Lecture 11: Convergence in probability

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Convergence in probability

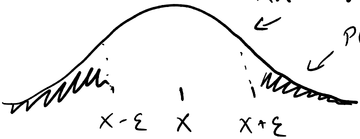
Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in probability* to a random variable X if, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0$$

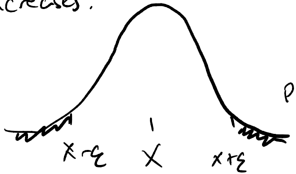
We write $X_n \xrightarrow{p} X$.

X_n distribution

$P(|X_n - X| \geq \varepsilon)$ i.e. $X_n \notin [X - \varepsilon, X + \varepsilon]$



n increases:



$P(|X_n - X| \geq \varepsilon)$ gets smaller

Example

Let $U \sim Uniform(0, 1)$, and let $X_n = \sqrt{n}\,\mathbb{I}\{U \leq 1/n\}$.

Then $X_n \xrightarrow{p} 0$.

$$X_n = \begin{cases} 0 & \text{if } U > \frac{1}{n} \\ \sqrt{n} & \text{if } U \leq \frac{1}{n} \end{cases}$$

WTS : $\forall\, \varepsilon > 0$,

$P(|X_n - 0| \geq \varepsilon) \to 0$ as $n \to \infty$

$$P(|X_n - 0| \geq \varepsilon) = P(\sqrt{n}\,\mathbb{1}\{U \leq \tfrac{1}{n}\} \geq \varepsilon)$$

$$= P(U \leq \tfrac{1}{n}) \quad \text{(for sufficiently large } n\text{)}$$

$$= \frac{1}{n} \to 0 \quad \text{as } n \to \infty$$

$\Rightarrow \quad P(|X_n - 0| \geq \varepsilon) \to 0$ as $n \to \infty$

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Weak Law of Large Numbers (WLLN)

Theorem: Let $X_1, X_2, \ldots$ be iid random variables with $\mathbb{E}[X_i] = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Then

$$\overline{X}_n \xrightarrow{p} \mu$$

Pf: WTS $P(|\bar{X}_n - \mu| \geq \varepsilon) \to 0$ as $n \to \infty$ $\forall \varepsilon > 0$

Let $\varepsilon > 0$

$$0 \leq P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{Var(\bar{X}_n)}{\varepsilon^2} \quad \text{(Chebyshev)}$$

$$Var(\bar{X}_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2} \sum_i Var(X_i)$$

$$= \frac{1}{n^2} \cdot n \cdot \sigma^2 = \frac{\sigma^2}{n}$$

$$0 \leq P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \to 0 \quad \text{as } n \to \infty$$

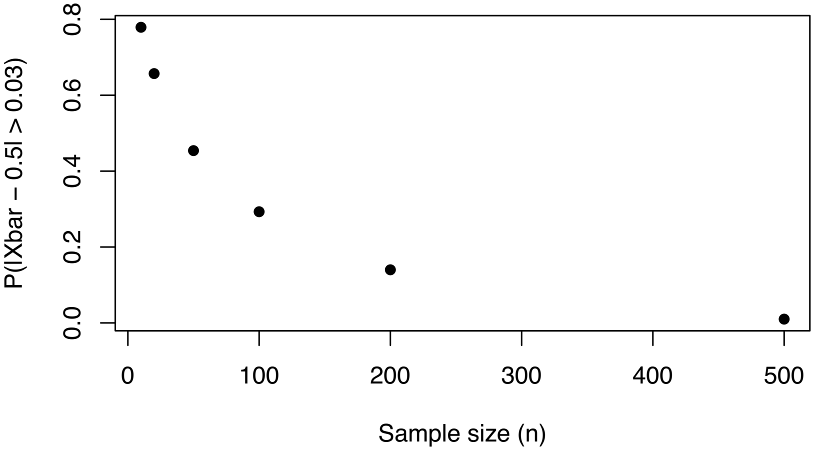
$$\Rightarrow P(|\bar{X}_n - \mu| \geq \varepsilon) \to 0 \quad \text{as } n \to \infty \qquad //$$

Activity Part I

Conduct a simulation to see the WLLN in action:

https://sta711-s25.github.io/class_activities/ca_lecture_11.html

Activity Part I



Another example

Suppose that $X_1, X_2, ... \overset{iid}{\sim} Uniform(0, 1)$, and let $X_{(n)} = \max\{X_1, ..., X_n\}$. Then $X_{(n)} \xrightarrow{p} 1$.

Activity Part II

Use a simulation to verify the Uniform example from the previous slide:

https://sta711-s25.github.io/class_activities/ca_lecture_11.html