Lecture 21: Neyman-Pearson lemma

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Neyman-Pearson test

Let $X_1, ..., X_n$ be a sample from some distribution with probability function f and parameter θ. To test

$$H_0 : \theta = \theta_0 \qquad H_A : \theta = \theta_1,$$

the **Neyman-Pearson** test rejects H_0 when

$$\frac{L(\theta_1|X)}{L(\theta_0|X)} = \frac{f(X_1, ..., X_n|\theta_1)}{f(X_1, ..., X_n|\theta_0)} > k$$

where k is chosen so that $\beta(\theta_0) = \alpha$.

Warm-up

Let $X_1, \ldots, X_n \overset{iid}{\sim} Poisson(\lambda)$, with pmf $f(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$. We want to test $H_0 : \lambda = \lambda_0$ vs. $H_A : \lambda = \lambda_1$, where $\lambda_1 > \lambda_0$. The Neyman-Pearson test rejects when

$$\frac{L(\lambda_1|\mathbf{X})}{L(\lambda_0|\mathbf{X})} > k.$$

want k st
$\beta(\lambda_0) = \alpha$

1. Calculate $\dfrac{L(\lambda_1|\mathbf{X})}{L(\lambda_0|\mathbf{X})}$
2. Rearrange the ratio to show that $\dfrac{L(\lambda_1|\mathbf{X})}{L(\lambda_0|\mathbf{X})} > k$ if and only if $\sum_i X_i > c$ for some c
3. Using the fact that $\sum_i X_i \sim Poisson(n\lambda)$, find c such that $\beta(\lambda_0) = \alpha$

$$L(\lambda | x) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda} \lambda^{\sum_i x_i}}{\prod_i x_i!}$$

$$\frac{L(\lambda_1 | x)}{L(\lambda_0 | x)} = \frac{e^{-n\lambda_1} \lambda_1^{\sum_i x_i}}{e^{-n\lambda_0} \lambda_0^{\sum_i x_i}} = e^{n(\lambda_0 - \lambda_1)} \left(\frac{\lambda_1}{\lambda_0}\right)^{\sum_i x_i} > K$$

$$\Leftrightarrow \quad \left(\sum_i x_i\right)(\log \lambda_1 - \log \lambda_0) + n(\lambda_0 - \lambda_1) > \log K$$

$$\Leftrightarrow \quad \underbrace{\sum_i x_i}_{\sim \text{Poisson}(n\lambda)} > \frac{\log K + n(\lambda_1 - \lambda_0)}{(\log \lambda_1 - \log \lambda_0)} = c$$

Under H_0, $\lambda = \lambda_0 \Rightarrow \sum_i X_i \sim \text{Poisson}(n\lambda_0)$

$\Rightarrow c =$ upper α quantile of $\text{Poisson}(n\lambda_0)$

Uniformly most powerful tests

Big idea: can't do better than the Neyman-Pearson test for two simple hypotheses!

What does it mean for one test to be "better" than another?

Definition: Consider testing $H_0 : \theta \in \Theta_0$ vs. $H_A : \theta \in \Theta_1$. Let $\mathcal{C}_\alpha$ be the set of level α tests for these hypotheses. A test in C_α is the **uniformly most powerful** level α test if:

It has power $\beta(\theta) \geq \beta^*(\theta) \quad \forall \theta \in \Theta_1$

power function for UMP test

power function for any other level α test of these hypotheses (i.e. $\beta^*(\theta) \leq \alpha \ \forall \theta \in \Theta_0$)

Neyman-Pearson setting: $H_0 : \theta = \theta_0 \qquad H_A : \theta = \theta_1$

UMP: $\beta(\theta_1) \geq \beta^*(\theta_1)$

Neyman-Pearson lemma

Lemma: The Neyman-Pearson test is a *uniformly most powerful* level α test of $H_0 : \theta = \theta_0$ vs. $H_A : \theta = \theta_1$.

Pf: N-P test rejects when $\dfrac{f(x|\theta_1)}{f(x|\theta_0)} > k$

where we choose k st $\beta_{NP}(\theta_0) = \alpha$

$\Rightarrow$ N-P test is a level α test ✓

Let β^* be power of another level α test of these hypotheses

$\Rightarrow \beta^*(\theta_0) \leq \alpha \quad \Rightarrow \beta_{NP}(\theta_0) - \beta^*(\theta_0) \geq 0$

WTS: $\beta_{NP}(\theta_1) \geq \beta^*(\theta_1) \quad \Rightarrow \beta_{NP}(\theta_1) - \beta^*(\theta_1) \geq 0$

In fact, we will show that

$$\beta_{NP}(\theta_1) - \beta^*(\theta_1) \geq k\left(\beta_{NP}(\theta_0) - \beta^*(\theta_0)\right) \geq 0$$

Neyman-Pearson lemma

Lemma: The Neyman-Pearson test is a *uniformly most powerful* level α test of $H_0 : \theta = \theta_0$ vs. $H_A : \theta = \theta_1$.

