STA 711 HW 1 Solutions

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1. Suppose that $X \sim Poisson(\lambda)$.

   (a) We use the fact that $e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$.

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} \\
&= e^{-\lambda} \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = e^{-\lambda} \lambda \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \\
&= e^{-\lambda} \lambda e^{\lambda} \\
&= \lambda
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \lambda e^{-\lambda} \sum_{x=1}^{\infty} x^2 \frac{\lambda^{x-1}}{x!} \\
&= \lambda e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^{x-1}}{(x-1)!} \\
&= \lambda e^{-\lambda} \left(\sum_{x=1}^{\infty} (x-1) \frac{\lambda^{x-1}}{(x-1)!} + \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right) \\
&= \lambda e^{-\lambda} \left(\lambda e^{\lambda} + e^{\lambda} \right) \\
&= \lambda^2 + \lambda
\end{aligned}$$

   Thus, $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X] = \lambda$.

(b)

$$\mathbb{E}[e^{tX}] = \sum_{x=0}^{\infty} e^{tx}\frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = \exp\{\lambda(e^t-1)\}$$

(c) From (b), $M_X(t) = \exp\{\lambda(e^t-1)\}$, so

$$M_X'(t) = \exp\{\lambda(e^t-1)\}\lambda e^t$$

and

$$M_X''(t) = \exp\{\lambda(e^t-1)\}\lambda^2 e^{2t} + \exp\{\lambda(e^t-1)\}\lambda$$

At $t = 0$, we get

$$\begin{aligned}\mathbb{E}[X] &= M_X'(0) = \lambda \\ \mathbb{E}[X^2] &= M_X''(0) = \lambda^2 + \lambda\end{aligned}$$

and thus $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X] = \lambda$.

2. Suppose that $X \sim Poisson(\lambda)$ and $Y \sim Poisson(\mu)$ are independent. Let $Z = X + Y$.

(a) We use the fact that the MGF for the sum of independent random variables is the product of their individual MGFs:

$$\begin{aligned}M_Z(t) &= M_X(t)M_Y(t) = \exp\{\lambda(e^t-1)\}\exp\{\mu(e^t-1)\} \\ &= \exp\{(\lambda+\mu)(e^t-1)\}\end{aligned}$$

This is the MGF of a $Poisson(\lambda+\mu)$ distribution, and so by the uniqueness of MGFs we conclude $X + Y \sim Poisson(\lambda+\mu)$.

(b)

$$\begin{aligned}P(X = x|Z = n) &= \frac{P(X = x, Z = n)}{P(Z = n)} = \frac{P(X = x, Y = n-x)}{P(Z = n)} \\ &= \frac{\frac{\lambda^x e^{-\lambda}}{x!}\cdot\frac{\mu^{n-x}e^{-\mu}}{(n-x)!}}{\frac{(\lambda+\mu)^n e^{-(\lambda+\mu)}}{n!}} \\ &= \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{\lambda+\mu}\right)^x\left(1-\frac{\lambda}{\lambda+\mu}\right)^{n-x}\end{aligned}$$

This is the pmf of a $Binomial\left(n, \frac{\lambda}{\lambda+\mu}\right)$ distribution.

3. The joint pdf is

$$f(x,y) = \begin{cases} c(x+2y) & 0 < y < 1, 0 < x < 2 \\ 0 & \text{else} \end{cases}$$

(a) To find c, we note that $f(x,y)$ must integrate to 1.

$$\begin{aligned} c\int_0^1\int_0^2 (x+2y)dxdy &= c\int_0^1\int_0^2 xdxdy + c\int_0^1\int_0^2 2ydxdy \\ &= 2c + 2c \end{aligned}$$

so we conclude that $c = 1/4$.

(b)

$$f(x) = \int_0^1 f(x,y)dy = \frac{1}{4}\int_0^1 (x+2y)dy = \frac{x+1}{4}$$

for $x \in (0,2)$.

(c)

$$\begin{aligned} P(Z < t) &= P(9 \le (1+X)^2 t) = P(\frac{3}{\sqrt{t}} - 1 \le X) = \int_{3/\sqrt{t}-1}^{1} (x+1)/4dx \\ &= \frac{1}{2} - \frac{9}{8t} \end{aligned}$$

Then, differentiating,

$$f_Z(t) = \frac{9}{8t^2}$$

for $t \in (1,9)$

4. (a)

$$\mathbb{E}[Y] = \int_0^\infty \frac{1}{\Gamma(k)\theta^k} y^k e^{-y/\theta} dy$$
$$= \frac{1}{\Gamma(k)\theta^k} \int_0^\infty y^k e^{-y/\theta} dy$$

