Lecture 28: Pivotal quantities

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Recap

Confidence set: Let $\theta \in \Theta$ be a parameter of interest, and $X_1, ..., X_n$ a sample. A set $C(X_1, ..., X_n) \subseteq \Theta$ is a $1 - \alpha$ **confidence set** for θ if

$$\inf_{\theta \in \Theta} P_\theta(\theta \in C(X_1, ..., X_n)) = 1 - \alpha$$

Inverting a test: Create a confidence set by inverting a test:

$$C(X_1, ..., X_n) = \{\theta_0 : (X_1, ..., X_n) \notin \mathcal{R}(\theta_0)\}$$

Using confidence sets to test hypotheses

Theorem: Let $\theta \in \Theta$ and let $C(X_1,\dots,X_n)$ be a $1-\alpha$ confidence set.

For any $\theta_0 \in \Theta$, let

$$R(\theta_0) = \{(X_1,\dots,X_n) : \theta_0 \notin C(X_1,\dots,X_n)\}$$

The test which rejects $H_0: \theta = \theta_0$ when $(X_1,\dots,X_n) \in R(\theta_0)$ is a α-level test

Pf:

$$P_{\theta_0}((X_1,\dots,X_n) \in R(\theta_0)) = P_{\theta_0}(\theta_0 \notin C(X_1,\dots,X_n))$$

$$= 1 - \underbrace{P_{\theta_0}(\theta_0 \in C(X_1,\dots,X_n))}_{\geq 1-\alpha}$$

$$\leq 1-(1-\alpha) = \alpha$$

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t interval

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. We want to construct a $1 - \alpha$ CI for μ.

Inverting t-test:

reject $H_0 : \mu = \mu_0$ when $\left| \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} \right| > t_{n-1, \frac{\alpha}{2}}$

$$\Rightarrow \quad 1-\alpha \quad CI = \left\{ \mu_0 : -t_{n-1,\frac{\alpha}{2}} \le \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} \le t_{n-1,\frac{\alpha}{2}} \right\}$$

$$= \left[\bar{X} - t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}}, \quad \bar{X} + t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \right]$$

Another way to look at this:

$$\underbrace{Q(X_1, \ldots, X_n, \mu)}_{\text{pivotal quantity}} = \frac{\sqrt{n}(\bar{X} - \mu)}{s} \sim t_{n-1} \leftarrow \text{Distribution does not depend on } \mu$$

t interval

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. We want to construct a $1-\alpha$ CI for μ.

$$Q(X_1, \ldots, X_n, \mu) = \frac{\sqrt{n}(\bar{X} - \mu)}{S} \sim t_{n-1}$$

To construct a $1-\alpha$ confidence set for μ:

Find a, b st

$$P_\mu(a \le Q(X_1, \ldots, X_n, \mu) \le b) = 1-\alpha$$

E.g. $Q(X_1, \ldots, X_n, \mu) \sim t_{n-1}$

$$a = -t_{n-1, \frac{\alpha}{2}} \qquad b = t_{n-1, \frac{\alpha}{2}}$$

don't depend on μ

$$\Rightarrow \quad P_\mu\left(-t_{n-1, \frac{\alpha}{2}} \le \frac{\sqrt{n}(\bar{X} - \mu)}{S} \le t_{n-1, \frac{\alpha}{2}}\right) = 1-\alpha$$

$$\Rightarrow \quad P_\mu\left(\bar{X} - t_{n-1, \frac{\alpha}{2}} \frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t_{n-1, \frac{\alpha}{2}} \frac{S}{\sqrt{n}}\right) = 1-\alpha$$

Pivotal quantities

Let $X_1, \ldots, X_n$ be a sample and θ be an unknown parameter. A function $Q(X_1, \ldots, X_n, \theta)$ is called a **pivot** if the distribution of $Q(X_1, \ldots, X_n, \theta)$ does not depend on θ.

To get a $1-\alpha$ confidence set for θ:

1) Find a, b st $P_\theta(a \le Q(X_1, \ldots, X_n, \theta) \le b) = 1-\alpha$

2) $1-\alpha$ confidence set:

$$\{\theta : a \le Q(X_1, \ldots, X_n, \theta) \le b\}$$

Example

Let $X_1, \ldots, X_n \overset{iid}{\sim} Uniform(0, \theta)$. We want to construct a $1 - \alpha$ CI for θ using a pivotal quantity.

