Asymptotic relative efficiency, delta method

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Last time

Suppose $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Three estimators of σ^2:

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 \quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 \quad \widetilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$$

Last time

Suppose $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Three estimators of σ^2:

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$$Var(\widehat{\sigma}^2) = \frac{2\sigma^4(n-1)}{n^2} < \frac{2\sigma^4}{n-1} = Var(s^2)$$

But asymptotically,

Asymptotic relative efficiency

Let $\theta \in \mathbb{R}$ be a parameter of interest, and let $\widehat{\theta}_{1,n}$ and $\widehat{\theta}_{2,n}$ be two estimators of θ such that

$$\sqrt{n}(\widehat{\theta}_{1,n} - \theta) \xrightarrow{d} N(0, \sigma_1^2) \qquad \sqrt{n}(\widehat{\theta}_{2,n} - \theta) \xrightarrow{d} N(0, \sigma_2^2)$$

The **asymptotic relative efficiency** (ARE) of $\widehat{\theta}_{1,n}$ compared to $\widehat{\theta}_{2,n}$ is:

Example: Normal variance estimators

Let $\theta \in \mathbb{R}$ be a parameter of interest, and let $\widehat{\theta}_{1,n}$ and $\widehat{\theta}_{2,n}$ be two estimators of θ such that

$$\sqrt{n}(\widehat{\theta}_{1,n} - \theta) \xrightarrow{d} N(0, \sigma_1^2) \qquad \sqrt{n}(\widehat{\theta}_{2,n} - \theta) \xrightarrow{d} N(0, \sigma_2^2)$$

$$ARE(\widehat{\theta}_{1,n}, \widehat{\theta}_{2,n}) = \frac{\sigma_2^2}{\sigma_1^2}$$

Example: sample mean vs. sample median

Let $\theta \in \mathbb{R}$ be a parameter of interest, and let $\widehat{\theta}_{1,n}$ and $\widehat{\theta}_{2,n}$ be two estimators of θ such that

$$\sqrt{n}(\widehat{\theta}_{1,n} - \theta) \xrightarrow{d} N(0, \sigma_1^2) \qquad \sqrt{n}(\widehat{\theta}_{2,n} - \theta) \xrightarrow{d} N(0, \sigma_2^2)$$

$$ARE(\widehat{\theta}_{1,n}, \widehat{\theta}_{2,n}) = \frac{\sigma_2^2}{\sigma_1^2}$$

Limiting distributions of functions of estimators?

Suppose $Y_1, ..., Y_n \overset{iid}{\sim} Exponential(\lambda)$, with pdf $f(y) = \lambda e^{-\lambda y}$ for $y > 0$.

Delta method

Example: exponential

Suppose $Y_1, ..., Y_n \overset{iid}{\sim} Exponential(\lambda)$, with pdf $f(y) = \lambda e^{-\lambda y}$ for $y > 0$.

CLT:

$$\sqrt{n}\left(\overline{Y} - \frac{1}{\lambda}\right) \overset{d}{\to} N\left(0, \frac{1}{\lambda^2}\right)$$

Example: exponential

Suppose $Y_1, ..., Y_n \overset{iid}{\sim} \text{Exponential}(\lambda)$, with pdf $f(y) = \lambda e^{-\lambda y}$ for $y > 0$.

$$\widehat{\lambda} = 1/\overline{Y} \quad \sqrt{n}(\widehat{\lambda} - \lambda) \overset{d}{\to} N(0, \lambda^2)$$

Notice: asymptotic variance depends on the parameter λ that we are trying to estimate!
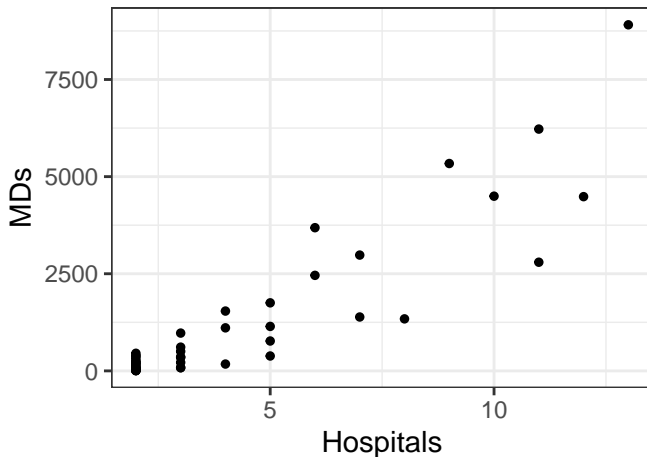
Example: Poisson

Suppose $Y_1, ..., Y_n \overset{iid}{\sim} Poisson(\lambda)$.

$$\widehat{\lambda} = \overline{Y} \qquad \sqrt{n}(\widehat{\lambda} - \lambda) \overset{d}{\to} N(0, \lambda)$$

Want: variance-stabilizing transformation g such that asymptotic variance of $\sqrt{n}(g(\widehat{\lambda}) - g(\lambda))$ does not depend on λ

Example: non-constant variance

Example: Data on the number of hospitals and number of doctors (MDs) in US counties

Question: How do we adjust for non-constant variance in a linear model?

Example: non-constant variance

Example: Data on the number of hospitals and number of doctors (MDs) in US counties