Lecture 13: Comparing types of convergence

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Recap: Convergence in probability

Definition: A sequence of random variables $X_1, X_2, ...$ *converges in probability* to a random variable X if, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0$$

We write $X_n \xrightarrow{p} X$.

Convergence in distribution

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in distribution* to a random variable X if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

at all points where $F_X(x)$ is continuous. We write $X_n \xrightarrow{d} X$.

Example

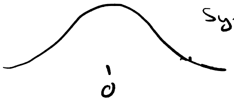
For symmetry around 0:
$P(X \le -t) = P(X \ge t)$

Suppose that $X \sim N(0,1)$, and let $X_n = -X$ for $n = 1, 2, 3, \ldots$.

Claim: $\underbrace{X_n \xrightarrow{d} X}_{(a)}$, but X_n $\underbrace{\text{does } \textit{not} \text{ converge}}_{(b)}$ in probability to X

(a) WTS $F_{X_n}(t) \to F_X(t)$ $\forall$ t where F_X is continuous

Intuition:



Symmetric around 0

X & $-X$ have same distribution

If X is symmetric around 0:
$P(X \le -t) = P(X \ge t)$

$F_{X_n}(t) = P(-X \le t) = P(X \ge -t)$
$= P(X \le t)$ //

(b) WTS $X_n \overset{P}{\not\to} X$

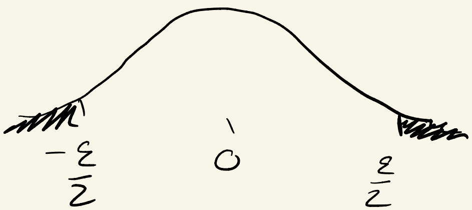
convergence in probability: $\forall \varepsilon > 0, \ P(|X_n - X| > \varepsilon) \to 0$ as $n \to \infty$

$X_n = -X$

$|X_n - X| = |-X - X| = 2|X|$

$P(|X_n - X| > \varepsilon) = P(2|X| > \varepsilon) = P\left(|X| > \frac{\varepsilon}{2}\right)$

$= P\left(X < -\frac{\varepsilon}{2}\right) + P\left(X > \frac{\varepsilon}{2}\right)$



$\lim\limits_{n \to \infty} P(|X_n - X| > \varepsilon) = P\left(X < -\frac{\varepsilon}{2}\right) + P\left(X > \frac{\varepsilon}{2}\right)$

> 0

$\Rightarrow X_n \overset{P}{\not\to} X$

Relationships between types of convergence

(a) If $X_n \xrightarrow{d} c$, where c is a constant, then $X_n \xrightarrow{p} c$

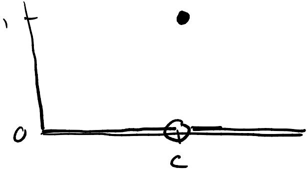
(b) If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$

(converge in probability is stronger)

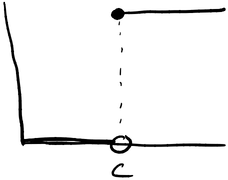
(a) constant c ← distribution is a point mass at c

pmf:

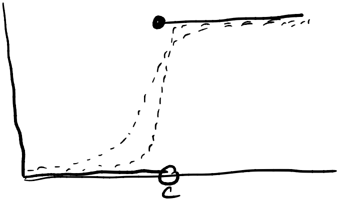
$P(c)=1$



cdf:



$X_n \xrightarrow{d} c$



Relationships between types of convergence

(a) If $X_n \xrightarrow{d} c$, where c is a constant, then $X_n \xrightarrow{p} c$

(b) If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$

<u>Pf of (a)</u>: WTS $X_n \xrightarrow{p} c$, i.e. $\forall \varepsilon > 0$ $P(|X_n - c| > \varepsilon) \to 0$

Let $\varepsilon > 0$. $P(|X_n - c| > \varepsilon) = 1 - P(|X_n - c| \leq \varepsilon)$

$= 1 - P(c - \varepsilon \leq X_n \leq c + \varepsilon)$

$= 1 - (F_{X_n}(c+\varepsilon) - F_{X_n}(c-\varepsilon))$

we know that $X_n \xrightarrow{d} c \Rightarrow F_{X_n}(t) \to F_c(t)$ $\forall t \neq c$

$\Rightarrow F_{X_n}(c+\varepsilon) \to F_c(c+\varepsilon) = P(c \leq c+\varepsilon) = 1$

$F_{X_n}(c-\varepsilon) \to F_c(c-\varepsilon) = P(c \leq c-\varepsilon) = 0$

$\Rightarrow \lim_{n\to\infty} P(|X_n - c| > \varepsilon) = \lim_{n\to\infty} (1 - (F_{X_n}(c+\varepsilon) - F_{X_n}(c-\varepsilon)))$

$= 1 - (1 - 0) = 0$ //

Relationships between types of convergence

(a) If $X_n \xrightarrow{d} c$, where c is a constant, then $X_n \xrightarrow{p} c$

(b) If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$

(b) WTS: $F_{X_n}(t) \to F_X(t)$ $\forall\ t$ where F_X is continuous

If F_X is continuous at t: $\lim_{\varepsilon \to 0} F_X(t-\varepsilon) = F_X(t) = \lim_{\varepsilon \to 0} F_X(t+\varepsilon)$

suffices to show that $\forall\ \varepsilon > 0$, $F_X(t-\varepsilon) \le \lim_{n\to\infty} F_{X_n}(t) \le F_X(t+\varepsilon)$

Let $\varepsilon > 0$, and let t be a continuity point of F_X

$$F_{X_n}(t) = P(X_n \le t) = \underbrace{P(X_n \le t,\ X \le t+\varepsilon)}_{\le P(X \le t+\varepsilon)} + \underbrace{P(X_n \le t,\ X > t+\varepsilon)}_{\substack{\text{if } X_n \le t,\ X > t+\varepsilon \\ \Rightarrow |X_n - X| > \varepsilon \\ \le P(|X_n - X| > \varepsilon)}}$$

$$\Rightarrow F_{X_n}(t) \le P(X \le t+\varepsilon) + P(|X_n - X| > \varepsilon)$$
$$= F_X(t+\varepsilon) + \underbrace{P(|X_n - X| > \varepsilon)}_{\to 0 \text{ as } n \to \infty}$$

Relationships between types of convergence

(a) If $X_n \xrightarrow{d} c$, where c is a constant, then $X_n \xrightarrow{p} c$

(b) If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$

(b) continued...

$$F_{X_n}(t) \leq F_X(t+\varepsilon) + P(|X_n - X| > \varepsilon)$$

$$\Rightarrow \lim_{n\to\infty} F_{X_n}(t) \leq F_X(t+\varepsilon) + 0 = F_X(t+\varepsilon)$$

Similarly, $F_X(t-\varepsilon) \leq \lim_{n\to\infty} F_{X_n}(t)$

So, $F_X(t-\varepsilon) \leq \lim_{n\to\infty} F_{X_n}(t) \leq F_X(t+\varepsilon) \quad \forall \varepsilon > 0$

$$\Rightarrow \lim_{\varepsilon\to 0} F_X(t-\varepsilon) \leq \lim_{n\to\infty} F_{X_n}(t) \leq \lim_{\varepsilon\to 0} F_X(t+\varepsilon)$$

$$\Rightarrow \lim_{n\to\infty} F_{X_n}(t) = F_X(t) \quad //$$