Lecture 29: Delta method and variance stabilizing transformations

Ciaran Evans

Motivating example: Exponential confidence interval

$$X_1, ..., X_n \overset{iid}{\sim} Exponential(\theta)$$

Last time: Used a pivotal quantity to find $1-\alpha$ CI for θ:

$$\left[\frac{a}{\sum_i X_i}, \frac{b}{\sum_i X_i}\right] = \left[a^* \hat{\theta}, \; b^* \hat{\theta}\right]$$

Alternative: Wald Confidence interval

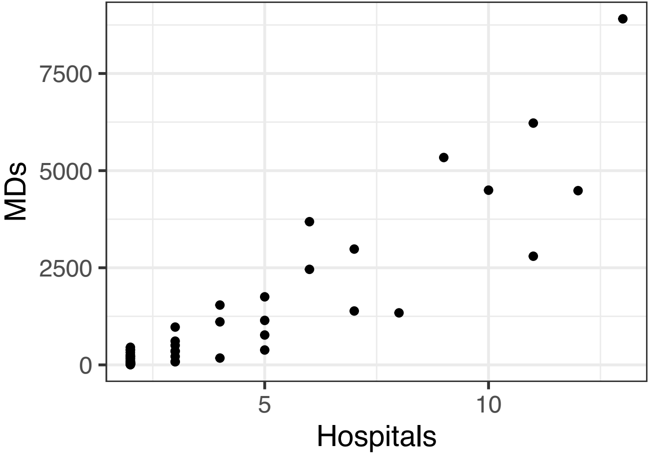
- MLE: $\widehat{\theta} = \frac{1}{\bar{X}} = \frac{n}{\sum_i X_i}$
- Asymptotic distribution: $\sqrt{n}(\widehat{\theta} - \theta) \approx N(0, \mathcal{I}_1^{-1}(\theta))$ $= N(0, \theta^2)$
- Wald CI:

$$\hat{\theta} \pm z_{\frac{\alpha}{2}} \frac{\hat{\theta}}{\sqrt{n}}$$

variance depends on the parameter we are trying to estimate!

Motivating example: non-constant variance

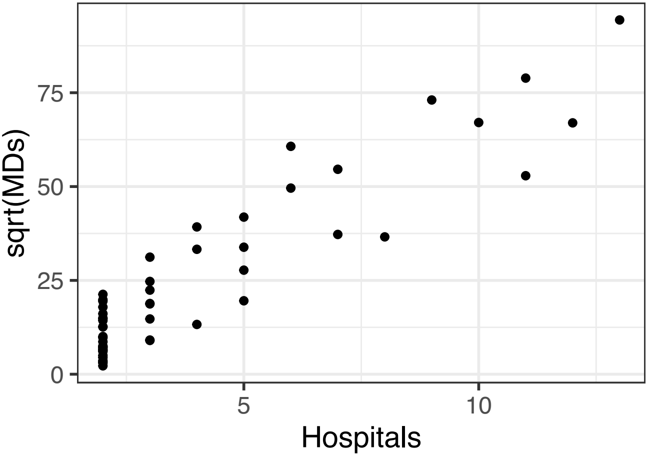
Example: Data on the number of hospitals and number of doctors (MDs) in US counties



Question: How do we adjust for non-constant variance in a linear model?

Motivating example: non-constant variance

Example: Data on the number of hospitals and number of doctors (MDs) in US counties



Goal: variance stabilizing transformation

Suppose $\widehat{\theta}$ is an estimator, and $Var(\widehat{\theta})$ depends on θ. Examples:

- Exponential: $\sqrt{n}(\widehat{\theta} - \theta) \approx N(0, \theta^2)$
- Poisson: $\sqrt{n}(\widehat{\lambda} - \lambda) \approx N(0, \lambda)$
- Bernoulli: $\sqrt{n}(\widehat{p} - p) \approx N(0, p(1-p))$

Goal: Find a transformation g such that $Var(g(\widehat{\theta}))$ does **not** depend on θ

variance stabilizing transformation

Delta method

Suppose $\hat{\theta}$ is an estimate of $\theta \in \mathbb{R}$, such that

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \sigma^2)$$

for some σ^2 (could be a function of θ). Let g be a continuously differentiable, with $g'(\theta) \neq 0$. Then

$$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \xrightarrow{d} N(0, \sigma^2 [g'(\theta)]^2)$$

