Lecture 12: Convergence in distribution

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Logistics

- Reminder: Exam 1 released February 21 (covers HW 1–4)
- Early-semester feedback form sent out

Recap: Convergence in probability

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in probability* to a random variable X if, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0$$

We write $X_n \xrightarrow{p} X$.

Example

Suppose that $X_1, X_2, \ldots \overset{iid}{\sim}$ *Uniform*$(0, 1)$, and let $X_{(n)} = \max\{X_1, \ldots, X_n\}$. Then $X_{(n)} \xrightarrow{p} 1$.

Pf: WTS $P(|X_{(n)} - 1| > \varepsilon) \to 0$ as $n \to \infty$ $\forall \varepsilon > 0$

Let $\varepsilon > 0$

$P(|X_{(n)} - 1| > \varepsilon) = 1 - P(|X_{(n)} - 1| \leq \varepsilon)$

$= 1 - P(1 - \varepsilon \leq X_{(n)} \leq 1 + \varepsilon)$

we know $X_{(n)} \leq 1 \Rightarrow X_{(n)} \leq 1 + \varepsilon$

$\Rightarrow P(1 - \varepsilon \leq X_{(n)} \leq 1 + \varepsilon) = P(1 - \varepsilon \leq X_{(n)})$

$P(1 - \varepsilon \leq X_{(n)}) = 1 - P(X_{(n)} \leq 1 - \varepsilon)$

$\Rightarrow P(|X_{(n)} - 1| > \varepsilon) = 1 - (1 - P(X_{(n)} \leq 1 - \varepsilon)) = P(X_{(n)} \leq 1 - \varepsilon)$

$= (P(X_i \leq 1 - \varepsilon))^n = (1 - \varepsilon)^n \to 0$ as $n \to \infty$

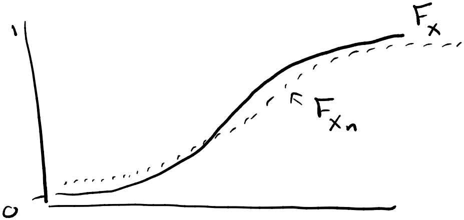
$\Rightarrow P(|X_{(n)} - 1| > \varepsilon) \to 0$ as $n \to \infty$ //

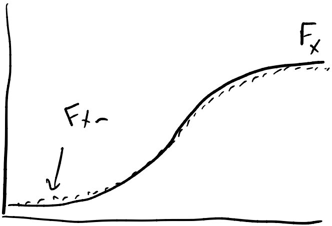
Convergence in distribution

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in distribution* to a random variable X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

at all points where $F_X(x)$ is continuous. We write $X_n \xrightarrow{d} X$.





↳ n increases

as n increases, F_{X_n} gets closer to F_X

Example

Exponential (θ) cdf $= 1 - e^{-\theta t}$

Suppose that $X_1, X_2, \ldots \overset{iid}{\sim} Uniform(0,1)$. Let $X_{(n)} = \max\{X_1, \ldots, X_n\}$. Then $n(1 - X_{(n)}) \xrightarrow{d} Y$, where $Y \sim Exp(1)$.

WTS: $F_{n(1-X_{(n)})}(t) \to F_Y(t)$ $\forall\, t$ where F_Y is continuous

$F_Y(t) = 1 - e^{-t}$

Pf: $F_{n(1-X_{(n)})}(t) = P(n(1-X_{(n)}) \le t)$

$= P(1 - X_{(n)} \le \frac{t}{n}) = P(X_{(n)} \ge 1 - \frac{t}{n})$

$= 1 - P(X_{(n)} \le 1 - \frac{t}{n})$

$= 1 - (1 - \frac{t}{n})^n$

$\lim_{n\to\infty} (1 - \frac{t}{n})^n = e^{-t}$

in general: $\lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x$

$\Rightarrow 1 - (1 - \frac{t}{n})^n \to 1 - e^{-t}$ as $n \to \infty$

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Convergence in distribution: Central Limit Theorem

Let $X_1, X_2, \ldots$ be iid random variables, whose mgf exists in a neighborhood of 0. Let $\mu = \mathbb{E}[X_i]$ and $\sigma^2 = Var(X_i) < \infty$. Then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z$$

where $Z \sim N(0, 1)$.

Intuition: $\bar{X}_n - \mu \xrightarrow{P} 0$

need to multiply by something increasing to "balance out" convergence to 0

why $\sqrt{n}$? $SD(\bar{X}_n) = \frac{\sigma}{\sqrt{n}}$

$SD(\sqrt{n}\,\bar{X}_n) = \sigma$ (not increasing or decreasing as $n \to \infty$)

Activity

Simulations to explore convergence in distribution:

https://sta711-s25.github.io/class_activities/ca_lecture_12.html