WTS $\quad \beta_{NP}(\theta_1) - \beta^*(\theta_1) \geq k\left(\beta_{NP}(\theta_0) - \beta^*(\theta_0)\right) \geq 0$

Let ϕ_{NP} denote N-P rejection function

$$\phi_{NP}(x) = \begin{cases} 1 & \text{N-P test rejects} \\ 0 & \text{N-P test fails to reject} \end{cases}$$

Likewise, ϕ^* rejection function for other test

$$\beta_{NP}(\theta) = P_\theta(\text{reject } H_0) = P_\theta(\phi_{NP}(x) = 1)$$

$$= \int_x \phi_{NP}(x) f(x \mid \theta)\, dx$$

$$\beta^*(\theta) = \int_x \phi^*(x) f(x \mid \theta)\, dx$$

Neyman-Pearson lemma

Lemma: The Neyman-Pearson test is a *uniformly most powerful* level α test of $H_0 : \theta = \theta_0$ vs. $H_A : \theta = \theta_1$.

$$\beta_{NP}(\theta_1) - \beta^*(\theta_1) = \int_x (\phi_{NP}(x) - \phi^*(x)) f(x|\theta_1)\, dx$$

$$k(\beta_{NP}(\theta_0) - \beta^*(\theta_0)) = \int_x (\phi_{NP}(x) - \phi^*(x))\, k f(x|\theta_0)\, dx$$

N.P test rejects when $f(x|\theta_1) > k f(x|\theta_0)$

$$\Rightarrow \quad f(x|\theta_1) - k f(x|\theta_0) \begin{cases} > 0 & \text{if } \phi_{NP}(x) = 1 \\ \leq 0 & \text{if } \phi_{NP}(x) = 0 \end{cases}$$

$$[\beta_{NP}(\theta_1) - \beta^*(\theta_1)] - k(\beta_{NP}(\theta_0) - \beta^*(\theta_0))$$

$$= \int_x (\phi_{NP}(x) - \phi^*(x))(f(x|\theta_1) - k f(x|\theta_0))\, dx$$

Neyman-Pearson lemma

Lemma: The Neyman-Pearson test is a *uniformly most powerful* level α test of $H_0 : \theta = \theta_0$ vs. $H_A : \theta = \theta_1$.

$$\left[\beta_{NP}(\theta_1) - \beta^*(\theta_1)\right] - k\left(\beta_{NP}(\theta_0) - \beta^*(\theta_0)\right)$$

$$= \int_x \underbrace{\left(\phi_{NP}(x) - \phi^*(x)\right)\left(f(x|\theta_1) - kf(x|\theta_0)\right)} \, dx$$

show: integrand is always ≥ 0

$$\int_x \underbrace{\left(\phi_{NP}(x) - \phi^*(x)\right)}\underbrace{\left(f(x|\theta_1) - kf(x|\theta_0)\right)} \, dx$$

> 0 if $\phi_{NP}(x) = 1$

≤ 0 if $\phi_{NP}(x) = 0$

$= 0$ if tests agree

$= 1$ NP rejects, other doesn't

$= -1$ other rejects, NP doesn't

≥ 0

Neyman-Pearson lemma

Lemma: The Neyman-Pearson test is a *uniformly most powerful* level α test of $H_0 : \theta = \theta_0$ vs. $H_A : \theta = \theta_1$.

integrand ≥ 0

$$\Rightarrow \quad [\beta_{NP}(\theta_1) - \beta^*(\theta_1)] - k\,(\beta_{NP}(\theta_0) - \beta^*(\theta_0)) \geq 0$$

$$\Rightarrow \quad \beta_{NP}(\theta_1) - \beta^*(\theta_1) \geq k\,\underbrace{(\beta_{NP}(\theta_0) - \beta^*(\theta_0))}_{\geq 0}$$

$$\Rightarrow \quad \beta_{NP}(\theta_1) \geq \beta^*(\theta_1)$$

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