Now, note that $y^k e^{-y/\theta}$ is the kernel of a $Gamma(k+1, \theta)$ distribution, and so we must have

$$\int_0^\infty y^k e^{-y/\theta} dy = \theta^{k+1}\Gamma(k+1)$$

Therefore,

$$\mathbb{E}[Y] = \theta\frac{\Gamma(k+1)}{\Gamma(k)}$$

It then only remains to show that $\frac{\Gamma(k+1)}{\Gamma(k)} = k$. We will use integration by parts; recall that $\int udv = uv - \int vdu$. Then, with $u = y^k$ and $v = -e^{-y}$,

$$\Gamma(k+1) = \int_0^\infty y^k e^{-y} dy = -y^k e^{-y}\Big|_0^\infty + \int_0^\infty e^{-y} k y^{k-1} dy = k\Gamma(k)$$

(b)

$$\mathbb{E}[Y^2] = \frac{1}{\Gamma(k)\theta^k} \int_0^\infty y^{k+1} e{-y/\theta} dy = \frac{\theta^{k+2}\Gamma(k+2)}{\theta^k\Gamma(k)} = \theta^2\frac{\Gamma(k+2)}{\Gamma(k)} = \theta^2(k+1)k,$$

where similar to part (a), we have recognized $y^{k+1}e{-y/\theta}$ as the kernel of a $Gamma(k+2, \theta)$ distribution. Then,

$$Var(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \theta^2 k^2 - \theta^2 k - \theta^2 k^2 = k\theta^2$$

(c)

$$\begin{aligned}
M_Y(t) = \mathbb{E}[e^{tY}] &= \frac{1}{\Gamma(k)\theta^k} \int_0^\infty e^{ty} y^{k-1} e^{-y/\theta} dy \\
&= \frac{1}{\Gamma(k)\theta^k} \int_0^\infty y^{k-1} \exp\left\{-\left(\frac{1}{\theta} - t\right) y\right\} dy \\
&= \frac{1}{\Gamma(k)\theta^k} \cdot \Gamma(k) \left(\frac{1}{\theta} - t\right)^{-k},
\end{aligned}$$

recognizing $y^{k-1} \exp\left\{-\left(\frac{1}{\theta} - t\right) y\right\}$ as the kernel of a Gamma distribution. Simplifying,

$$M_Y(t) = (1 - \theta t)^{-k}$$

(d) Using part (c), the mgf for one of the variables is $M(t) = (1 - 2t)^{-1/2}$. Since we have a sum of independent variables, the mgf for the sum is

$$M_{\sum_i Y_i}(t) = \prod_i (1 - 2t)^{-1/2} = (1 - 2t)^{-n/2}$$

which is the mgf for a $Gamma(n/2, 2)$ distribution (also known as a χ^2_n distribution!)

5. (a)

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|\lambda]] = \mathbb{E}[\lambda] = \frac{1}{\psi} \cdot \mu\psi = \mu,$$

using the fact that λ follows a Gamma distribution, and the mean of a Gamma distribution calculated in 4(a).

(b)

$$\begin{aligned}
Var(Y) &= \mathbb{E}[Var(Y|\lambda)] + Var(\mathbb{E}[Y|\lambda]) \\
&= \mathbb{E}[\lambda] + Var(\lambda) \\
&= \mu + \mu^2\psi
\end{aligned}$$

(c) (It helps to use the fact that $\Gamma(y+1) = y!$)

$$
\begin{aligned}
P(Y=y) &= \int_0^\infty P(Y=y|\lambda) f(\lambda) d\lambda \\
&= \int_0^\infty \frac{\lambda^y e^{-\lambda}}{y!} \cdot \frac{\lambda^{\frac{1}{\psi}-1} e^{\frac{-\lambda}{\mu\psi}}}{\Gamma(1/\psi)(\mu\psi)^{1/\psi}} d\lambda \\
&= \int_0^\infty \frac{\lambda^y e^{-\lambda}}{\Gamma(y+1)} \cdot \frac{\lambda^{\frac{1}{\psi}-1} e^{\frac{-\lambda}{\mu\psi}}}{\Gamma(1/\psi)(\mu\psi)^{1/\psi}} d\lambda
\end{aligned}
$$

6. (a) Since F_X is a monotonic, continuous cdf, then F_X and F_X^{-1} are both strictly increasing, and

$$P(X \leq t) = P(F_X^{-1}(U) \leq t) = P(U \leq F_X(t)) = F_X(t),$$

where the final step follows from the cdf of a $Uniform(0,1)$ distribution.

(b) See the R part below.

