-1)}{n^2} = \frac{(2n-1)\sigma^4}{n^2} < \frac{2\sigma^4}{n-1}$$

$$\text{MSE}(S^2) = \frac{2\sigma^4}{n-1}$$

Best unbiased estimators

Recall: $\hat{\theta}$ is unbiased if $\mathbb{E}[\hat{\theta}] = \theta$

Def: $\hat{\theta}$ is a best unbiased estimator of θ

if $\underbrace{MSE(\hat{\theta})}_{Var(\hat{\theta})} \leq \underbrace{MSE(\hat{\theta}^*)}_{Var(\hat{\theta}^*)}$

for any other unbiased estimator $\hat{\theta}^*$

Goal could be to find a best unbiased estimator

Cramér - Rao lower band (CRLB)

<u>Thm</u>: Let $X_1, \dots, X_n$ be a sample from a distribution with probability function $f(x \mid \theta)$, and $\theta \in \mathbb{R}$. And let $\hat{\theta}$ be an unbiased estimator of θ. Under regularity conditions,

$$\operatorname{Var}(\hat{\theta}) \geq \frac{1}{\underset{\sim}{I}(\theta)}$$