Lecture 11: Probability inequalities

What we need to do

Markov's inequality

Theorem: Let Y be a non-negative random variable, and suppose that $\mathbb{E}[Y]$ exists. Then for any $t > 0$,

$$P(Y \geq t) \leq \frac{\mathbb{E}[Y]}{t}$$

Chebyshev's inequality

Theorem: Let Y be a random variable, and let $\mu = \mathbb{E}[Y]$ and $\sigma^2 = Var(Y)$. Then

$$P(|Y - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

With your neighbor, apply Markov's inequality to prove Chebyshev's inequality.

Cauchy-Schwarz inequality

Theorem: For any two random variables X and Y,

$$|\mathbb{E}[XY]| \leq \mathbb{E}|XY| \leq (\mathbb{E}[X^2])^{1/2}(\mathbb{E}[Y^2])^{1/2}$$

Example: The *correlation* between X and Y is defined by

$$\varrho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$$

Using the Cauchy-Schwarz inequality, we can show that
$-1 \leq \varrho(X, Y) \leq 1$.