Convergence of random variables

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Recap: convergence

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in probability* to a random variable X if, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0$$

We write $X_n \xrightarrow{p} X$.

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in distribution* to a random variable X if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

at all points where $F_X(x)$ is continuous. We write $X_n \xrightarrow{d} X$.

Central Limit Theorem

Let $X_1, X_2, \ldots$ be iid random variables, with $\mu = \mathbb{E}[X_i]$ and $0 < \sigma^2 = Var(X_i) < \infty$. Then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z$$

where $Z \sim N(0, 1)$.

Other key results

Continuous mapping theorem: Let $X_1, X_2, ...$ be a sequence of random variables, and g a continuous function.

Slutsky's theorem: Let $\{X_n\}, \{Y_n\}$ be sequences of random variables, and suppose that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$, where c is a constant. Then:

Asymptotic normality with sample standard deviation

Let $X_1, X_2, ...$ be iid random variables, with $\mu = \mathbb{E}[X_i]$ and $0 < \sigma^2 = Var(X_i) < \infty$. Let $\widehat{\sigma}^2$ be an estimate of σ^2 such that $\widehat{\sigma}^2 \xrightarrow{p} \sigma^2$. Then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\widehat{\sigma}} \xrightarrow{d} Z \sim N(0, 1)$$

Relationships between types of convergence

(a) If $X_n \stackrel{d}{\to} c$, where c is a constant, then $X_n \stackrel{p}{\to} c$

(b) If $X_n \stackrel{p}{\to} X$, then $X_n \stackrel{d}{\to} X$