7. (a)

$$
\begin{aligned}
Cov\left(\sum_{i=1}^m a_i X_i, \sum_{j=1}^n b_j Y_j\right) &= \mathbb{E}\left[\left(\sum_{i=1}^m a_i X_i - \sum_{i=1}^m a_i \mathbb{E}[X_i]\right)\left(\sum_{j=1}^n b_j Y_j - \sum_{j=1}^n b_j \mathbb{E}[Y_j]\right)\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^m a_i (X_i - \mathbb{E}[X_i])\right)\left(\sum_{j=1}^n b_j (Y_j - \mathbb{E}[Y_j])\right)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^m \sum_{j=1}^n a_i b_j (X_i - \mathbb{E}[X_i])(Y_j - \mathbb{E}[Y_j])\right] \\
&= \sum_{i=1}^m \sum_{j=1}^n a_i b_j \mathbb{E}\left[(X_i - \mathbb{E}[X_i])(Y_j - \mathbb{E}[Y_j])\right] \\
&= \sum_{i=1}^m \sum_{j=1}^n a_i b_j Cov(X_i, Y_j)
\end{aligned}
$$

(b)

$$
\begin{aligned}
Var\left(\sum_{i=1}^{m} X_i\right) &= Cov\left(\sum_{i=1}^{m} X_i, \sum_{i=1}^{m} X_i\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} Cov(X_i, X_j) \\
&= \sum_{i=1}^{m} Var(X_i) + \sum_{i=1}^{m}\sum_{j\neq i} Cov(X_i, X_j) \\
&= \sum_{i=1}^{m} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)
\end{aligned}
$$

8. $X_1, ..., X_n \overset{iid}{\sim} Uniform(0, 1)$, and $Y = \max\{X_1, ..., X_n\}$

   (a) For $t \in [0, 1]$:

$$
\begin{aligned}
P(Y \leq t) = P(X_1, ..., X_n \leq t) = \prod_{i=1}^{n} P(X_i \leq t) \text{ (independence)} \\
= t^n
\end{aligned}
$$

   Then, the pdf is

$$
f_Y(t) = nt^{n-1}
$$

   (b)

$$
\mathbb{E}[Y] = \int_0^1 t \cdot nt^{n-1} dt = \frac{n}{n+1} t^{n+1}\Big|_0^1 = \frac{n}{n+1}
$$

   (c) We want to find the pdf of $X_{(k)}$; we begin by finding the cdf, then differentiate. To find $P(X_{(k)} \leq t)$, note that $X_{(k)} \leq t$ requires *at least* k of $X_1, ..., X_n$ to be $\leq t$ – and, possibly, all n are $\leq t$. That is,

$$
\begin{aligned}
P(X_{(k)} \leq t) &= \sum_{i=k}^{n} P(\text{exactly } i \text{ of } X_1, \ldots, X_n \leq t) \\
&= \sum_{i=k}^{n} \binom{n}{i} F_X(t)^i (1 - F_X(t))^{n-i} \\
&= \sum_{i=k}^{n} \binom{n}{i} t^i (1-t)^{n-i} \quad \text{(Uniform distribution)}
\end{aligned}
$$

Differentiating,

$$
\begin{aligned}
f(t) &= \sum_{i=k}^{n} \left[\binom{n}{i} i t^{i-1} (1-t)^{n-i} - \binom{n}{i} (n-i) t^i (1-t)^{n-i-1} \right] \\
&= \binom{n}{k} k t^{k-1} (1-t)^{n-k} + \sum_{i=k+1}^{n} \binom{n}{i} i t^{i-1} (1-t)^{n-i} - \sum_{i=k}^{n} \binom{n}{i} (n-i) t^i (1-t)^{n-i-1}
\end{aligned}
$$

Since $n - i = 0$ when $i = n$, then we have

$$
\begin{aligned}
f(t) &= \binom{n}{k} k t^{k-1} (1-t)^{n-k} + \sum_{i=k+1}^{n} \binom{n}{i} i t^{i-1} (1-t)^{n-i} - \sum_{i=k}^{n-1} \binom{n}{i} (n-i) t^i (1-t)^{n-i-1} \\
&= \binom{n}{k} k t^{k-1} (1-t)^{n-k} + \sum_{i=k}^{n-1} \binom{n}{i+1} (i+1) t^i (1-t)^{n-i-1} - \sum_{i=k}^{n-1} \binom{n}{i} (n-i) t^i (1-t)^{n-i-1}
\end{aligned}
$$

Since $\binom{n}{i+1}(i+1) = \binom{n}{i}(n-i)$, the second and third terms cancel out, and we are left with

$$
f(t) = \binom{n}{k} k t^{k-1} (1-t)^{n-k}
$$