# Best unbiased estimators and the Cramer-Rao lower bound

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# Best unbiased estimators

Suppose we restrict ourselves to **unbiased** estimators.

**Definition** (best unbiased estimator):

Let $\theta$ be a parameter of interest, and $\hat{\theta}$ be an estimator of $\theta$. If, for all values of $\theta$,

$\mathbb{E}[\hat{\theta}] = \theta$, and

$$Var(\hat{\theta}) \leq Var(\hat{\theta}^*)$$

for all other unbiased estimators $\hat{\theta}^*$, then $\hat{\theta}$ is a best unbiased estimator of $\theta$

(aka "uniform minimum variance unbiased estimator" UMVUE)

# Cramer-Rao lower bound

Let $Y_1, \dots, Y_n$ be a sample from a distribution with probability function $f(y|\theta)$, and let $\widehat{\theta}$ be an estimator of $\theta \in \mathbb{R}$. Under regularity conditions,

expectation if $\theta$ is true value of parameter

variance if $\theta$ is true value of parameter

$$Var_\theta(\widehat{\theta}) \geq \frac{\left(\frac{d}{d\theta}\mathbb{E}_\theta(\widehat{\theta})\right)^2}{\underbrace{\mathbb{E}_\theta\left[\left(\underbrace{\frac{d}{d\theta}\log f(Y_1, \dots, Y_n|\theta)}_{\text{score}}\right)^2\right]}_{\text{variance of score, because expectation of score} = 0 \text{ under regularity conditions}}}$$

If data are iid: $\mathbb{E}_\theta\left[\left(\frac{d}{d\theta}\log f(Y_1, \dots, Y_n|\theta)\right)^2\right] = n\,\mathcal{I}_1(\theta) = \mathcal{I}_n(\theta)$

If $\hat{\theta}$ is unbiased: $\frac{d}{d\theta}\mathbb{E}_\theta(\hat{\theta}) = \frac{d}{d\theta}\theta = 1$

So, in the iid and unbiased setting,

$$Var_\theta(\hat{\theta}) \geq \frac{1}{n\,\mathcal{I}_1(\theta)} = \frac{1}{n}\mathcal{I}_1^{-1}(\theta)$$

# Example

Suppose $Y_1, \dots, Y_n \overset{iid}{\sim} Poisson(\lambda)$

MLE: $\hat{\lambda}_{MLE} = \bar{Y}$ $\quad E[\hat{\lambda}_{MLE}] = E[Y_i] = \lambda$ (unbiased)

$$Var(\hat{\lambda}_{MLE}) = Var(\bar{Y}) \underset{(iid)}{=} \frac{Var(Y_i)}{n} = \frac{\lambda}{n}$$

CRLB: Previously, found $\mathcal{I}_1(\lambda) = \frac{1}{\lambda}$

$\Rightarrow$ CRLB for unbiased estimator: $\frac{\lambda}{n}$

$\Rightarrow$ $Var(\hat{\lambda}_{MLE}) = \frac{\lambda}{n} = \text{CRLB}$

$\Rightarrow$ $\hat{\lambda}_{MLE} = \bar{Y}$ attains CRLB

so $\hat{\lambda}_{MLE}$ is a best unbiased estimator of $\lambda$

# Your assignment

Suppose $Y_1, ..., Y_n \overset{iid}{\sim} Bernoulli(p)$.

- Find the Cramer-Rao lower bound for an unbiased estimator of $p$
- The MLE $\widehat{p} = \overline{Y}$ is unbiased for $p$. Does the MLE attain the CRLB?

Solution on next page!

$Y_1, \dots, Y_n \overset{iid}{\sim}$ Bernoulli$(p)$

MLE: $\hat{p} = \bar{Y}$ $\qquad \mathbb{E}[\hat{p}] = \mathbb{E}[Y_i] = p$

$$\mathrm{Var}(\hat{p}) = \frac{\mathrm{Var}(Y_i)}{n} = \frac{p(1-p)}{n}$$

CRLB:

First, find Fisher information. For a single $Y \sim$ Bernoulli$(p)$,

$$f(Y|p) = p^Y (1-p)^{1-Y}$$

$$\log f(Y|p) = Y \log(p) + (1-Y)\log(1-p)$$

$$\frac{d}{dp} \log f(Y|p) = \frac{Y}{p} - \frac{(1-Y)}{1-p} = \frac{Y(1-p) - (1-Y)p}{p(1-p)}$$

$$= \frac{Y - Yp - p + Yp}{p(1-p)} = \frac{Y-p}{p(1-p)}$$

$$\mathcal{I}_1(p) = \mathrm{Var}\left(\frac{d}{dp} \log f(Y|p)\right) = \frac{1}{p^2(1-p)^2} \mathrm{Var}(Y) = \frac{1}{p(1-p)}$$

$$\Rightarrow \quad \mathrm{CRLB} = \frac{1}{n} \mathcal{I}_1^{-1}(p) = \frac{p(1-p)}{n}$$

$\Rightarrow$ $\hat{p}$ attains CRLB, so is <u>best unbiased estimator</u>