Convergence of random variables

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Warmup

Work on the warmup activity (handout), then we will discuss as a class.

Warmup

Let $X_1, X_2, \ldots$ be iid random variables with $\mathbb{E}[X_i] = \mu$ and $Var(X_i) = \sigma^2 < \infty$.

$$\mathbb{E}[\overline{X}_n] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu$$

(independence)

$$Var(\overline{X}_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n}$$

(chebyshev)

$$P(|\overline{X}_n - \mu| \geq \varepsilon) \leq \frac{\mathbb{E}[(\overline{X}_n - \mu)^2]}{\varepsilon^2} = \frac{Var(\overline{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}$$

$$0 \leq P(|\overline{X}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}$$

$$\Rightarrow \lim_{n\to\infty} P(|\overline{X}_n - \mu| \geq \varepsilon) = \lim_{n\to\infty}\frac{\sigma^2}{n\varepsilon^2} = 0 \quad //$$

Convergence in probability

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in probability* to a random variable X if, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0$$

We write $X_n \xrightarrow{p} X$.

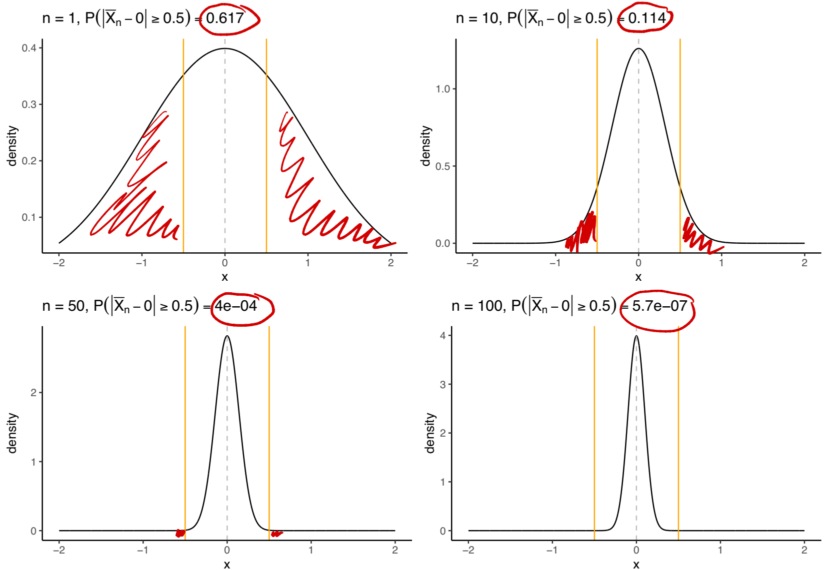
Weak Law of Large Numbers (WLLN): Let $X_1, X_2, \ldots$ be iid random variables with $\mathbb{E}[X_i] = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Then

$$\overline{X}_n \xrightarrow{p} \mu$$

Example: WLLN

Intuition: $\bar{X}_n$ "concentrates" around μ

Suppose $X_1, X_2, \ldots \overset{iid}{\sim} N(0,1)$. Then $\overline{X}_n \sim N(0, \frac{1}{n})$. Let $\varepsilon = 0.5$.



Example

$\sqrt{n}\,\mathbb{1}\{u \le \frac{1}{n}\} = \begin{cases} 0 & \text{if } u > \frac{1}{n} \\ \sqrt{n} & \text{if } u \le \frac{1}{n} \end{cases}$

if $n > \varepsilon^2 \Rightarrow \sqrt{n} > \varepsilon \Rightarrow \sqrt{n}\,\mathbb{1}\{u \le \frac{1}{n}\} \geq \varepsilon$
as long as $\mathbb{1}\{u \le \frac{1}{n}\} = 1$

Let $U \sim Uniform(0,1)$, and let $X_n = \sqrt{n}\,\mathbb{I}\{U \le 1/n\}$.

Then $X_n \xrightarrow{p} 0$.

$X_n = \begin{cases} 0 & \text{if } u > \frac{1}{n} \\ \sqrt{n} & \text{if } u \le \frac{1}{n} \end{cases}$

want to show: $\forall \varepsilon > 0$, $P(|X_n - 0| \geq \varepsilon) \to 0$ as $n \to \infty$

Let $\varepsilon > 0$.

$P(|X_n - 0| \geq \varepsilon) = P(X_n \geq \varepsilon)$ $(X_n \geq 0)$

$= P(\sqrt{n}\,\mathbb{1}\{u \le \frac{1}{n}\} \geq \varepsilon)$

$= P(u \le \frac{1}{n})$ (for all $n \geq \varepsilon^2$)

$n \geq \varepsilon^2 \Leftrightarrow \sqrt{n} \geq \varepsilon$
$\Rightarrow \sqrt{n}\,\mathbb{1}\{u \le \frac{1}{n}\} \geq \varepsilon \Leftrightarrow \mathbb{1}\{u \le \frac{1}{n}\} = 1$

$= \frac{1}{n} \to 0$ as $n \to \infty$ //

Class activity, Part I

Work on the class activity (handout), then we will discuss as a class.