Asymptotic distribution of the LRT

Suppose we observe iid data $X_1, ..., X_n$ from a distribution with parameter $\theta \in \mathbb{R}$, and we wish to test $H_0 : \theta = \theta_0$ vs. $H_A : \theta \neq \theta_0$.

Theorem: Under H_0,

$$2 \log \left(\frac{L(\widehat{\theta}_{MLE}|\mathbf{X})}{L(\theta_0|\mathbf{X})} \right) \xrightarrow{d} \chi^2_1$$

Key proof pieces

1. Let $\ell(\theta) = \log L(\theta|\mathbf{X})$ denote the log-likelihood. Using a second-order Taylor expansion of $\ell(\theta_0)$ around $\widehat{\theta}$, argue that if $\widehat{\theta}_{MLE}$ is close to θ_0 then

$$2\ell(\widehat{\theta}) - 2\ell(\theta_0) \approx -\ell''(\widehat{\theta})(\widehat{\theta} - \theta_0)^2 = -\frac{1}{n}\ell''(\widehat{\theta})(\sqrt{n}(\widehat{\theta} - \theta_0))^2$$

2. Using results previously derived (when proving the asymptotic normality of the MLE), find the limits for the following two quantities when H_0 is true:

   - $-\frac{1}{n}\ell''(\widehat{\theta}) \xrightarrow{p}$

   - $\sqrt{n}(\widehat{\theta} - \theta_0) \xrightarrow{d}$

3. Apply Slutsky's theorem and the continuous mapping theorem to argue that

$$2\ell(\widehat{\theta}) - 2\ell(\theta_0) \xrightarrow{d} \chi^2_1$$