Lecture 19: Hypothesis testing framework

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General framework for hypothesis tests

Definition: Let $\theta \in \mathbb{R}^p$ be some parameter of interest. A **hypothesis** is a statement about θ. The goal of a hypothesis test is to compare two competing hypotheses:

$$H_0 : \theta \in \Theta_0 \qquad H_A : \theta \in \Theta_1$$

Outcomes

$$H_0 : \theta \in \Theta_0 \qquad H_A : \theta \in \Theta_1$$

The test either **rejects** H_0 or **fails to reject** H_0. Possible outcomes:

Goal

	H_0 is true	H_0 is false
fail to reject	correct decision	type II error
reject	type I error	correct decision

Goal: Minimize type II error rate, subject to control of type I error rate

Constructing a test

$$H_0 : \theta \in \Theta_0 \qquad H_A : \theta \in \Theta_1$$

Constructing a test

Given observed data $X_1, ..., X_n$:

1. Calculate a test statistic $T_n = T(X_1, ..., X_n)$
2. Choose a rejection region $\mathcal{R} = \{(x_1, ..., x_n) : \text{ reject } H_0\}$
3. Reject H_0 if $(X_1, ..., X_n) \in \mathcal{R}$

Example: $X_1, ..., X_n$ iid with mean μ and variance σ^2

Power function

Suppose we reject H_0 when $(X_1, ..., X_n) \in R$. The **power function** $\beta(\theta)$ is

$$\beta(\theta) = P_\theta((X_1, ..., X_n) \in R)$$

Example

$X_1, ..., X_n$ iid from a population with mean μ and variance σ^2.

$$H_0 : \mu = \mu_0 \qquad H_A : \mu \neq \mu_0$$

Rejecting H_0

$$H_0 : \theta \in \Theta_0 \qquad H_A : \theta \in \Theta_1$$

Question: A hypothesis test rejects H_0 if $(X_1, ..., X_n)$ is in the rejection region $\mathcal{R}$. Are there any issues if we only use a rejection region to test hypotheses?

p-values

$$H_0 : \theta \in \Theta_0 \qquad H_A : \theta \in \Theta_1$$

Given α, we construct a rejection region $\mathcal{R}$ and reject H_0 when $(X_1, ..., X_n) \in \mathcal{R}$. Let $(x_1, ..., x_n)$ be an observed set of data.

Definition: The **p-value** for the observed data $(x_1, ..., x_n)$ is the smallest α for which we reject H_0.

p-values

Suppose we have a test which rejects H_0 when $T(X_1, ..., X_n) > c_\alpha$, where c_α is chosen so that

$$\sup_{\theta \in \Theta_0} \beta(\theta) = \sup_{\theta \in \Theta_0} P_\theta(T(X_1, ..., X_n) > c_\alpha) = \alpha$$

Let $x_1, ..., x_n$ be a set of observed data.

Theorem: The p-value for the set of observed data $x_1, ..., x_n$ is

$$p = \sup_{\theta \in \Theta_0} P_\theta(T(X_1, ..., X_n) > T(x_1, ..., x_n))$$

Proof of theorem