STA 711 Exam 1 Review

Instructions: Submit your work as a single PDF. You may choose to either hand-write your work and submit a PDF scan, or type your work using LaTeX and submit the resulting PDF. See the course website for a homework template file and instructions on getting started with LaTeX and Overleaf.

Questions

1. Let $Y_1, ..., Y_n \overset{iid}{\sim} Uniform(a, b)$, where a and b are unknown and $a < b$. Recall that a uniform distribution has pdf

$$f(y) = \begin{cases} \frac{1}{b-a} & a \leq y \leq b \\ 0 & \text{else} \end{cases}$$

   (a) Find the maximum likelihood estimators $\widehat{a}$ and $\widehat{b}$.

   (b) Let $\tau = \mathbb{E}[Y_1]$. Find the MLE $\widehat{\tau}$.

2. Let $Y_1, ..., Y_n$ be iid from a distribution with pdf

$$f(y) = \frac{2}{\lambda\sqrt{2\pi}} e^y \exp\left\{\frac{-(e^y - 1)^2}{2\lambda^2}\right\},$$

   where $y > 0$ and $\lambda > 0$. Find the MLE of λ.

3. Let $Y_1, ..., Y_n$ be an iid sample from a continuous distribution with pdf

$$f(y) = \frac{1}{2}\exp\{-|y - \theta|\},$$

   where $-\infty < y < \infty$ and $-\infty < \theta < \infty$. Find the maximum likelihood estimator of θ. *Hint:* avoid calculus

4. Let $Y_1, ..., Y_n$ be iid from a distribution with pdf

$$f(y) = a^\theta \theta y^{-\theta - 1}$$

   where $\theta > 0$, $y \geq a$, and a is a known constant. Find the MLE of θ.

5. Let $Y_1, ..., Y_n$ be iid from a distribution with pdf

$$f(y) = \begin{cases} \dfrac{1}{\theta_1 + \theta_2} \exp\left\{\dfrac{-y}{\theta_1}\right\} & y > 0 \\ \dfrac{1}{\theta_1 + \theta_2} \exp\left\{\dfrac{y}{\theta_2}\right\} & y \leq 0 \end{cases}$$

   with $\theta_1, \theta_2 > 0$. Show that the maximum likelihood estimators of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are given by

$$\widehat{\theta}_1 = T_1 + \sqrt{T_1 T_2} \qquad \widehat{\theta}_2 = T_2 + \sqrt{T_1 T_2}$$

   where

$$T_1 = \frac{1}{n}\sum_{i=1}^n Y_i \mathbb{1}\{Y_i > 0\} \qquad T_2 = -\frac{1}{n}\sum_{i=1}^n Y_i \mathbb{1}\{Y_i \leq 0\}.$$

   You are not required to check a second derivative for this problem.

6. The exponential distribution with parameter λ has pdf

$$f(y|\lambda) = \frac{1}{\lambda}\exp\left\{-\frac{y}{\lambda}\right\}, \qquad y > 0.$$

Let $Y_i > 0$ be a continuous, positive response variable of interest, and let $\mathbf{x}_i \in \mathbb{R}^{k+1}$ be a vector of covariates. Suppose we observe independent samples $(\mathbf{x}_1, Y_1), ..., (\mathbf{x}_n, Y_n)$ from the following model:

$$\begin{aligned} Y_i|\mathbf{x}_i &\sim Exponential(\lambda_i) \\ -\frac{1}{\lambda_i} &= \mathbf{x}_i^T \boldsymbol{\beta} \end{aligned}$$

where the distribution of $\mathbf{x}_i$ does not depend on $\boldsymbol{\beta}$. Find the score function $U(\boldsymbol{\beta})$ and the Hessian $\mathbf{H}(\boldsymbol{\beta})$ in matrix form.

7. Suppose that $Y_1, Y_2, ...$ are identically distributed with $\mathbb{E}[Y_i] = \mu$, $Var(Y_i) = \sigma^2$, and covariances

$$Cov(Y_i, Y_{i+j}) = \begin{cases} \rho\sigma^2 & |j| \leq 2 \\ 0 & |j| > 2 \end{cases},$$

where $\rho \in [-1, 1]$ and $\rho \neq 0$. Show that $\overline{Y}_n \xrightarrow{p} \mu$ as $n \to \infty$. (Note: you may not directly use the version of the WLLN stated in class, because it assumes iid data).

8. Let $X_n \sim Pareto(1, n)$. This means that $X_n \geq 1$, and the pdf of X_n is

$$f_{X_n}(x) = \frac{n}{x^{n+1}}.$$

   (a) As $n \to \infty$, $X_n \xrightarrow{p} c$. Find c, and prove the convergence in probability. Show all work.

   (b) Find a sequence a_n such that $nX_n - a_n$ converges in distribution. Show all work.

9. Suppose that $Y_1, ..., Y_n$ are an iid sample from the *log-normal* distribution, with pdf

$$f(y|\mu, \sigma^2) = \frac{1}{y\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma^2}(\log(y) - \mu)^2\right\}.$$

   (a) Show that $\log(Y_i) \sim N(\mu, \sigma^2)$.

   (b) Show that the geometric mean $\left(\prod\limits_{i=1}^{n} Y_i\right)^{1/n} \xrightarrow{p} \exp(\mu)$.

10. Let $X_1, ..., X_n$ be an iid sample from a population with mean μ_1 and variance σ^2, and $Y_1, ..., Y_m$ an iid sample from a population with mean μ_2 and variance σ^2. Furthermore, assume that the common variance σ^2 for the two populations is finite. We wish to test the hypotheses $H_0 : \mu_1 = \mu_2$ vs. $H_A : \mu_1 \neq \mu_2$. Consider the test statistic

$$T = \frac{\overline{X} - \overline{Y}}{\sqrt{s_p^2\left(\frac{1}{n} + \frac{1}{m}\right)}}$$

where

$$s_p^2 = \frac{\sum\limits_{i=1}^{n}(X_i - \overline{X})^2 + \sum\limits_{j=1}^{m}(Y_j - \overline{Y})^2}{n + m - 2}$$

(a) Show that if H_0 is true and $\mu_1 = \mu_2 = \mu$, you can rewrite T as

$$T = \left(\sqrt{1-\lambda_{n,m}}\sqrt{n}(\overline{X}-\mu) - \sqrt{\lambda_{n,m}}\sqrt{m}(\overline{Y}-\mu)\right)/s_p$$

where $\lambda_{n,m} = \dfrac{n}{n+m}$.

(b) Suppose that $n, m \to \infty$ such that $\lambda_{n,m} \to \lambda \in (0,1)$. Using the fact that

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 \xrightarrow{p} \sigma^2 \qquad \frac{1}{m}\sum_{j=1}^{m}(Y_j - \overline{Y})^2 \xrightarrow{p} \sigma^2$$

show that $s_p^2 \xrightarrow{p} \sigma^2$.

(c) Suppose that H_0 is true and $n, m \to \infty$ such that $\lambda_{n,m} \to \lambda \in (0,1)$. Show that $T \xrightarrow{d} N(0,1)$.