Activity: MSE

MSE

Let $X_1, ..., X_n \overset{iid}{\sim} Uniform(0, \theta)$. The method of moments estimator of θ is $\hat{\theta}_{\text{MoM}} = 2\overline{X}$, and the maximum likelihood estimator is $\hat{\theta}_{\text{MLE}} = X_{(n)}$.

We would like to compute and compare the MSE for both estimators. It will help to know that $\mathbb{E}[X_i] = \frac{\theta}{2}$, $Var(X_i) = \frac{\theta^2}{12}$, $\mathbb{E}[X_{(n)}] = \frac{n}{n+1}\theta$, and $Var(X_{(n)}) = \frac{n\theta^2}{(n+1)^2(n+2)}$.

1. Calculate $MSE(\hat{\theta}_{\text{MoM}})$ and $MSE(\hat{\theta}_{\text{MLE}})$. Which estimator has a smaller MSE?

MoM: $\mathbb{E}[2\bar{x}] = 2\mathbb{E}[\bar{x}] = 2\left(\frac{\theta}{2}\right) = \theta \Rightarrow \text{Bias}(\hat{\theta}_{mom}) = 0$

$Var(2\bar{X}) = 4Var(\bar{X}) = 4\frac{Var(X_i)}{n} = \frac{4\theta^2}{12n} = \frac{\theta^2}{3n}$

$MSE(\hat{\theta}_{mom}) = \text{Bias}^2(\hat{\theta}_{mom}) + Var(\hat{\theta}_{mom}) = \frac{\theta^2}{3n}$

MLE: $\mathbb{E}[X_{(n)}] = \frac{n}{n+1}\theta \Rightarrow \text{Bias}(\hat{\theta}_{MLE}) = -\frac{\theta}{n+1}$ (tend to underestimate)

$Var(X_{(n)}) = \frac{n\theta^2}{(n+1)^2(n+2)}$

$\Rightarrow MSE(\hat{\theta}_{MLE}) = \frac{\theta^2}{(n+1)^2} + \frac{n\theta^2}{(n+1)^2(n+2)} = \frac{\theta^2(n+2) + n\theta^2}{(n+1)^2(n+2)} = \frac{2\theta^2(n+1)}{(n+1)^2(n+2)} = \frac{2\theta^2}{(n+1)(n+2)}$

$MSE(\hat{\theta}_{MLE}) = \frac{2\theta^2}{(n+1)(n+2)} < \frac{\theta^2}{3n} = MSE(\hat{\theta}_{mom})$ for $n > 2$

2. The maximum likelihood estimator $\hat{\theta}_{\text{MLE}} = X_{(n)}$ is a *biased* estimator of θ. We may wonder if we can improve the MSE by *unbiasing* the MLE. Find a simple transformation $g(\hat{\theta}_{\text{MLE}})$ such that $\mathbb{E}[g(\hat{\theta}_{\text{MLE}})] = \theta$, then calculate $MSE(g(\hat{\theta}_{\text{MLE}}))$ and compare to $MSE(\hat{\theta}_{\text{MLE}})$.

$\mathbb{E}[\hat{\theta}_{MLE}] = \frac{n}{n+1}\theta \Rightarrow \mathbb{E}\left[\frac{n+1}{n}\hat{\theta}_{MLE}\right] = \theta$

$Var\left(\frac{n+1}{n}\hat{\theta}_{MLE}\right) = \left(\frac{n+1}{n}\right)^2 Var(\hat{\theta}_{MLE})$

$= \left(\frac{n+1}{n}\right)^2 \cdot \frac{n\theta^2}{(n+1)^2(n+2)}$

$= \frac{\theta^2}{n(n+2)}$

$\Rightarrow MSE\left(\frac{n+1}{n}\hat{\theta}_{MLE}\right) = \frac{\theta^2}{n(n+2)} < \frac{2\theta^2}{(n+1)(n+2)} = MSE(\hat{\theta}_{MLE})$

So, $\left(\frac{n+1}{n}\right)X_{(n)}$ is an unbiased estimator of θ, and has a lower MSE than either $\hat{\theta}_{mom}$ or $\hat{\theta}_{MLE}$