STA 711 Exam 1

Due: Monday, March 3, 10:00pm on Canvas.

Instructions: Submit your work as a single PDF. For this assignment, you may include written work by scanning it and incorporating it into the PDF. Include all R code needed to reproduce your results in your submission.

Mastery: To master this exam, you will need to master at least 5 of the 7 questions. (You may, of course, attempt all 7).

Rules: This is an open-book, open-notes exam. You may:

- Use any resources from the course (the textbook, the course website, class notes, previous assignments, etc.)
- Email me, or come to office hours, with specific questions (I may be somewhat less helpful than for regular assignments)
- Use one or two days from your bank of extension days, if you need more time on the exam

You may *not*:

- Use the internet to look up any questions on the exam
- Use any resources outside of the course (other textbooks, any textbook solution manuals, notes from other courses or universities, etc.)
- Use WolframAlpha or any generative AI to help solve the problems
- Discuss the exam with anyone else

Maximum likelihood questions

1. Let $Y_1, ..., Y_n \overset{iid}{\sim} Uniform(a, b)$, where a and b are unknown and $a < b$. Recall that a uniform distribution has pdf

$$f(y|a,b) = \begin{cases} \frac{1}{b-a} & a \le y \le b \\ 0 & \text{else} \end{cases}$$

   (a) Find the maximum likelihood estimators $\widehat{a}$ and $\widehat{b}$.

   (b) Let $\tau = \mathbb{E}[Y_1]$. Find the MLE $\widehat{\tau}$.

2. Let $Y_1, ..., Y_n$ be iid from a distribution with pdf

$$f(y|\lambda) = \frac{2}{\lambda\sqrt{2\pi}} e^y \exp\left\{\frac{-(e^y - 1)^2}{2\lambda^2}\right\},$$

   where $y > 0$ and $\lambda > 0$. Find the MLE of λ.

3. Let $Y_1, ..., Y_n$ be an iid sample from a continuous distribution with pdf

$$f(y|\theta) = \frac{1}{2} \exp\{-|y - \theta|\},$$

   where $-\infty < y < \infty$ and $-\infty < \theta < \infty$. Find the maximum likelihood estimator of θ. You are not required to check a second derivative for this problem.

4. Let $Y_1, ..., Y_n$ be iid from a distribution with pdf

$$f(y|\theta) = a^\theta \theta y^{-\theta-1}$$

   where $\theta > 0$, $y \ge a$, and a is a known constant. Find the MLE of θ.

5. Let $Y_1, ..., Y_n$ be iid from a distribution with pdf

$$f(y|\theta_1, \theta_2) = \begin{cases} \dfrac{1}{\theta_1 + \theta_2} \exp\left\{\dfrac{-y}{\theta_1}\right\} & y > 0 \\ \dfrac{1}{\theta_1 + \theta_2} \exp\left\{\dfrac{y}{\theta_2}\right\} & y \le 0 \end{cases}$$

   with $\theta_1, \theta_2 > 0$. Show that the maximum likelihood estimators of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are given by

$$\widehat{\theta}_1 = T_1 + \sqrt{T_1 T_2} \qquad \widehat{\theta}_2 = T_2 + \sqrt{T_1 T_2}$$

   where

$$T_1 = \frac{1}{n}\sum_{i=1}^n Y_i \mathbb{1}\{Y_i > 0\} \qquad T_2 = -\frac{1}{n}\sum_{i=1}^n Y_i \mathbb{1}\{Y_i \le 0\}.$$

   You are not required to check a second derivative for this problem.

Exponential regression

The exponential distribution with parameter λ has pdf

$$f(y|\lambda) = \frac{1}{\lambda} \exp\left\{-\frac{y}{\lambda}\right\}, \qquad y > 0.$$

6. (a) Show that this exponential distribution is an example of an exponential dispersion model (EDM), by finding the canonical parameter θ, the dispersion parameter ϕ, and the cumulant function $\kappa(\theta)$.

   (b) Suppose $Y \sim Exponential(\lambda)$ with the pdf above. Using properties of the cumulant function κ, calculate $\mathbb{E}[Y]$ and $Var(Y)$. (Your answer should use κ to calculate these values; do not integrate the pdf or derive the mgf).

   (c) Let $Y_i > 0$ be a continuous, positive response variable of interest, and let $X_i = (1, X_{i,1}, ..., X_{i,p})^T \in \mathbb{R}^{p+1}$ be a vector of covariates. Suppose we observe independent samples $(X_1, Y_1), ..., (X_n, Y_n)$ from the following model:

$$\begin{aligned} Y_i &\sim Exponential(\lambda_i) \\ -\frac{1}{\lambda_i} &= \beta^T X_i \end{aligned}$$

   Write down the score function $U(\beta|X, Y)$ and the Hessian $\mathbf{H}(\beta|X, Y)$ in matrix form. Your answer should involve the λ_i (i.e. do not leave the answer in terms of μ_i, θ_i, $Var(Y_i)$, etc.)

7. Now we apply the model from Question 6 to real data. A factory is interested in the relationship between the amount of stress applied to a piece of steel, and the time it takes until that steel breaks. We use the following model:

$$\begin{aligned} time_i &\sim Exponential(\lambda_i) \\ -\frac{1}{\lambda_i} &= \beta_0 + \beta_1 stress_i \end{aligned}$$

   The raw data contains $n = 40$ observations $(stress_1,\ time_1), ..., (stress_{40},\ time_{40})$.

   You can load the data into R by

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steel <- read.csv("https://sta711-s25.github.io/exams/steel.csv")
```

   Use Newton's method to calculate $\widehat{\beta} = (\widehat{\beta}_0, \widehat{\beta}_1)^T$. Begin at

$$\beta^{(0)} = \begin{pmatrix} -\frac{1}{\overline{Y}} \\ 0 \end{pmatrix} = \begin{pmatrix} -15.18397 \\ 0 \end{pmatrix}$$

   For the purpose of this question, stop when

$$\max\{|\beta_0^{(r+1)} - \beta_0^{(r)}|,\ |\beta_1^{(r+1)} - \beta_1^{(r)}|\} < 0.0001$$