Lecture 15: Asymptotic normality and beginning testing

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Convergence of the MLE

Suppose that we observe $Y_1, Y_2, Y_3, ...$ iid from a distribution with probability function $f(y|\theta)$, where $\theta \in \mathbb{R}^d$ is the parameter(s) we are trying to estimate. Let

$$\ell_n(\theta) = \sum_{i=1}^n \log f(Y_i|\theta)$$
 $\widehat{\theta}_n = \operatorname{argmax}_{\theta} \ell_n(\theta)$

$$\mathcal{I}_1(heta) = -\mathbb{E}\left[rac{\partial^2}{\partial heta^2} \log f(Y_i| heta)
ight]$$

Theorem: Under certain regularity conditions,

(a)
$$\widehat{\theta}_n \stackrel{p}{\to} \theta$$

(b) $\sqrt{n}(\widehat{\theta}_n - \theta) \stackrel{d}{\to} N(0, \mathcal{I}_1^{-1}(\theta))$

Proof of asymptotic normality

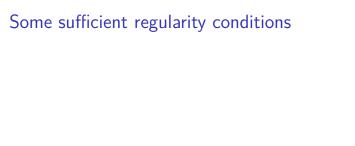
Slutsky's theorem

Let $\{X_n\}$, $\{Y_n\}$ be sequences of random variables, and suppose that $X_n \stackrel{d}{\to} X$ and $Y_n \stackrel{p}{\to} c$ for some constant c. Then:

$$\triangleright$$
 $X_n + Y_n \rightarrow$

$$ightharpoonup X_n Y_n
ightarrow$$

$$ightharpoonup \frac{X_n}{Y_n}
ightharpoonup$$



A counterexample

Suppose $Y_1, Y_2, ... \stackrel{\textit{iid}}{\sim} \textit{Uniform}(0, \theta)$.

A counterexample

Suppose $Y_1, Y_2, ... \stackrel{iid}{\sim} Bernoulli(p)$.

$$\triangleright \widehat{p} =$$

▶ If p = 0 or p = 1, what is $\sqrt{n}(\hat{p} - p)$?

Where we are going

So far:

- ▶ How can we estimate parameters/ fit a model?
- Asymptotic properties of MLEs

Next:

How can we use our estimates for inference?

Future:

- General hypothesis testing framework
- Other methods for testing hypotheses
- Confidence intervals

Let $X_1, X_2, ...$ be an iid sample from a population with mean μ and variance σ^2 . We want to test

$$H_0: \mu = \mu_0$$
 $H_A: \mu \neq \mu_0$

Test statistic:

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Test statistic:
$$Z_n = \frac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}}$$

What if σ is unknown?

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Test statistic:
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Rejecting: Should we reject H_0 when Z_n is close to 0, or when Z_n is far away from 0?

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Test statistic:
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Rejecting: Reject H_0 when $|Z_n| > z_{\alpha/2}$

p-value: How do we calculate a p-value?

