Lecture 24: Likelihood ratio tests

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Likelihood ratio test

Let $X_1, ..., X_n$ be a sample from a distribution with parameter $\theta \in \mathbb{R}^d$. We wish to test $H_0 : \theta \in \Theta_0$ vs. $H_A : \theta \in \Theta_1$.

The **likelihood ratio test** (LRT) rejects H_0 when

$$\frac{\sup\limits_{\theta \in \Theta_1} L(\theta|\mathbf{X})}{\sup\limits_{\theta \in \Theta_0} L(\theta|\mathbf{X})} > k,$$

where k is chosen such that $\sup\limits_{\theta \in \Theta_0} \beta_{LR}(\theta) \leq \alpha$.

Example: linear regression with normal data

Suppose we observe $(X_1, Y_1), ..., (X_n, Y_n)$, where $Y_i = \beta^T X_i + \varepsilon_i$ and $\varepsilon_i \stackrel{iid}{\sim} N(0, \sigma^2)$. Partition $\beta = (\beta_{(1)}, \beta_{(2)})^T$. We wish to test $H_0 : \beta_{(2)} = 0$ vs. $H_A : \beta_{(2)} \neq 0$.

Asymptotics of the LRT

Suppose we observe iid data $X_1, ..., X_n$ from a distribution with parameter $\theta \in \mathbb{R}$, and we wish to test $H_0 : \theta = \theta_0$ vs. $H_A : \theta \neq \theta_0$.

Theorem: Under H_0,

$$2 \log \left(\frac{L(\widehat{\theta}_{MLE} | \mathbf{X})}{L(\theta_0 | \mathbf{X})} \right) \xrightarrow{d} \chi^2_1$$

Generalization to higher dimensions

Suppose we observe iid data $X_1, ..., X_n$ with parameter $\theta \in \mathbb{R}^d$. Partition $\theta = (\theta_{(1)}, \theta_{(2)})^T$, with $\theta_{(2)} \in \mathbb{R}^q$. We wish to test

$$H_0 : \theta_{(2)} = \mathbf{0} \qquad H_A : \theta_{(2)} \neq \mathbf{0}$$

Theorem: Under H_0,

$$2 \log \left(\frac{\sup_\theta L(\theta|\mathbf{X})}{\sup\limits_{\theta:\theta_{(2)}=0} L(\theta|\mathbf{X})} \right) \xrightarrow{d} \chi^2_q$$