Convergence of random variables

Ciaran Evans

Recap: convergence

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in probability* to a random variable X if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

We write $X_n \xrightarrow{p} X$.

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in distribution* to a random variable X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

at all points where $F_X(x)$ is continuous. We write $X_n \xrightarrow{d} X$.

Central Limit Theorem

Let $X_1, X_2, \ldots$ be iid random variables, with $\mu = \mathbb{E}[X_i]$ and $0 < \sigma^2 = Var(X_i) < \infty$. Then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z$$

where $Z \sim N(0,1)$.

Intuition:

$\bar{X}_n \xrightarrow{P} \mu$ (WLLN)

$\bar{X}_n - \mu \xrightarrow{P} 0$ $\quad Var(\bar{X}_n) = \frac{\sigma^2}{n} \to 0$

$Var\left(\frac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu)\right) = 1$

Other key results

Continuous mapping theorem: Let $X_1, X_2, \ldots$ be a sequence of random variables, and g a continuous function.

- If $X_n \xrightarrow{d} X$, then $g(X_n) \xrightarrow{d} g(X)$
- If $X_n \xrightarrow{p} X$, then $g(X_n) \xrightarrow{p} g(X)$ ← proof: HW

Slutsky's theorem: Let $\{X_n\}, \{Y_n\}$ be sequences of random variables, and suppose that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$, where c is a constant. Then:

- $X_n + Y_n \xrightarrow{d} X + c$
- $X_n Y_n \xrightarrow{d} cX$
- $\dfrac{X_n}{Y_n} \xrightarrow{d} \dfrac{X}{c}$, if $c \neq 0$

Asymptotic normality with sample standard deviation

Let $X_1, X_2, \ldots$ be iid random variables, with $\mu = \mathbb{E}[X_i]$ and $0 < \sigma^2 = Var(X_i) < \infty$. Let $\widehat{\sigma}^2$ be an estimate of σ^2 such that $\widehat{\sigma}^2 \xrightarrow{p} \sigma^2$. Then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\widehat{\sigma}} \xrightarrow{d} Z \sim N(0,1)$$

① $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$ $\quad$ $\hat{\sigma} = \sqrt{\hat{\sigma}^2} \xrightarrow{p} \sigma$ $\quad$ (continuous mapping)

$\Rightarrow \dfrac{\sigma}{\hat{\sigma}} \xrightarrow{p} 1$

② By CLT, $\dfrac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z$

③ $\dfrac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}} = \dfrac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \cdot \dfrac{\sigma}{\hat{\sigma}} \xrightarrow{d} Z \cdot 1 = Z$

(by Slutsky's theorem)

Relationships between types of convergence

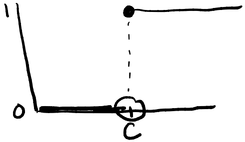
(a) If $X_n \xrightarrow{d} c$, where c is a constant, then $X_n \xrightarrow{p} c$

(b) If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$

$P(c=c) = 1$

(a) point mass at c:

pmf: $f_c(x) = \begin{cases} 1 & x = c \\ 0 & x \neq c \end{cases}$



cdf: $F_c(x) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$

Suppose $X_n \xrightarrow{d} c$, and let $\varepsilon > 0$.

$P(|X_n - c| \geq \varepsilon) = 1 - P(|X_n - c| < \varepsilon) = 1 - P(c - \varepsilon < X_n < c + \varepsilon)$

$= 1 - (F_{X_n}(c+\varepsilon) - F_{X_n}(c-\varepsilon))$

Since $X_n \xrightarrow{d} c$, $F_{X_n}(t) \to F_c(t) \quad \forall t \neq c$

$F_{X_n}(c+\varepsilon) \to F_c(c+\varepsilon) = 1$

$F_{X_n}(c-\varepsilon) \to F_c(c-\varepsilon) = 0$

$\Rightarrow \lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) = 1 - \lim_{n\to\infty} (F_{X_n}(c+\varepsilon) - F_{X_n}(c-\varepsilon))$

$= 1 - (1 - 0) = 0$ //

Relationships between types of convergence

(a) If $X_n \xrightarrow{d} c$, where c is a constant, then $X_n \xrightarrow{p} c$

(b) If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$

(b) WTS: $F_{X_n}(t) \to F_X(t)$ $\forall t$ where F_X is continuous

If F_X is continuous at t: $\lim_{\varepsilon \to 0^+} F_X(t-\varepsilon) = F_X(t) = \lim_{\varepsilon \to 0^+} F_X(t+\varepsilon)$

it suffices to show that $\forall \varepsilon > 0$, $F_X(t-\varepsilon) \leq \lim_{n\to\infty} F_{X_n}(t) \leq F_X(t+\varepsilon)$

Let $\varepsilon > 0$, and let t be a continuity point of F_X

$$F_{X_n}(t) = P(X_n \leq t) = \underbrace{P(X_n \leq t,\ X \leq t+\varepsilon)}_{\leq P(X \leq t+\varepsilon)} + \underbrace{P(X_n \leq t,\ X > t+\varepsilon)}_{\leq P(|X_n - X| \geq \varepsilon)}$$

$$\Rightarrow F_{X_n}(t) \leq F_X(t+\varepsilon) + P(|X_n - X| \geq \varepsilon)$$

$$\Rightarrow \lim_{n\to\infty} F_{X_n}(t) \leq F_X(t+\varepsilon) + \underbrace{\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon)}_{=0 \text{ b/c } X_n \xrightarrow{p} X}$$

$$\leq F_X(t+\varepsilon)$$

Relationships between types of convergence

(a) If $X_n \xrightarrow{d} c$, where c is a constant, then $X_n \xrightarrow{p} c$

(b) If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$

(b) continued...

So: $\lim_{n\to\infty} F_{X_n}(t) \le F_X(t+\varepsilon)$

Similarly, $F_X(t-\varepsilon) \le \lim_{n\to\infty} F_{X_n}(t)$

$\Rightarrow \quad F_X(t-\varepsilon) \le \lim_{n\to\infty} F_{X_n}(t) \le F_X(t+\varepsilon) \qquad \forall \varepsilon > 0$

$\Rightarrow \quad \lim_{\varepsilon\to 0^+} F_X(t-\varepsilon) \le \lim_{n\to\infty} F_{X_n}(t) \le \lim_{\varepsilon\to 0^+} F_X(t+\varepsilon)$

$\Rightarrow \quad F_X(t) \le \lim_{n\to\infty} F_{X_n}(t) \le F_X(t)$

$\Rightarrow \quad \lim_{n\to\infty} F_{X_n}(t) = F_X(t)$

So, $X_n \xrightarrow{d} X$ //