$\hat{\theta} = X_{(n)}$, so maybe we can use $X_{(n)}$ to create a confidence set

$P(X_{(n)} \leq t) = (P(X_i \leq t))^n = \left(\frac{t}{\theta}\right)^n \qquad t \in [0, \theta]$

$Q(X_1, \ldots, X_n, \theta) = Q(X_{(n)}, \theta) = \frac{X_{(n)}}{\theta}$

$\Rightarrow P(Q(X_{(n)}, \theta) \leq t) = P\left(\frac{X_{(n)}}{\theta} \leq t\right) = P(X_{(n)} \leq \theta t)$

$= \left(\frac{\theta t}{\theta}\right)^n = t^n$

no θ!

$\Rightarrow$ Distribution of $Q(X_{(n)}, \theta) = \frac{X_{(n)}}{\theta}$

does not depend on θ

$\Rightarrow$ pivotal quantity!

Example

Let $X_1, \ldots, X_n \overset{iid}{\sim} Uniform(0, \theta)$. We want to construct a $1-\alpha$ CI for θ using a pivotal quantity.

Pivotal quantity: $\frac{X_{(n)}}{\theta}$

Find a, b st $P\left(a \le \frac{X_{(n)}}{\theta} \le b\right) = 1-\alpha$

$P\left(\frac{X_{(n)}}{\theta} \le t\right) = t^n$

$\Rightarrow P\left(a \le \frac{X_{(n)}}{\theta} \le b\right) = b^n - a^n$ $\quad (a, b \in [0,1])$

So we want a, b st $b^n - a^n = 1-\alpha$

E.g. $b = 1 \Rightarrow 1 - a^n = 1-\alpha \Rightarrow a = \alpha^{\frac{1}{n}}$

$\Rightarrow P\left(\alpha^{\frac{1}{n}} \le \frac{X_{(n)}}{\theta} \le 1\right) = 1-\alpha$

$\Rightarrow P\left(X_{(n)} \le \theta \le \frac{X_{(n)}}{\alpha^{\frac{1}{n}}}\right) = 1-\alpha \Rightarrow$ CI: $\left[X_{(n)}, \frac{X_{(n)}}{\alpha^{1/n}}\right]$

Example

Let $X_1, \ldots, X_n \overset{iid}{\sim} Exponential(\theta)$, with pdf $f(x|\theta) = \theta e^{-\theta x}$. Find a pivotal quantity $Q(X_1, \ldots, X_n, \theta)$ and construct a $1 - \alpha$ confidence interval for θ using that quantity.

- Begin with the MLE, $\widehat{\theta} = \dfrac{n}{\sum_{i=1}^n X_i}$
- If $X \sim Exponential(\theta)$, then $cX \sim Exponential\left(\dfrac{\theta}{c}\right)$

· MLE suggests we should consider functions involving $\sum_{i=1}^{n} X_i$

· If $X \sim$ Exponential (θ), $\theta X \sim$ Exponential (1)

$\Rightarrow \quad \theta \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, 1) \quad \leftarrow$ doesn't depend on θ!

So $Q(X_1, \ldots, X_n, \theta) = \theta \sum_{i=1}^{n} X_i$ is a pivotal quantity

(next page) $\longrightarrow$

$$Q(X_1, \dots, X_n, \theta) = \theta \sum_{i=1}^{n} X_i$$

Now want a, b st $P_\theta\left(a \leq \theta \sum_i X_i \leq b\right) = 1-\alpha$

e.g. $a = \text{Gamma}(n,1)_{1-\frac{\alpha}{2}}$ (lower $\frac{\alpha}{2}$ quantile)

$b = \text{Gamma}(n,1)_{\frac{\alpha}{2}}$ (upper $\frac{\alpha}{2}$ quantile)

$$a \leq \theta \sum_i X_i \leq b$$

$$\iff \frac{a}{\sum_i X_i} \leq \theta \leq \frac{b}{\sum_i X_i}$$

So the $1-\alpha$ CI is

$$\left[\frac{a}{\sum_i X_i}, \frac{b}{\sum_i X_i}\right]$$