# Lecture 29: Delta method and variance stabilizing transformations

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# Motivating example: Exponential confidence interval

$$X_1, ..., X_n \overset{iid}{\sim} Exponential(\theta)$$

**Last time:** Used a pivotal quantity to find $1 - \alpha$ CI for $\theta$:
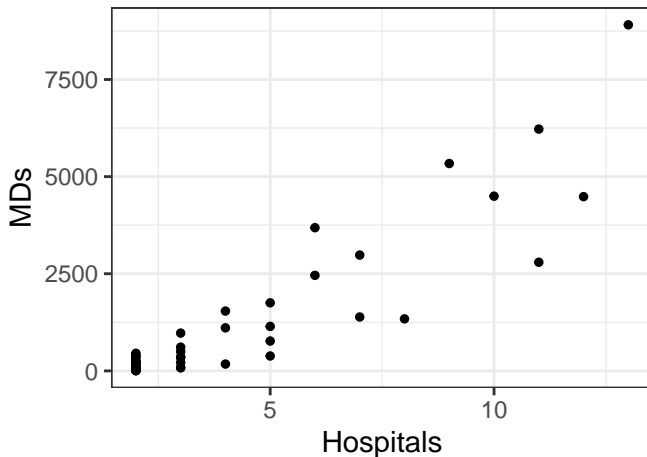
$$\left[ \frac{a}{\sum_i X_i}, \frac{b}{\sum_i X_i} \right]$$

**Alternative:** Wald Confidence interval

▶ MLE: $\widehat{\theta} =$

▶ Asymptotic distribution: $\sqrt{n}(\widehat{\theta} - \theta) \approx$

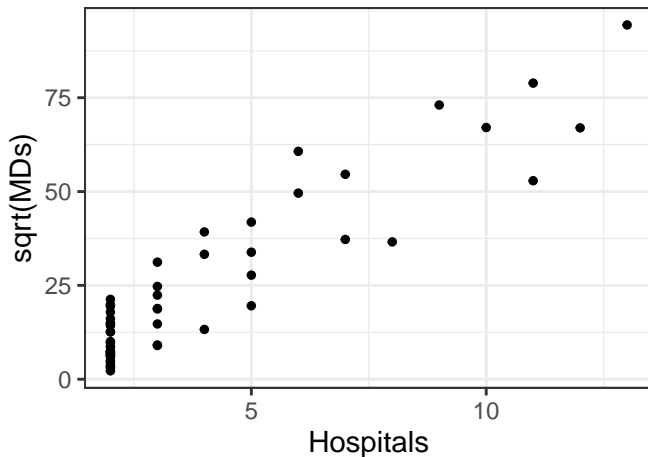▶ Wald CI:

# Motivating example: non-constant variance

**Example:** Data on the number of hospitals and number of doctors (MDs) in US counties



**Question:** How do we adjust for non-constant variance in a linear model?

# Motivating example: non-constant variance

**Example:** Data on the number of hospitals and number of doctors (MDs) in US counties

# Goal: variance stabilizing transformation

Suppose $\widehat{\theta}$ is an estimator, and $Var(\widehat{\theta})$ depends on $\theta$. Examples:

- Exponential: $\sqrt{n}(\widehat{\theta} - \theta) \approx N(0, \theta^2)$

- Poisson: $\sqrt{n}(\widehat{\lambda} - \lambda) \approx N(0, \lambda)$

- Bernoulli: $\sqrt{n}(\widehat{p} - p) \approx N(0, p(1 - p))$

**Goal:** Find a transformation $g$ such that $Var(g(\widehat{\theta}))$ does **not** depend on $\theta$

# Delta method

# Delta method

Suppose that $\widehat{\theta}$ is an estimate of $\theta \in \mathbb{R}$, such that

$$\sqrt{n}(\widehat{\theta} - \theta) \xrightarrow{d} N(0, \sigma^2)$$

for some $\sigma^2$, and $g$ is a continuously differentiable function with $g'(\theta) \neq 0$. Then

$$\sqrt{n}(g(\widehat{\theta}) - g(\theta)) \xrightarrow{d} N(0, \sigma^2[g'(\theta)]^2)$$

# Example: Exponential

$$\sqrt{n}(\widehat{\theta} - \theta) \approx N(0, \theta^2)$$

# Example: Poisson

$$\sqrt{n}(\widehat{\lambda} - \lambda) \approx N(0, \lambda)$$

# Example: Bernoulli

$$\sqrt{n}(\widehat{p} - p) \approx N(0, p(1 - p))$$

# Activity

Verify in simulations that a variance stabilizing transformation works
(produces intervals with the desired coverage):

https://sta711-s25.github.io/class_activities/ca_lecture_29.html