Lecture 23: Likelihood ratio tests

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Course logistics

- HW 6 due today, HW 7 on course website
- Exam 1 done (woo!)
- Exam 2 plan: released on April 4, due April 11
    - Focus on convergence, hypothesis testing, maybe confidence intervals
    - Would cover HW 5 – HW 8

Last time: binary classification and classification error

explanatory variables

response

Consider data (X, Y) with $X \in \mathbb{R}^d$ and $Y \in \{0, 1\}$. Fit a model to estimate

$$p(x) = P(Y = 1 | X = x)$$

Our binary predictions are

$$\widehat{Y} = \begin{cases} 1 & p(x) \geq h \\ 0 & p(x) < h \end{cases}$$

The **classification error** is given by $P(\widehat{Y} \neq Y)$.

Result: For any binary classifier, $h = 0.5$ minimizes classification error.

Changing the threshold

Tradeoff: increase threshold
sens↓ , spec↑

Threshold of 0.7:

		Actual	
		$Y = 0$	$Y = 1$
Predicted	$\widehat{Y} = 0$	412	136
	$\widehat{Y} = 1$	12	154

sens: $\frac{154}{290}$

spec: $\frac{412}{424}$

Threshold of 0.3:

		Actual	
		$Y = 0$	$Y = 1$
Predicted	$\widehat{Y} = 0$	309	49
	$\widehat{Y} = 1$	115	241

sens: $\frac{241}{290}$

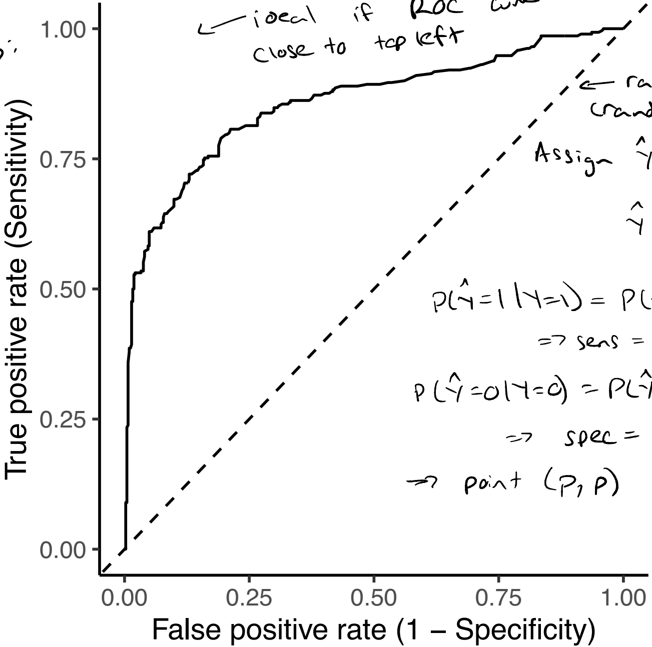
spec: $\frac{309}{424}$

← receiver operating characteristic

ROC curve: consider all thresholds

summarizing: Area under curve (AUC)

$0.5 \le AUC \le 1$



← ideal if ROC curve is close to top left

← random classifier (random guessing)

Assign $\hat{Y} = 1$ w/ prob p

$\hat{Y} = 0$ w/ prob $1-p$

$P(\hat{Y}=1 \mid Y=1) = P(\hat{Y}=1) = p$

$\Rightarrow$ sens $= p$

$P(\hat{Y}=0 \mid Y=0) = P(\hat{Y}=0) = 1-p$

$\Rightarrow$ spec $= p$

$\Rightarrow$ point (p, p) for $p \in [0,1]$

Binary classification vs. hypothesis testing

- Both binary classification and hypothesis testing involve deciding between two options
- Error metrics for both involve looking at correct decisions, false positives (type I errors), false negatives (type II errors)

Question: How do binary classification and hypothesis testing *differ*?

Binary classification vs. hypothesis testing

Binary classification:

- Can use training data to estimate performance and so choose a threshold
- Thresholds are chosen to maximize some combination of sensitivity and specificity

Hypothesis testing:

- Conceptually a two-step approach: control type I error, then hope to have good power (i.e., don't consider tests which have high type I error)
- Only see one test result; don't get to estimate type I error or power from a single test
- Want theoretical guarantees that (if assumptions are met) type I error can be controlled at desired level

Binary classification vs. hypothesis testing

- Usual approach to binary classification: maximize some combination of sensitivity and specificity
- Neyman-Pearson classification[1]: control probability of false positives (1 - specificity) at desired level, then try to maximize sensitivity

Question: Why might you choose one of these approaches over the other?

[1] Scott, C., & Nowak, R. (2005). A Neyman-Pearson approach to statistical learning. *IEEE Transactions on Information Theory*, 51(11), 3806-3819.

