Lecture 37: FWER and Holm's procedure

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Recap: FWER and Bonferroni correction

Definition: Suppose we test m null hypotheses $H_{0,1}, ..., H_{0,m}$. The *family-wise error rate* is the probability of making *at least one* type I error:

$$FWER = P\left(\bigcup_{i:H_{0,i} \text{ is true}} \{\text{reject } H_{0,i}\}\right)$$

Bonferroni correction: To control FWER at level α,

reject H_{0i} if $p_i < \frac{\alpha}{m}$

Holm's procedure

Bonferroni correction: union bound →

reject if $p_i < \alpha^*$ then $FWER \le m_0 \alpha^*$

want: reject if $p_i < \frac{\alpha}{m_0}$ but m_0 is unknown

Suppose we test 5 hypotheses, and observe p-values 0.4, 0.01, 0, 0, 0. Does it still seem reasonable to use the Bonferroni cutoff $\alpha/5$ for each test?

"ideal" Bonferroni correction: $\frac{\alpha}{m_0}$

Since we don't know m_0, use $\frac{\alpha}{m} \le \frac{\alpha}{m_0}$

↑ more stringent cutoff

⇒ less power if H_A is true

idea: order p-values $p_{(1)} \le p_{(2)} \le \dots \le p_{(m)}$

First test: if $p_{(1)} < \frac{\alpha}{m}$ (Bonferroni threshold)

if we reject $H_{0(1)}$, consider $p_{(2)}$

reject $H_{0(2)}$ if (m-1 tests remaining)

$$p_{(2)} < \frac{\alpha}{m-1}$$

continue: reject $H_{0(i)}$ if $p_{(i)} < \frac{\alpha}{m-i+1}$

As soon as $p_{(i)} > \frac{\alpha}{m-i+1}$, stop (fail to reject $H_{0(i)}, H_{0(i+1)}, \dots$)

Holm's procedure

Suppose we test m null hypotheses $H_{0,1}, \ldots, H_{0,m}$. Let p_i be the corresponding p-value for test i.

- Order the p-values $p_{(1)} \leq p_{(2)} \leq \cdots \leq p_{(m)}$
- Let $i^* = \min\left\{i : p_{(i)} > \frac{\alpha}{m-i+1}\right\}$
- Reject $H_{0,(i)}$ for all $i < i^*$

Claim: Holm's procedure controls FWER at level α

Pf: Let $I_0 = \{i : H_{0(i)}$ is true$\}$. Let $m_0 = |I_0|$ (# true nulls)

Let $j = \min(I_0)$ (index of true null hyp. w/ smallest p-value)

($\forall i < j$, $H_{0(i)}$ is false)

Holm's procedure compares $p_{(j)}$ to $\frac{\alpha}{m-j+1}$

If $p_{(j)} > \frac{\alpha}{m-j+1}$, fail to reject all of true nulls

$P($fail to reject all true nulls$) \geq P\left(p_{(j)} > \frac{\alpha}{m-j+1}\right)$

$\Rightarrow P($at least one type I error$) \leq P\left(p_{(j)} < \frac{\alpha}{m-j+1}\right)$

Holm's procedure

Suppose we test m null hypotheses $H_{0,1}, \ldots, H_{0,m}$. Let p_i be the corresponding p-value for test i.

- Order the p-values $p_{(1)} \leq p_{(2)} \leq \cdots \leq p_{(m)}$
- Let $i^* = \min\left\{i : p_{(i)} > \frac{\alpha}{m-i+1}\right\}$
- Reject $H_{0,(i)}$ for all $i < i^*$

Claim: Holm's procedure controls FWER at level α

FWER = P(at least one type I error) $\leq P\left(p_{(j)} < \frac{\alpha}{m-j+1}\right)$

There are m_0 elements in I_0 and

$j = \min(I_0) \Rightarrow m-j+1 \geq m_0$

$\Rightarrow P\left(p_{(j)} < \frac{\alpha}{m-j+1}\right) \leq P\left(p_{(j)} \leq \frac{\alpha}{m_0}\right)$

$= P\left(\min_{i \in I_0} p_i < \frac{\alpha}{m_0}\right)$ (union bound)

$= P\left(\bigcup_{i \in I_0} \left\{p_i < \frac{\alpha}{m_0}\right\}\right) \leq m_0\left(\frac{\alpha}{m_0}\right) = \alpha$

$\Rightarrow$ FWER $\leq \alpha$ //