STA 711 Exam 1 Review

Information: The first exam will cover maximum likelihood estimation (including in the context of regression models), and convergence. This matches with material from the first four homework assignments. The questions below are not completely comprehensive, but will give you a sense of the kinds of questions I could ask on the exam.

Questions

1. Let $Y_1, ..., Y_n \overset{iid}{\sim} Uniform(a, b)$, where a and b are unknown and $a < b$. Recall that a uniform distribution has pdf

$$f(y) = \begin{cases} \frac{1}{b-a} & a \leq y \leq b \\ 0 & \text{else} \end{cases}$$

   (a) Find the maximum likelihood estimators $\widehat{a}$ and $\widehat{b}$.

   (b) Let $\tau = \mathbb{E}[Y_1]$. Find the MLE $\widehat{\tau}$.

2. Let $Y_1, ..., Y_n$ be iid from a distribution with pdf

$$f(y) = \frac{2}{\lambda\sqrt{2\pi}} e^y \exp\left\{\frac{-(e^y - 1)^2}{2\lambda^2}\right\},$$

   where $y > 0$ and $\lambda > 0$. Find the MLE of λ.

3. Let $Y_1, ..., Y_n$ be an iid sample from a continuous distribution with pdf

$$f(y) = \frac{1}{2}\exp\{-|y - \theta|\},$$

   where $-\infty < y < \infty$ and $-\infty < \theta < \infty$. Find the maximum likelihood estimator of θ. *Hint:* avoid calculus

4. Let $Y_1, ..., Y_n$ be iid from a distribution with pdf

$$f(y) = a^\theta \theta y^{-\theta-1}$$

   where $\theta > 0$, $y \geq a$, and a is a known constant. Find the MLE of θ.

5. Let $Y_1, ..., Y_n$ be iid from a distribution with pdf

$$f(y) = \begin{cases} \dfrac{1}{\theta_1 + \theta_2} \exp\left\{\dfrac{-y}{\theta_1}\right\} & y > 0 \\ \dfrac{1}{\theta_1 + \theta_2} \exp\left\{\dfrac{y}{\theta_2}\right\} & y \leq 0 \end{cases}$$

   with $\theta_1, \theta_2 > 0$. Show that the maximum likelihood estimators of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are given by

$$\widehat{\theta}_1 = T_1 + \sqrt{T_1 T_2} \qquad \widehat{\theta}_2 = T_2 + \sqrt{T_1 T_2}$$

   where

$$T_1 = \frac{1}{n}\sum_{i=1}^n Y_i \mathbb{1}\{Y_i > 0\} \qquad T_2 = -\frac{1}{n}\sum_{i=1}^n Y_i \mathbb{1}\{Y_i \leq 0\}.$$

   You are not required to check a second derivative for this problem.

6. The exponential distribution with parameter λ has pdf

$$f(y|\lambda) = \frac{1}{\lambda} \exp\left\{-\frac{y}{\lambda}\right\}, \qquad y > 0.$$

Let $Y_i > 0$ be a continuous, positive response variable of interest, and let $\mathbf{x}_i \in \mathbb{R}^{k+1}$ be a vector of covariates. Suppose we observe independent samples $(\mathbf{x}_1, Y_1), ..., (\mathbf{x}_n, Y_n)$ from the following model:

$$\begin{aligned} Y_i|\mathbf{x}_i &\sim \textit{Exponential}(\lambda_i) \\ -\frac{1}{\lambda_i} &= \mathbf{x}_i^T \boldsymbol{\beta} \end{aligned}$$

where the distribution of $\mathbf{x}_i$ does not depend on $\boldsymbol{\beta}$. Find the score function $U(\boldsymbol{\beta})$ and the Hessian $\mathbf{H}(\boldsymbol{\beta})$ in matrix form.

7. We have seen that fitting a linear regression model with ordinary least squares is equivalent to maximum likelihood estimation if we assume that the distribution of the error term ε_i is Normal with a constant variance. What if we don't have normal errors – that is, what if $Y_i|\mathbf{x}_i$ is not normal?

   A common approach is to *transform* our response so that the transformed variable looks more Normal. That is, choose some function g such that $g(Y_i)|\mathbf{x}_i$ is approximately Normal. Often, we choose a transformation based on exploration of the data and residual plots, but another approach is the *Box-Cox transformation.*

   Suppose that we have independent data $(\mathbf{x}_i, Y_i)$ from some model, with all $Y_i > 0$. The Box-Cox transformation, with parameter λ, is

$$Y_{\lambda i} = \frac{Y_i^\lambda - 1}{\lambda}$$

   The goal is to find λ such that $Y_{\lambda i}|\mathbf{x}_i \approx N(\mathbf{x}_i^T\boldsymbol{\beta}, \sigma^2)$. To estimate the parameters $\lambda, \boldsymbol{\beta}$, and σ^2, the method of maximum likelihood is used.

