

# Convergence of random variables

Ciaran Evans

## Recap: convergence

**Definition:** A sequence of random variables  $X_1, X_2, \dots$  *converges in probability* to a random variable  $X$  if, for every  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} P(|X_n - X| \geq \varepsilon) = 0$$

We write  $X_n \xrightarrow{p} X$ .

**Definition:** A sequence of random variables  $X_1, X_2, \dots$  *converges in distribution* to a random variable  $X$  if

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x)$$

at all points where  $F_X(x)$  is continuous. We write  $X_n \xrightarrow{d} X$ .

# Central Limit Theorem

Let  $X_1, X_2, \dots$  be iid random variables, with  $\mu = \mathbb{E}[X_i]$  and  $\sigma^2 = \text{Var}(X_i) < \infty$ . Then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z$$

where  $Z \sim N(0, 1)$ .

## Other key results

**Continuous mapping theorem:** Let  $X_1, X_2, \dots$  be a sequence of random variables, and  $g$  a continuous function.

**Slutsky's theorem:** Let  $\{X_n\}, \{Y_n\}$  be sequences of random variables, and suppose that  $X_n \xrightarrow{d} X$  and  $Y_n \xrightarrow{p} c$ , where  $c$  is a constant. Then:

## Asymptotic normality with sample standard deviation

Let  $X_1, X_2, \dots$  be iid random variables, with  $\mu = \mathbb{E}[X_i]$  and  $\sigma^2 = \text{Var}(X_i) < \infty$ . Let  $\hat{\sigma}^2$  be an estimate of  $\sigma^2$  such that  $\hat{\sigma}^2 \xrightarrow{P} \sigma^2$ . Then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}} \xrightarrow{d} Z \sim N(0, 1)$$

## Relationships between types of convergence

- (a) If  $X_n \xrightarrow{d} c$ , where  $c$  is a constant, then  $X_n \xrightarrow{p} c$
- (b) If  $X_n \xrightarrow{p} X$ , then  $X_n \xrightarrow{d} X$