Lecture 20: p-values and Neyman-Pearson test

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p-values

$$H_0 : \theta \in \Theta_0 \qquad H_A : \theta \in \Theta_1$$

Given α, we construct a rejection region $\mathcal{R}$ and reject H_0 when $(X_1, ..., X_n) \in \mathcal{R}$. Let $(x_1, ..., x_n)$ be an observed set of data.

Definition: The **p-value** for the observed data $(x_1, ..., x_n)$ is the smallest α for which we reject H_0.

p-values

Suppose we have a test which rejects H_0 when $T(X_1, ..., X_n) > c_\alpha$, where c_α is chosen so that

$$\sup_{\theta \in \Theta_0} \beta(\theta) = \sup_{\theta \in \Theta_0} P_\theta(T(X_1, ..., X_n) > c_\alpha) = \alpha$$

Let $x_1, ..., x_n$ be a set of observed data.

Theorem: The p-value for the set of observed data $x_1, ..., x_n$ is

$$p = \sup_{\theta \in \Theta_0} P_\theta(T(X_1, ..., X_n) > T(x_1, ..., x_n))$$

Proof of theorem

Recap: hypothesis testing and power function

$$H_0 : \theta \in \Theta_0 \qquad H_A : \theta \in \Theta_1$$

Given observed data $X_1, ..., X_n$:

1. Calculate a test statistic $T_n = T(X_1, ..., X_n)$
2. Choose a rejection region $\mathcal{R} = \{(x_1, ..., x_n) : \text{ reject } H_0\}$
3. Reject H_0 if $(X_1, ..., X_n) \in \mathcal{R}$

The **power function** $\beta(\theta)$ is

$$\beta(\theta) = P_\theta((X_1, ..., X_n) \in \mathcal{R})$$

Goal: maximize power for $\theta \in \Theta_1$, subject to control of power for $\theta \in \Theta_0$

Wald test for normal mean

Let $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \qquad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$. The Wald test rejects if

Wald test for normal mean

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$$H_0 : \mu = \mu_0 \qquad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$. The Wald test rejects if

$$\overline{X}_n > \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$$

We know that $\beta(\mu_0) = \alpha$ for this test.

Our question: Is there a *better* test for these hypotheses?

- To answer this question, we will need to introduce the *Neyman-Pearson* test

Rearranging

Rearranging

Let $\mathbf{X} = X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \qquad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$.

The Wald test rejects if $\overline{X}_n > c_0$, which is equivalent to rejecting when

$$\frac{L(\mu_1|\mathbf{X})}{L(\mu_0|\mathbf{X})} = \frac{f(X_1, ..., X_n|\mu_1)}{f(X_1, ..., X_n|\mu_0)} > k_0$$

Neyman-Pearson test

Let $X_1, ..., X_n$ be a sample from some distribution with probability function f and parameter θ. To test

$$H_0 : \theta = \theta_0 \qquad H_A : \theta = \theta_1,$$

the **Neyman-Pearson** test rejects H_0 when

$$\frac{L(\theta_1|X)}{L(\theta_0|X)} = \frac{f(X_1, ..., X_n|\theta_1)}{f(X_1, ..., X_n|\theta_0)} > k$$

where k is chosen so that $\beta(\theta_0) = \alpha$.

Example

Let $\mathbf{X} = X_1, ..., X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \qquad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$.

The Wald test rejects when

$$\frac{L(\mu_1|\mathbf{X})}{L(\mu_0|\mathbf{X})} > k,$$

where k is chosen such that $\beta(\mu_0) = \alpha$.

Example

Let $X_1, ..., X_n \overset{iid}{\sim} \textit{Exponential}(\theta)$, with pdf $f(x|\theta) = \theta e^{-\theta x}$. We want to test

$$H_0 : \theta = \theta_0 \qquad H_A : \theta = \theta_1,$$

where $\theta_1 < \theta_0$. The Neyman-Pearson test rejects when

$$\frac{L(\theta_1|\mathbf{X})}{L(\theta_0|\mathbf{X})} > k.$$

1. Calculate $\dfrac{L(\theta_1|\mathbf{X})}{L(\theta_0|\mathbf{X})}$
2. Rearrange the ratio to show that $\dfrac{L(\theta_1|\mathbf{X})}{L(\theta_0|\mathbf{X})} > k$ if and only if $\sum_i X_i > c$ for some c
3. Using the fact that $\sum_i X_i \sim \textit{Gamma}(n, \theta)$, find c such that $\beta(\theta_0) = \alpha$