Lecture 18: t-tests

Ciaran Evans

Previously: Wald tests for a population mean

Suppose $X_1, ..., X_n$ are an iid sample from a population with mean μ and variance σ^2. We wish to test

$$H_0 : \mu = \mu_0 \qquad H_A : \mu \neq \mu_0$$

- If σ^2 is known:
- If σ^2 is unknown:

Issue: Wald tests with small n

The Wald test for a population mean μ relies on

$$Z_n = \frac{\sqrt{n}(\overline{X}_n - \mu)}{s} \approx N(0,1)$$

- $Z_n \xrightarrow{d} N(0,1)$ as $n \to \infty$
- But for small n, Z_n is not normal, even if $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

What is the exact distribution of $\frac{\sqrt{n}(\overline{X}_n - \mu)}{s}$?

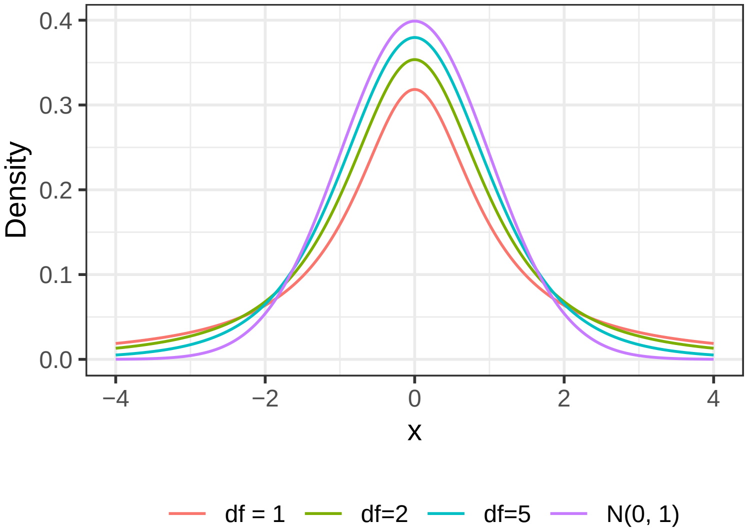
t-tests

If $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \sim N(0, 1)$$

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{s} \sim t_{n-1}$$

t-distribution



t distribution

Definition: Let $Z \sim N(0,1)$ and $V \sim \chi^2_d$ be independent. Then

$$T = \frac{Z}{\sqrt{V/d}} \sim t_d$$

Claim: If $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{s} \sim t_{n-1}$$

What we want to show

$$(n-1)\frac{s^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$(n-1)\frac{s^2}{\sigma^2} \perp\!\!\!\perp \frac{\sqrt{n}(\overline{X}-\mu)}{\sigma}$$

Decomposing the sum of squares

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 = \sum_{i=1}^{n} \left(\frac{X_i - \overline{X}}{\sigma} \right)^2 + n \left(\frac{\overline{X} - \mu}{\sigma} \right)^2$$

Cochran's theorem

Let $Z_1, ..., Z_n \overset{iid}{\sim} N(0, 1)$, and let $Z = [Z_1, ..., Z_n]^T$. Let $A_1, ..., A_k \in \mathbb{R}^{n \times n}$ be symmetric matrices such that $Z^T Z = \sum_{i=1}^{k} Z^T A_i Z$, and let $r_i = rank(A_i)$. Then the following are equivalent:

- $r_1 + \cdots + r_k = n$
- The $Z^T A_i Z$ are independent
- Each $Z^T A_i Z \sim \chi^2_{r_i}$

Application to t-tests