Previously: Neyman-Pearson test

Example: Let $X_1, ..., X_n \overset{iid}{\sim} \textit{Exponential}(\theta)$, with pdf $f(x|\theta) = \theta e^{-\theta x}$. We want to test

$$H_0 : \theta = \theta_0 \qquad H_A : \theta = \theta_1,$$

where $\theta_1 < \theta_0$. The Neyman-Pearson test rejects when

$$\frac{L(\theta_1|\mathbf{X})}{L(\theta_0|\mathbf{X})} > k.$$

Question: What should I do if I want to test the hypotheses

$$H_0 : \theta = \theta_0 \qquad H_A : \theta \neq \theta_0$$

Likelihood ratio test

Let $X_1, ..., X_n$ be a sample from a distribution with parameter $\theta \in \mathbb{R}^d$. We wish to test $H_0 : \theta \in \Theta_0$ vs. $H_A : \theta \in \Theta_1$.

The **likelihood ratio test** (LRT) rejects H_0 when

greatest likelihood under H_A

$$\frac{\sup_{\theta \in \Theta_1} L(\theta|\mathbf{X})}{\sup_{\theta \in \Theta_0} L(\theta|\mathbf{X})} > k,$$

greatest likelihood under H_0

where k is chosen such that $\sup_{\theta \in \Theta_0} \beta_{LR}(\theta) \leq \alpha$. ← control type I error

Example

Let $X_1, ..., X_n \overset{iid}{\sim} Exponential(\theta)$, with pdf $f(x|\theta) = \theta e^{-\theta x}$. We want to test

$$H_0 : \theta = \theta_0 \qquad H_A : \theta \neq \theta_0$$

LRT:
$$\frac{\sup_{\theta \in \Theta_1} L(\theta|x)}{\sup_{\theta \in \Theta_0} L(\theta|x)} = \frac{\sup_{\theta \neq \theta_0} L(\theta|x)}{L(\theta_0|x)} = \frac{\sup_{\theta} L(\theta|x)}{L(\theta_0|x)}$$

$$L(\theta|x) = \theta^n e^{-\theta \sum_i x_i}$$

$$\sup_{\theta} L(\theta|x) = L(\underset{\substack{\uparrow \\ \text{MLE}}}{\hat{\theta}}|x)$$

MLE: $\hat{\theta} = \frac{1}{\bar{X}}$

$$\frac{L(\hat{\theta}|x)}{L(\theta_0|x)} = \left(\frac{\hat{\theta}}{\theta_0}\right)^n \exp\left\{\theta_0 \sum_i X_i - \hat{\theta} \sum_i X_i\right\}$$

$$= (\theta_0 \bar{X})^{-n} \exp\left\{\theta_0 n \bar{X} - n\right\}$$

Example

Let $X_1, ..., X_n \overset{iid}{\sim} Exponential(\theta)$, with pdf $f(x|\theta) = \theta e^{-\theta x}$. We want to test

$$H_0 : \theta = \theta_0 \qquad H_A : \theta \neq \theta_0$$

Reject if $(\theta_0 \bar{X})^{-n} \exp\{\theta_0 n \bar{X} - n\} > k$

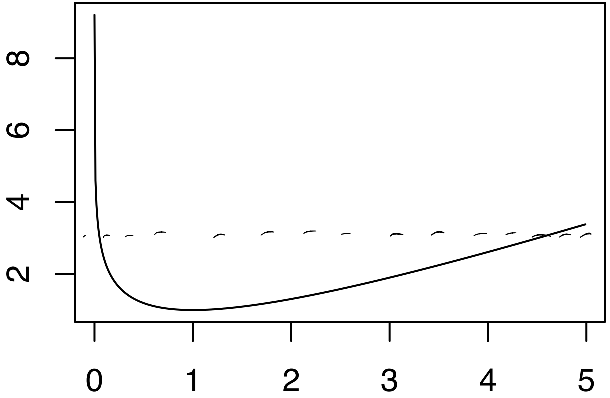
$\Rightarrow -n \log(\theta_0 \bar{X}) + n\theta_0 \bar{X} - n > \log(k)$

$\Rightarrow \theta_0 \bar{X} - \log(\theta_0 \bar{X}) > \frac{\log(k)}{n} + 1$

next week: asymptotic distribution of LRT statistic

Example

Plot of $\theta_0\overline{X} - \log(\theta_0\overline{X})$:



minimum when $\theta_0 \bar{X} = 1$ i.e. $\hat{\theta} = \frac{1}{\bar{X}} = \theta_0$

reject when $\theta_0 \bar{X}$ is far from 1 $\left(= \frac{\theta_0}{\hat{\theta}}\right)$

- no nice closed form, but could get a numerical solution
- use fact that $n\bar{X} = \sum_i X_i \sim \text{Gamma}(n, \theta_0)$ under H_0

Example: linear regression with normal data

Suppose we observe $(X_1, Y_1), ..., (X_n, Y_n)$, where $Y_i = \beta^T X_i + \varepsilon_i$ and $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$. Partition $\beta = (\beta_{(1)}, \beta_{(2)})^T$. We wish to test $H_0 : \beta_{(2)} = 0$ vs. $H_A : \beta_{(2)} \neq 0$.