Lecture 10: Probability inequalities

Ciaran Evans

Last time

- Wald tests for single coefficients:

$$Z = \frac{\widehat{\beta}_j - 0}{\widehat{SE}(\widehat{\beta}_j)} \qquad \text{under } H_0,\ Z \approx N(0,1)$$

- Tests for nested models:

$$G = 2(\log L_{\text{full}} - \log L_{\text{reduced}}) \qquad \text{under } H_0,\ G \approx \chi^2_q$$

What we need

We need to show that

$$\widehat{\beta} \approx N(\beta, \mathcal{I}^{-1}(\beta))$$

This requires:

- a notion of convergence of random variables
- asymptotic results about MLEs
- hypothesis testing fundamentals

Roadmap:

1. Preliminary machinery – probability inequalities, types of convergence, theorems about convergence
2. Properties of MLEs – consistency and asymptotic normality
3. Hypothesis testing theory – types of hypotheses, types of error, and types of hypothesis test (Neyman-Pearson, Wald, Likelihood ratio)

Markov's inequality

Theorem: Let Y be a non-negative random variable, and suppose that $\mathbb{E}[Y]$ exists. Then for any $t > 0$,

$$P(Y \geq t) \leq \frac{\mathbb{E}[Y]}{t}$$

Chebyshev's inequality

Theorem: Let Y be a random variable, and let $\mu = \mathbb{E}[Y]$ and $\sigma^2 = Var(Y)$. Then

$$P(|Y - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

With your neighbor, apply Markov's inequality to prove Chebyshev's inequality.

Cauchy-Schwarz inequality

Theorem: For any two random variables X and Y,

$$|\mathbb{E}[XY]| \leq \mathbb{E}|XY| \leq (\mathbb{E}[X^2])^{1/2}(\mathbb{E}[Y^2])^{1/2}$$

Example: The *correlation* between X and Y is defined by

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

Using the Cauchy-Schwarz inequality, show that
$-1 \leq \rho(X,Y) \leq 1$.

Jensen's inequality

Theorem: For any random variable Y, if g is a convex function, then

$$\mathbb{E}[g(Y)] \geq g(\mathbb{E}[Y])$$