   (a) Show that the log-likelihood $\ell(\lambda, \boldsymbol{\beta}, \sigma^2|\mathbf{y}, \mathbf{X})$ is (up to a constant) given by

$$-\frac{n}{2}\log(\sigma^2) - \frac{1}{\sigma^2}\sum_{i=1}^n (Y_{\lambda i} - \mathbf{x}_i^T\boldsymbol{\beta})^2 + (\lambda - 1)\sum_{i=1}^n \log(Y_i)$$

   (b) Given a value of λ, find maximum likelihood estimators for $\boldsymbol{\beta}$ and σ^2.

8. Suppose that $Y_1, Y_2, ...$ are identically distributed with $\mathbb{E}[Y_i] = \mu$, $Var(Y_i) = \sigma^2$, and covariances

$$Cov(Y_i, Y_{i+j}) = \begin{cases} \rho\sigma^2 & |j| \leq 2 \\ 0 & |j| > 2 \end{cases},$$

   where $\rho \in [-1, 1]$ and $\rho \neq 0$. Show that $\overline{Y}_n \xrightarrow{p} \mu$ as $n \to \infty$. (Note: you may not directly use the version of the WLLN stated in class, because it assumes iid data).

9. Let $X_n \sim \textit{Pareto}(1, n)$. This means that $X_n \geq 1$, and the pdf of X_n is

$$f_{X_n}(x) = \frac{n}{x^{n+1}}.$$

   (a) As $n \to \infty$, $X_n \xrightarrow{p} c$. Find c, and prove the convergence in probability. Show all work.

(b) Find a sequence a_n such that $nX_n - a_n$ converges in distribution. Show all work.

10. Suppose that $Y_1, ..., Y_n$ are an iid sample from the *log-normal* distribution, with pdf

$$f(y|\mu, \sigma^2) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(\log(y) - \mu)^2\right\}.$$

(a) Show that $\log(Y_i) \sim N(\mu, \sigma^2)$.

(b) Show that the geometric mean $\left(\prod_{i=1}^{n} Y_i\right)^{1/n} \xrightarrow{p} \exp(\mu)$.

11. Let $X_1, ..., X_n$ be an iid sample from a population with mean μ_1 and variance σ^2, and $Y_1, ..., Y_m$ an iid sample from a population with mean μ_2 and variance σ^2. Furthermore, assume that the common variance σ^2 for the two populations is finite. We wish to test the hypotheses $H_0 : \mu_1 = \mu_2$ vs. $H_A : \mu_1 \neq \mu_2$. Consider the test statistic

$$T = \frac{\overline{X} - \overline{Y}}{\sqrt{s_p^2\left(\frac{1}{n} + \frac{1}{m}\right)}}$$

where

$$s_p^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2 + \sum_{j=1}^{m}(Y_j - \overline{Y})^2}{n + m - 2}$$

(a) Show that if H_0 is true and $\mu_1 = \mu_2 = \mu$, you can rewrite T as

$$T = \left(\sqrt{1 - \lambda_{n,m}}\sqrt{n}(\overline{X} - \mu) - \sqrt{\lambda_{n,m}}\sqrt{m}(\overline{Y} - \mu)\right)/s_p$$

where $\lambda_{n,m} = \dfrac{n}{n + m}$.

(b) Suppose that $n, m \to \infty$ such that $\lambda_{n,m} \to \lambda \in (0, 1)$. Using the fact that

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 \xrightarrow{p} \sigma^2 \qquad \frac{1}{m}\sum_{j=1}^{m}(Y_j - \overline{Y})^2 \xrightarrow{p} \sigma^2$$

show that $s_p^2 \xrightarrow{p} \sigma^2$.

(c) Suppose that H_0 is true and $n, m \to \infty$ such that $\lambda_{n,m} \to \lambda \in (0, 1)$. Show that $T \xrightarrow{d} N(0, 1)$. For this, you may use the fact that if $\{U_n\}$ and $\{V_n\}$ are two independent sequences such that $U_n \xrightarrow{d} U$ and $V_n \xrightarrow{d} V$, and U and V are also independent, then $U_n + V_n \xrightarrow{d} U + V$.