MSE for estimating Normal variance

Suppose $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Consider two estimates of σ^2:

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 \qquad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Our goal is to find the MSE for these two estimators. We will use the following key facts:

- $\frac{1}{\sigma^2}\sum\limits_{i=1}^{n}(X_i - \overline{X})^2 \sim \chi^2_{n-1}$
- If $V \sim \chi^2_\nu$, then $\mathbb{E}[V] = \nu$ and $Var(V) = 2\nu$

Questions

1. Using the information above, show that $\mathbb{E}[s^2] = \sigma^2$ (that is, the sample variance s^2 is an unbiased estimator).

2. Now calculate $\mathbb{E}[\widehat{\sigma}^2]$. Does $\widehat{\sigma}^2$ tend to overestimate or underestimate σ^2?

3. Using the information above, compute $Var(s^2)$ and $Var(\widehat{\sigma}^2)$.

4. Which estimator has a lower MSE?

5. You are currently showing on homework that the Fisher information (for a single observation) for the variance of a normal distribution is $\mathcal{I}_1(\sigma^2) = \dfrac{1}{2\sigma^4}$. Does s^2 attain the CRLB?

6. Now **suppose we know the mean** μ, and consider the estimator $\widetilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$. Show that this estimator is unbiased for σ^2. (*Hint*: If $Z \sim N(0,1)$ then $Z^2 \sim \chi^2_1$)

7. Does $\widetilde{\sigma}^2$ from the previous question attain the CRLB?