Activity: counterexamples

Convergence in distribution but *not* in probability

Suppose $X \sim N(0, 1)$, and let $X_n = -X$ for $n = 1, 2, 3, ...$ We will show that $X_n \xrightarrow{d} X$ but X_n does *not* converge in probability.

1. Using the fact that X is *symmetric* around 0, show that $F_{X_n}(t) = F_X(t)$ for all n and t. Conclude that $X_n \xrightarrow{d} X$.

2. Show that for any $\varepsilon > 0$, $\lim\limits_{n\to\infty} P(|X_n - X| \geq \varepsilon) > 0$. Conclude that X_n does not converge in probability to X.

Convergence in probability does not imply convergence of moments

Let $X_1, X_2, X_3, ...$ be a sequence such that each $X_n \in \{0, n\}$, with $P(X_n = n) = \frac{1}{n}$ and $P(X_n = 0) = 1 - \frac{1}{n}$.

1. Show that $X_n \xrightarrow{p} 0$.

2. Show that $\mathbb{E}[X_n] \not\to 0$.