Convergence of random variables

Ciaran Evans

Last time: Class activity

Suppose that $X_1, X_2, ... \overset{iid}{\sim} Uniform(0,1)$, and let $X_{(n)} = \max\{X_1, ..., X_n\}$. Then $X_{(n)} \xrightarrow{p} 1$.

<u>Pf</u> : WTS $\forall \varepsilon > 0$, $P(|X_{(n)} - 1| \geq \varepsilon) \to 0$ as $n \to \infty$

let $0 < \varepsilon < 1$

$$P(|X_{(n)} - 1| \geq \varepsilon) = P(X_{(n)} \leq 1 - \varepsilon) + \underbrace{P(X_{(n)} \geq 1 + \varepsilon)}_{\substack{=0 \\ (X_{(n)} \leq 1)}}$$

$$= P(X_{(n)} \leq 1 - \varepsilon)$$

$$= P(X_1 \leq 1-\varepsilon, X_2 \leq 1-\varepsilon, ..., X_n \leq 1-\varepsilon)$$

$$= P(X_1 \leq 1-\varepsilon) P(X_2 \leq 1-\varepsilon) \cdots P(X_n \leq 1-\varepsilon)$$

$$= (1-\varepsilon)^n$$

$$\to 0 \quad \text{as } n \to \infty$$

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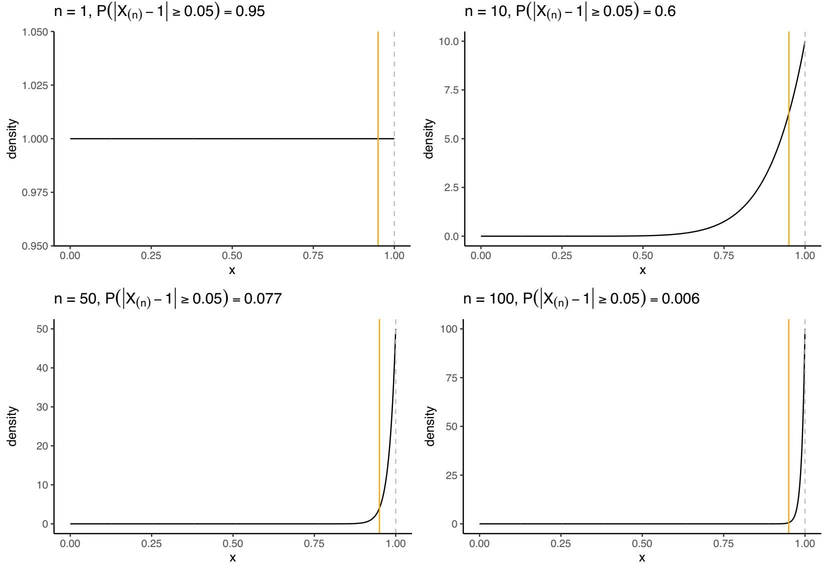
if $\varepsilon \geq 1$

$$P(|X_{(n)} - 1| \geq \varepsilon) \leq P(|X_{(n)} - 1| \geq 1) \to 0$$

Last time: Class activity

pdf: nx^{n-1}

Suppose that $X_1, X_2, \ldots \overset{iid}{\sim} Uniform(0,1)$. Then $X_{(n)} \sim Beta(n,1)$



Warmup

Work on the warmup activity (handout), then we will discuss as a class.

Warmup

Suppose that $X_1, X_2, \ldots \overset{iid}{\sim} Uniform(0,1)$, $X_{(n)} = \max\{X_1, \ldots, X_n\}$, and let $Y_n = n(1 - X_{(n)})$.

$$F_{Y_n}(t) = P(Y_n \leq t) = P(n(1 - X_{(n)}) \leq t)$$

$$= P(1 - X_{(n)} \leq \tfrac{t}{n})$$

$$= P(X_{(n)} \geq 1 - \tfrac{t}{n})$$

$$= 1 - P(X_{(n)} \leq 1 - \tfrac{t}{n})$$

$$= 1 - (P(X_i \leq 1 - \tfrac{t}{n}))^n$$

$$= 1 - (1 - \tfrac{t}{n})^n$$

$$(1 + \tfrac{x}{n})^n \to e^x$$

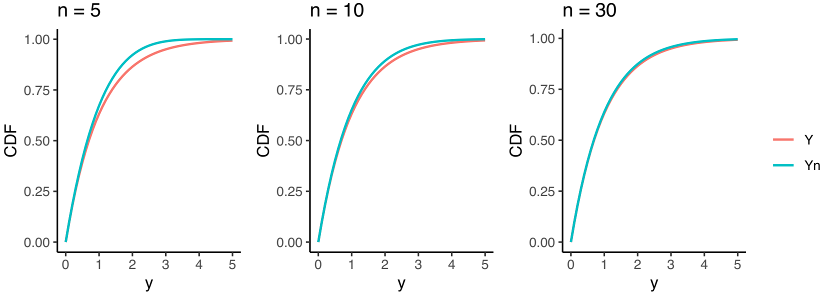
$$\lim_{n\to\infty} F_{Y_n}(t) = \lim_{n\to\infty} 1 - (1 - \tfrac{t}{n})^n$$

$$= 1 - e^{-t}$$

(weibull distribution)

Warmup

$$F_{Y_n}(t) = 1 - \left(1 - \frac{t}{n}\right)^n \to F_Y(t) = 1 - e^{-t}$$



Convergence in distribution

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in distribution* to a random variable X if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

at all points where $F_X(x)$ is continuous. We write $X_n \xrightarrow{d} X$.

Class activity

Work on the class activity (handout), then we will discuss as a class.

Class activity

Suppose that $X_1, X_2, \ldots$ are iid random variables with cdf

$$F(x) = \begin{cases} 1 - \left(\frac{1}{x}\right)^{\alpha} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

Let $X_{(n)} = \max\{X_1, \ldots, X_n\}$, and $Y_n = n^{-1/\alpha} X_{(n)}$.

$$F_{Y_n}(t) = P(n^{-1/\alpha} X_{(n)} \leq t) = P(X_{(n)} \leq n^{\frac{1}{\alpha}} t)$$

$$= \left(1 - \left(\frac{1}{n^{\frac{1}{\alpha}} t}\right)^{\alpha}\right)^n$$

$$= \left(1 - \frac{1}{t^{\alpha} n}\right)^n$$

$$= \left(1 - \frac{t^{-\alpha}}{n}\right)^n$$

$$\to e^{-t^{-\alpha}}$$

(Frechet distribution)

Convergence in distribution: Central Limit Theorem

Let $X_1, X_2, \ldots$ be iid random variables, with $\mu = \mathbb{E}[X_i]$ and $0 < \sigma^2 = Var(X_i) < \infty$. Then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z$$

where $Z \sim N(0, 1)$.

what if we don't know σ? $\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}$

Does $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}} \xrightarrow{d} Z$?

Yes! Key results:

- $\hat{\sigma}^2 \xrightarrow{P} \sigma^2$ (HW)
- Continuous mapping theorem: if g is continuous,
  $X_n \xrightarrow{d} X \Rightarrow g(X_n) \xrightarrow{d} g(X)$
  $X_n \xrightarrow{P} X \Rightarrow g(X_n) \xrightarrow{P} g(X)$