Lecture 12: Convergence in distribution

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Logistics

- Reminder: Exam 1 released February 21 (covers HW 1–4)
- Early-semester feedback form sent out

Recap: Convergence in probability

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in probability* to a random variable X if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

We write $X_n \xrightarrow{p} X$.

Example

Suppose that $X_1, X_2, \ldots \overset{iid}{\sim} \mathit{Uniform}(0, 1)$, and let $X_{(n)} = \max\{X_1, \ldots, X_n\}$. Then $X_{(n)} \xrightarrow{p} 1$.

Convergence in distribution

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in distribution* to a random variable X if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

at all points where $F_X(x)$ is continuous. We write $X_n \xrightarrow{d} X$.

Example

Suppose that $X_1, X_2, ... \overset{iid}{\sim} Uniform(0,1)$. Let $X_{(n)} = \max\{X_1, ..., X_n\}$. Then $n(1 - X_{(n)}) \xrightarrow{d} Y$, where $Y \sim Exp(1)$.

Convergence in distribution: Central Limit Theorem

Let $X_1, X_2, ...$ be iid random variables, whose mgf exists in a neighborhood of 0. Let $\mu = \mathbb{E}[X_i]$ and $\sigma^2 = Var(X_i) < \infty$. Then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z$$

where $Z \sim N(0, 1)$.

Activity

Simulations to explore convergence in distribution:

https://sta711-s25.github.io/class_activities/ca_lecture_12.html