

STA 711 HW 1 Solutions

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1. Suppose that $X \sim \text{Poisson}(\lambda)$.

(a) We use the fact that $e^\lambda = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$.

$$\begin{aligned}\mathbb{E}[X] &= \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} \\ &= e^{-\lambda} \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = e^{-\lambda} \lambda \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \\ &= e^{-\lambda} \lambda e^\lambda \\ &= \lambda\end{aligned}$$

$$\begin{aligned}\mathbb{E}[X^2] &= \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \lambda e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^{x-1}}{(x-1)!} \\ &= \lambda e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^{x-1}}{(x-1)!} \\ &= \lambda e^{-\lambda} \left(\sum_{x=1}^{\infty} (x-1) \frac{\lambda^{x-1}}{(x-1)!} + \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right) \\ &= \lambda e^{-\lambda} (\lambda e^\lambda + e^\lambda) \\ &= \lambda^2 + \lambda\end{aligned}$$

Thus, $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda$.

(b)

$$\mathbb{E}[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = \exp\{\lambda(e^t - 1)\}$$

(c) From (b), $M_X(t) = \exp\{\lambda(e^t - 1)\}$, so

$$M'_X(t) = \exp\{\lambda(e^t - 1)\} \lambda e^t$$

and

$$M''_X(t) = \exp\{\lambda(e^t - 1)\} \lambda^2 e^{2t} + \exp\{\lambda(e^t - 1)\} \lambda$$

At $t = 0$, we get

$$\mathbb{E}[X] = M'_X(0) = \lambda$$

$$\mathbb{E}[X^2] = M''_X(0) = \lambda^2 + \lambda$$

and thus $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda$.

2. Suppose that $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are independent. Let $Z = X + Y$.

(a) We use the fact that the MGF for the sum of independent random variables is the product of their individual MGFs:

$$\begin{aligned} M_Z(t) &= M_X(t)M_Y(t) = \exp\{\lambda(e^t - 1)\} \exp\{\mu(e^t - 1)\} \\ &= \exp\{(\lambda + \mu)(e^t - 1)\} \end{aligned}$$

This is the MGF of a $\text{Poisson}(\lambda + \mu)$ distribution, and so by the uniqueness of MGFs we conclude $X + Y \sim \text{Poisson}(\lambda + \mu)$.

(b)

$$\begin{aligned} P(X = x | Z = n) &= \frac{P(X = x, Z = n)}{P(Z = n)} = \frac{P(X = x, Y = n - x)}{P(Z = n)} \\ &= \frac{\frac{\lambda^x e^{-\lambda}}{x!} \cdot \frac{\mu^{n-x} e^{-\mu}}{(n-x)!}}{\frac{(\lambda + \mu)^n e^{-(\lambda + \mu)}}{n!}} \\ &= \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{\lambda + \mu} \right)^x \left(1 - \frac{\lambda}{\lambda + \mu} \right)^{n-x} \end{aligned}$$

This is the pmf of a $\text{Binomial}\left(n, \frac{\lambda}{\lambda + \mu}\right)$ distribution.

3. The joint pdf is

$$f(x, y) = \begin{cases} c(x + 2y) & 0 < y < 1, 0 < x < 2 \\ 0 & \text{else} \end{cases}$$

(a) To find c , we note that $f(x, y)$ must integrate to 1.

$$\begin{aligned} c \int_0^1 \int_0^2 (x + 2y) dx dy &= c \int_0^1 \int_0^2 x dx dy + c \int_0^1 \int_0^2 2y dx dy \\ &= 2c + 2c \end{aligned}$$

so we conclude that $c = 1/4$.

(b)

$$f(x) = \int_0^1 f(x, y) dy = \frac{1}{4} \int_0^1 (x + 2y) dy = \frac{x + 1}{4}$$

(c)

$$\begin{aligned} P(Z < t) &= P(9 \leq (1 + X)^2 t) = P\left(\frac{3}{\sqrt{t}} - 1 \leq X\right) = \int_{3/\sqrt{t}-1}^1 (x + 1)/4 dx \\ &= \frac{1}{2} - \frac{9}{8t} \end{aligned}$$

Then, differentiating,

$$f_Z(t) = \frac{9}{8t^2}$$

for $t \in [9/4, 9]$

4. (a)

$$\begin{aligned} \mathbb{E}[Y] &= \int_0^\infty \frac{1}{\Gamma(k)\theta^k} y^k e^{-y/\theta} dy \\ &= \frac{1}{\Gamma(k)\theta^k} \int_0^\infty y^k e^{-y/\theta} dy \end{aligned}$$

