Lecture 15: Asymptotic normality and beginning testing

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Convergence of the MLE

Suppose that we observe $Y_1, Y_2, Y_3, \ldots$ iid from a distribution with probability function $f(y|\theta)$, where $\theta \in \mathbb{R}^d$ is the parameter(s) we are trying to estimate. Let

$$\ell_n(\theta) = \sum_{i=1}^{n} \log f(Y_i|\theta)$$

$$\widehat{\theta}_n = \text{argmax}_\theta \; \ell_n(\theta)$$

$$\mathcal{I}_1(\theta) = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2} \log f(Y_i|\theta)\right]$$

Theorem: Under certain regularity conditions,

(a) $\widehat{\theta}_n \xrightarrow{p} \theta$
(b) $\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{d} N(0, \mathcal{I}_1^{-1}(\theta))$

Proof of asymptotic normality

Slutsky's theorem

Let $\{X_n\}, \{Y_n\}$ be sequences of random variables, and suppose that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$ for some constant c. Then:

- $X_n + Y_n \rightarrow$

- $X_n Y_n \rightarrow$

- $\dfrac{X_n}{Y_n} \rightarrow$

Some sufficient regularity conditions

A counterexample

Suppose $Y_1, Y_2, ... \overset{iid}{\sim} \mathit{Uniform}(0, \theta)$.

A counterexample

Suppose $Y_1, Y_2, \ldots \overset{iid}{\sim} Bernoulli(p)$.

- $\widehat{p} =$
- If $p = 0$ or $p = 1$, what is $\sqrt{n}(\widehat{p} - p)$?

Where we are going

So far:

- How can we estimate parameters/ fit a model?
- Asymptotic properties of MLEs

Next:

- How can we use our estimates for inference?

Future:

- General hypothesis testing framework
- Other methods for testing hypotheses
- Confidence intervals

Hypothesis tests for a population mean

Let $X_1, X_2, ...$ be an iid sample from a population with mean μ and variance σ^2. We want to test

$$H_0 : \mu = \mu_0 \qquad H_A : \mu \neq \mu_0$$

Test statistic:

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Test statistic: $Z_n = \dfrac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}}$

What if σ is unknown?

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Test statistic: $Z_n = \dfrac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}}$

Rejecting: Should we reject H_0 when Z_n is close to 0, or when Z_n is far away from 0?

Hypothesis tests for a population mean

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Test statistic: $Z_n = \dfrac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}}$

Rejecting: Reject H_0 when $|Z_n| > z_{\alpha/2}$

p-value: How do we calculate a p-value?

Wald test for one parameter