STA 711 Homework 5

Due: Friday, February 21, 10:00am on Canvas.

Instructions: Submit your work as a single PDF. You may choose to either hand-write your work and submit a PDF scan, or type your work using LaTeX and submit the resulting PDF. See the course website for a homework template file and instructions on getting started with LaTeX and Overleaf.

Convergence of random variables

In this section, you will practice proving limits for sequences of random variables. As a reminder, here are some of the common techniques for proving convergence:

- For convergence in probability: if you have a sequence of means, try to apply the WLLN
- For convergence in probability: if you can easily calculate a mean or variance, try bounding probabilities with Markov's or Chebyshev's inequality
- For convergence in probability: if calculating means or variances is hard, try calculating the probabilities directly for the convergence
- For convergence in distribution to a normal or χ^2: check if the central limit theorem applies
- For convergence in distribution: if CLT is not the right strategy, try calculating the cdfs directly

1. For each of the following sequences $\{Y_n\}$, show that $Y_n \xrightarrow{p} 1$. Then write a simulation in R demonstrating the convergence.

   (a) $Y_n = 1 + nX_n$, where $X_n \sim Bernoulli(1/n)$

   (b) $Y_n = \frac{1}{n} \sum\limits_{i=1}^{n} X_i^2$, where $X_i \overset{iid}{\sim} N(0, 1)$

2. Suppose that $Y_1, Y_2, ... \overset{iid}{\sim} Beta(1, \beta)$. Find a value of ν such that $n^\nu(1 - Y_{(n)})$ converges in distribution. Then write a simulation in R demonstrating the convergence. (*Hint:* if you are struggling to find ν, starting with simulations may be helpful)

3. In this problem, we will prove part of the continuous mapping theorem. Let $\{Y_n\}$ be a sequence of real-valued random variables such that $Y_n \xrightarrow{p} Y$ for some random variable Y. Let g be a continuous function; recall that g is continuous if for all $\varepsilon > 0$, there exists some $\delta > 0$ such that $|g(x) - g(y)| < \varepsilon$ whenever $|x - y| < \delta$. Prove that $g(Y_n) \xrightarrow{p} g(Y)$.

4. Consider the simple linear regression model

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

where the X_i are known constants, and the ε_i are iid with $\mathbb{E}[\varepsilon_i] = 0$ and $Var(\varepsilon_i) = \sigma^2$. It can be shown that the least squares estimate of β_1 is

$$\widehat{\beta}_1 = \beta_1 + \frac{\sum_{i=1}^{n}(X_i - \overline{X}_n)\varepsilon_i}{\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}.$$

Show that if $\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 \to \infty$ as $n \to \infty$, then $\widehat{\beta}_1 \xrightarrow{p} \beta$. (Note: no distribution for ε_i or Y_i has been assumed, so $\widehat{\beta}_1$ cannot be treated as a maximum likelihood estimator).