Now, note that $y^k e^{-y/\theta}$ is the kernel of a $Gamma(k+1, \theta)$ distribution, and so we must have

$$\int_0^{\infty} y^k e^{-y/\theta} dy = \theta^{k+1} \Gamma(k+1)$$

Therefore,

$$\mathbb{E}[Y] = \theta \frac{\Gamma(k+1)}{\Gamma(k)}$$

It then only remains to show that $\frac{\Gamma(k+1)}{\Gamma(k)} = k$. We will use integration by parts; recall that $\int u dv = uv - \int v du$. Then, with $u = y^k$ and $v = -e^{-y}$,

$$\Gamma(k+1) = \int_0^{\infty} y^k e^{-y} dy = -y^k e^{-y} \Big|_0^{\infty} + \int_0^{\infty} e^{-y} k y^{k-1} dy = k \Gamma(k)$$

(b)

$$\mathbb{E}[Y^2] = \frac{1}{\Gamma(k)\theta^k} \int_0^{\infty} y^{k+1} e^{-y/\theta} dy = \frac{\theta^{k+2} \Gamma(k+2)}{\theta^k \Gamma(k)} = \theta^2 \frac{\Gamma(k+2)}{\Gamma(k)} = \theta^2 (k+1)k,$$

where similar to part (a), we have recognized $y^{k+1} e^{-y/\theta}$ as the kernel of a $Gamma(k+2, \theta)$ distribution. Then,

$$Var(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \theta^2 k^2 - \theta^2 k - \theta^2 k^2 = k\theta^2$$

(c)

$$\begin{aligned} M_Y(t) &= \mathbb{E}[e^{tY}] = \frac{1}{\Gamma(k)\theta^k} \int_0^{\infty} e^{ty} y^{k-1} e^{-y/\theta} dy \\ &= \frac{1}{\Gamma(k)\theta^k} \int_0^{\infty} y^{k-1} \exp \left\{ - \left(\frac{1}{\theta} - t \right) y \right\} dy \\ &= \frac{1}{\Gamma(k)\theta^k} \cdot \Gamma(k) \left(\frac{1}{\theta} - t \right)^{-k}, \end{aligned}$$

recognizing $y^{k-1} \exp \left\{ - \left(\frac{1}{\theta} - t \right) y \right\}$ as the kernel of a Gamma distribution. Simplifying,

$$M_Y(t) = (1 - \theta t)^{-k}$$

- (d) Using part (c), the mgf for one of the variables is $M(t) = (1 - 2t)^{-1/2}$. Since we have a sum of independent variables, the mgf for the sum is

$$M_{\sum_i Y_i}(t) = \prod_i (1 - 2t)^{-1/2} = (1 - 2t)^{-n/2}$$

which is the mgf for a $Gamma(n/2, 2)$ distribution (also known as a χ_n^2 distribution!)

5. (a)

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|\lambda]] = \mathbb{E}[\lambda] = \frac{1}{\psi} \cdot \mu\psi = \mu,$$

using the fact that λ follows a Gamma distribution, and the mean of a Gamma distribution calculated in 4(a).

(b)

$$\begin{aligned} Var(Y) &= \mathbb{E}[Var(Y|\lambda)] + Var(\mathbb{E}[Y|\lambda]) \\ &= \mathbb{E}[\lambda] + Var(\lambda) \\ &= \mu + \mu^2\psi \end{aligned}$$

(c) (It helps to use the fact that $\Gamma(y + 1) = y!$)

$$\begin{aligned} P(Y = y) &= \int_0^\infty P(Y = y|\lambda) f(\lambda) d\lambda \\ &= \int_0^\infty \frac{\lambda^y e^{-\lambda}}{y!} \cdot \frac{\lambda^{\frac{1}{\psi}-1} e^{-\frac{\lambda}{\mu\psi}}}{\Gamma(1/\psi)(\mu\psi)^{1/\psi}} d\lambda \\ &= \int_0^\infty \frac{\lambda^y e^{-\lambda}}{\Gamma(y+1)} \cdot \frac{\lambda^{\frac{1}{\psi}-1} e^{-\frac{\lambda}{\mu\psi}}}{\Gamma(1/\psi)(\mu\psi)^{1/\psi}} d\lambda \end{aligned}$$

6. (a) Since F_X is a monotonic, continuous cdf, then F_X and F_X^{-1} are both strictly increasing, and

$$P(X \leq t) = P(F_X^{-1}(U) \leq t) = P(U \leq F_X(t)) = F_X(t),$$

where the final step follows from the cdf of a *Uniform*(0, 1) distribution.

- (b) See the R part below.

