Lecture 14: Continuing convergence of random variables

Relationships between types of convergence

- If $X_n \xrightarrow{a.s.} X$, then $X_n \xrightarrow{p} X$
- If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$
- If $X_n \xrightarrow{d} c$, where c is a constant, then $X_n \xrightarrow{p} c$

Proof $(X_n \xrightarrow{P} \Rightarrow X_n \xrightarrow{d} X)$

Proof: WTS $F_{X_n}(t) \to F_X(t)$ $\forall t$ where F_X is continuous, as $n \to \infty$

Let $\varepsilon > 0$

$$F_X(t-\varepsilon) \leq F_X(t) \leq F_X(t+\varepsilon) \quad \text{(cdf is non-decreasing)}$$

If F_X is continuous at t, $\lim_{\varepsilon \to 0} F_X(t-\varepsilon) = F_X(t) = \lim_{\varepsilon \to 0} F_X(t+\varepsilon)$

It suffices to show that $\forall \varepsilon > 0$, $F_X(t-\varepsilon) \leq \lim_{n \to \infty} F_{X_n}(t) \leq F_X(t+\varepsilon)$

Formal proof: Let $\varepsilon > 0$, and let t be an arbitrary continuity point of F_X.

$$F_{X_n}(t) = P(X_n \leq t) = \underbrace{P(X_n \leq t, X \leq t+\varepsilon)}_{\leq P(X \leq t+\varepsilon)} + \underbrace{P(X_n \leq t, X > t+\varepsilon)}_{\leq P(|X_n - X| > \varepsilon)}$$

$$\Rightarrow F_{X_n}(t) \leq F_X(t+\varepsilon) + \underbrace{P(|X_n - X| > \varepsilon)}_{\to 0 \text{ as } n \to \infty}$$ b/c $X_n \xrightarrow{P} X$ by assumption

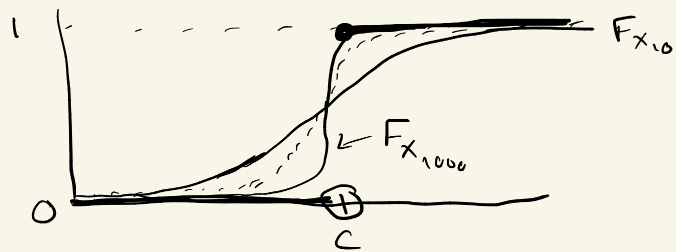
Similarly, $F_X(t-\varepsilon) - P(|X_n - X| > \varepsilon) \leq F_{X_n}(t)$

$$\Rightarrow F_X(t-\varepsilon) - \underbrace{P(|X_n - X| > \varepsilon)}_{\to 0} \leq F_{X_n}(t) \leq F_X(t+\varepsilon) + \underbrace{P(|X_n - X| > \varepsilon)}_{\to 0}$$

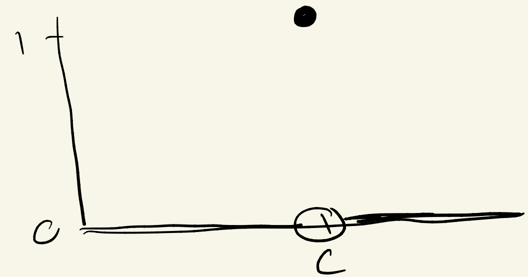
$\lim n \to \infty$: $\Rightarrow F_X(t-\varepsilon) \leq F_{X_n}(t) \leq F_X(t+\varepsilon)$ $\forall \varepsilon > 0$ //

Proof of (c) (If $X_n \xrightarrow{d} c$, where c is a constant, then $X_n \xrightarrow{P} c$)

cdf of point mass at c



pmf:



WTS $X_n \xrightarrow{P} c$ i.e. WTS $\forall\, \varepsilon > 0$, $P(|X_n - c| > \varepsilon) \to 0$

Proof: Let $\varepsilon > 0$.

$$
\begin{aligned}
P(|X_n - c| > \varepsilon) &= P(X_n < c - \varepsilon \text{ or } X_n > c + \varepsilon) \\
&= P(X_n < c - \varepsilon) + P(X_n > c + \varepsilon) \\
&\le P(X_n \le c - \varepsilon) + P(X_n > c + \varepsilon) \to 0 \\
&= \underbrace{F_{X_n}(c - \varepsilon)}_{\to F_c(c-\varepsilon) = 0} + \underbrace{(1 - F_{X_n}(c + \varepsilon))}_{\to 1 - F_c(c+\varepsilon) = 1 - 1 = 0}
\end{aligned}
$$

(convergence in distribution)

$\Rightarrow P(|X_n - c| > \varepsilon) \to 0$ as $n \to \infty$ //

$X_n \xrightarrow{d} c$ means $\forall\, t$ where F_c is continuous $F_{X_n}(t) \xrightarrow{} F_c(t)$

Practice question

Suppose that $X_1, X_2, \ldots \overset{\text{iid}}{\sim} \text{Uniform}(0,1)$. Then $X_{(n)} \xrightarrow{p} 1$.

$X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$

WTS $\quad P(|X_{(n)} - 1| > \varepsilon) \to 0 \quad$ as $n \to \infty \quad \forall\, \varepsilon > 0$

Pf: Let $\varepsilon > 0$ (and $\varepsilon < 1$)

$$P(|X_{(n)} - 1| > \varepsilon) = 1 - P(-\varepsilon \le X_{(n)} - 1 \le \varepsilon)$$
$$= 1 - P(1-\varepsilon \le X_{(n)} \le 1+\varepsilon)$$
$$= 1 - P(1-\varepsilon \le X_{(n)})$$

(we know $X_{(n)} \le 1$)

$$P(1-\varepsilon \le X_{(n)}) = 1 - P(X_{(n)} \le 1-\varepsilon)$$
$$= 1 - P(X_1 \le 1-\varepsilon)\,P(X_2 \le 1-\varepsilon) \cdots P(X_n \le 1-\varepsilon)$$
$$= 1 - (1-\varepsilon)^n$$

$$P(|X_{(n)} - 1| > \varepsilon) = 1 - (1 - (1-\varepsilon)^n) = (1-\varepsilon)^n \to 0$$

as $n \to \infty$ //