Pf: Taylor expansion:

$$g(\hat{\theta}) \approx g(\theta) + g'(\theta)(\hat{\theta} - \theta)$$

$$\Rightarrow \sqrt{n}(g(\hat{\theta}) - g(\theta)) \approx \underbrace{g'(\theta) \underbrace{\sqrt{n}(\hat{\theta} - \theta)}_{\xrightarrow{d} N(0,\sigma^2)}}_{}$$

$$\xrightarrow{d} g'(\theta) N(0, \sigma^2)$$

$$= N(0, \sigma^2 [g'(\theta)]^2)$$

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Delta method

Suppose that $\widehat{\theta}$ is an estimate of $\theta \in \mathbb{R}$, such that

$$\sqrt{n}(\widehat{\theta} - \theta) \xrightarrow{d} N(0, \sigma^2)$$

for some σ^2, and g is a continuously differentiable function with $g'(\theta) \neq 0$. Then

$$\sqrt{n}(g(\widehat{\theta}) - g(\theta)) \xrightarrow{d} N(0, \sigma^2 [g'(\theta)]^2)$$

Variance stabilizing transformation: find g s.t.

$\sigma^2 [g'(\theta)]^2$

does not depend on θ

Example: Exponential

$$\sqrt{n}(\widehat{\theta} - \theta) \approx N(0, \theta^2)$$

$$\sqrt{n}\,(g(\hat{\theta}) - g(\theta)) \xrightarrow{d} N(0, \theta^2 [g'(\theta)]^2)$$

$\Rightarrow$ want $\theta^2 [g'(\theta)]^2$ <u>constant</u>

$$\Rightarrow \quad g'(\theta) \propto \frac{1}{\theta}$$

$$\Rightarrow \quad g(\theta) \propto \log(\theta)$$

$$\sqrt{n}\,(\log(\hat{\theta}) - \log(\theta)) \xrightarrow{d} N(0, 1)$$

$1-\alpha$ CI for $\log(\theta)$: $\log(\hat{\theta}) \pm z_{\frac{\alpha}{2}} \cdot \frac{1}{\sqrt{n}}$

$1-\alpha$ CI for θ :

$$\left[\exp\left\{ \log(\hat{\theta}) - z_{\frac{\alpha}{2}} \cdot \frac{1}{\sqrt{n}} \right\},\ \exp\left\{ \log(\hat{\theta}) + z_{\frac{\alpha}{2}} \cdot \frac{1}{\sqrt{n}} \right\} \right]$$

Example: Poisson

$$\sqrt{n}(\widehat{\lambda} - \lambda) \approx N(0, \lambda)$$

$\sqrt{n}\left(g(\hat{\lambda}) - g(\lambda)\right) \xrightarrow{d} N\left(0, \lambda [g'(\lambda)]^2\right)$

$\Rightarrow$ want $g'(\lambda) \propto \lambda^{-\frac{1}{2}}$

$\Rightarrow$ $g(\lambda) \propto \sqrt{\lambda}$

$g(\lambda) = \sqrt{\lambda}$

$\Rightarrow g'(\lambda) = \frac{1}{2\sqrt{\lambda}}$

$\Rightarrow [g'(\lambda)]^2 = \frac{1}{4\lambda}$

$\Rightarrow \sqrt{n}\left(\sqrt{\hat{\lambda}} - \sqrt{\lambda}\right) \xrightarrow{d} N\left(0, \frac{1}{4}\right)$

Example: Bernoulli

$$\frac{\partial}{\partial x} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\partial}{\partial x} \arcsin(\sqrt{x}) = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\sqrt{n}(\hat{p} - p) \approx N(0, p(1-p))$$

$$\sqrt{n}\left(g(\hat{p}) - g(p)\right) \xrightarrow{d} N\left(0,\ p(1-p)\,[g'(p)]^2\right)$$

$$\Rightarrow \quad g'(p) \propto \frac{1}{\sqrt{p(1-p)}} = \frac{1}{\sqrt{p}\sqrt{1-p}}$$

$$g(p) = \arcsin(\sqrt{p})$$

$$\sqrt{n}\left(\arcsin(\sqrt{\hat{p}}) - \arcsin(\sqrt{p})\right) \xrightarrow{d} N\left(0, \frac{1}{4}\right)$$

$1-\alpha$ CI for $\arcsin(\sqrt{p}) = \arcsin(\sqrt{\hat{p}}) \pm z_{\frac{\alpha}{2}} \cdot \frac{1}{2\sqrt{n}}$

$1-\alpha$ CI for p

$$\left[\ \sin\left(\arcsin(\sqrt{\hat{p}}) - z_{\frac{\alpha}{2}} \cdot \frac{1}{2\sqrt{n}}\right)^2,\ \sin\left(\arcsin(\sqrt{\hat{p}}) + z_{\frac{\alpha}{2}} \cdot \frac{1}{2\sqrt{n}}\right)^2\ \right]$$

Activity

Verify in simulations that a variance stabilizing transformation works (produces intervals with the desired coverage):

https://sta711-s25.github.io/class_activities/ca_lecture_29.html