7. (a)

$$\begin{aligned} \text{Cov}\left(\sum_{i=1}^m a_i X_i, \sum_{j=1}^n b_j Y_j\right) &= \mathbb{E}\left[\left(\sum_{i=1}^m a_i X_i - \sum_{i=1}^m a_i \mathbb{E}[X_i]\right)\left(\sum_{j=1}^n b_j Y_j - \sum_{j=1}^n b_j \mathbb{E}[Y_j]\right)\right] \\ &= \mathbb{E}\left[\left(\sum_{i=1}^m a_i (X_i - \mathbb{E}[X_i])\right)\left(\sum_{j=1}^n b_j (Y_j - \mathbb{E}[Y_j])\right)\right] \\ &= \mathbb{E}\left[\sum_{i=1}^m \sum_{j=1}^n a_i b_j (X_i - \mathbb{E}[X_i])(Y_j - \mathbb{E}[Y_j])\right] \\ &= \sum_{i=1}^m \sum_{j=1}^n a_i b_j \mathbb{E}[(X_i - \mathbb{E}[X_i])(Y_j - \mathbb{E}[Y_j])] \\ &= \sum_{i=1}^m \sum_{j=1}^n a_i b_j \text{Cov}(X_i, Y_j) \end{aligned}$$

- (b)

$$\begin{aligned} \text{Var}\left(\sum_{i=1}^m X_i\right) &= \text{Cov}\left(\sum_{i=1}^m X_i, \sum_{i=1}^m X_i\right) \\ &= \sum_{i=1}^m \sum_{j=1}^m \text{Cov}(X_i, X_j) \\ &= \sum_{i=1}^m \text{Var}(X_i) + \sum_{i=1}^m \sum_{j \neq i}^m \text{Cov}(X_i, X_j) \\ &= \sum_{i=1}^m \text{Var}(X_i) + 2 \sum_{i < j}^m \text{Cov}(X_i, X_j) \end{aligned}$$

8. $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Uniform}(0, 1)$, and $Y = \max\{X_1, \dots, X_n\}$

(a) For $t \in [0, 1]$:

$$\begin{aligned} P(Y \leq t) &= P(X_1, \dots, X_n \leq t) = \prod_{i=1}^n P(X_i \leq t) \text{ (independence)} \\ &= t^n \end{aligned}$$

Then, the pdf is

$$f_Y(t) = nt^{n-1}$$

(b)

$$\mathbb{E}[Y] = \int_0^1 t \cdot nt^{n-1} dt = \frac{n}{n+1} t^{n+1} \Big|_0^1 = \frac{n}{n+1}$$

(c) We want to find the pdf of $X_{(k)}$; we begin by finding the cdf, then differentiate.

To find $P(X_{(k)} \leq t)$, note that $X_{(k)} \leq t$ requires *at least* k of X_1, \dots, X_n to be $\leq t$ – and, possibly, all n are $\leq t$. That is,

$$\begin{aligned} P(X_{(k)} \leq t) &= \sum_{i=k}^n P(\text{exactly } i \text{ of } X_1, \dots, X_n \leq t) \\ &= \sum_{i=k}^n \binom{n}{i} F_X(t)^i (1 - F_X(t))^{n-i} \\ &= \sum_{i=k}^n \binom{n}{i} t^i (1 - t)^{n-i} \text{ (Uniform distribution)} \end{aligned}$$

Differentiating,

$$\begin{aligned} f(t) &= \sum_{i=k}^n \left[\binom{n}{i} i t^{i-1} (1 - t)^{n-i} - \binom{n}{i} (n - i) t^i (1 - t)^{n-i-1} \right] \\ &= \binom{n}{k} k t^{k-1} (1 - t)^{n-k} + \sum_{i=k+1}^n \binom{n}{i} i t^{i-1} (1 - t)^{n-i} - \sum_{i=k}^n \binom{n}{i} (n - i) t^i (1 - t)^{n-i-1} \end{aligned}$$

Since $n - i = 0$ when $i = n$, then we have

$$\begin{aligned} f(t) &= \binom{n}{k} k t^{k-1} (1-t)^{n-k} + \sum_{i=k+1}^n \binom{n}{i} i t^{i-1} (1-t)^{n-i} - \sum_{i=k}^{n-1} \binom{n}{i} (n-i) t^i (1-t)^{n-i-1} \\ &= \binom{n}{k} k t^{k-1} (1-t)^{n-k} + \sum_{i=k}^{n-1} \binom{n}{i+1} (i+1) t^i (1-t)^{n-i-1} - \sum_{i=k}^{n-1} \binom{n}{i} (n-i) t^i (1-t)^{n-i-1} \end{aligned}$$

Since $\binom{n}{i+1} (i+1) = \binom{n}{i} (n-i)$, the second and third terms cancel out, and we are left with

$$f(t) = \binom{n}{k} k t^{k-1} (1-t)^{n-k}$$