Activity: counterexamples

Convergence in distribution but *not* in probability

Suppose $X \sim N(0,1)$, and let $X_n = -X$ for $n = 1, 2, 3, \ldots$. We will show that $X_n \xrightarrow{d} X$ but X_n does *not* converge in probability.

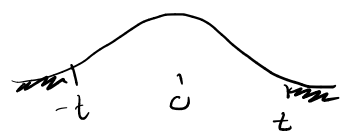
1. Using the fact that X is *symmetric* around 0, show that $F_{X_n}(t) = F_X(t)$ for all n and t. Conclude that $X_n \xrightarrow{d} X$.

X is symmetric around $0 \Rightarrow \forall t,\ P(X \geq t) = P(X \leq -t)$

Let $t \in \mathbb{R}, n \in \mathbb{N}$.

$P(X_n \leq t) = P(-X \leq t) = P(X \geq -t)$

$= P(X \leq t)$ (by symmetry)



$\Rightarrow F_{X_n}(t) = F_X(t)$ for all t, for all n.

$\Rightarrow F_{X_n}(t) \to F_X(t)$ for all t, so $X_n \xrightarrow{d} X$.

2. Show that for any $\varepsilon > 0$, $\lim\limits_{n\to\infty} P(|X_n - X| \geq \varepsilon) > 0$. Conclude that X_n does not converge in probability to X.

Let $\varepsilon > 0$.

$P(|X_n - X| \geq \varepsilon) = P(|-X - X| \geq \varepsilon) = P(|X| \geq \frac{\varepsilon}{2})$

$= P(X \geq \frac{\varepsilon}{2}) + P(X \leq -\frac{\varepsilon}{2})$

$= 2F_X(-\frac{\varepsilon}{2})$ (by symmetry)

$\Rightarrow \lim\limits_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 2F_X(-\frac{\varepsilon}{2})$

$X \sim N(0,1)$, and the support of a Normal distribution is the whole real line.

So, $2F_X(-\frac{\varepsilon}{2}) > 0$ for any ε

$\Rightarrow \lim\limits_{n\to\infty} P(|X_n - X| \geq \varepsilon) > 0$ so $X_n \not\xrightarrow{P} X$

Convergence in probability does not imply convergence of moments

Let $X_1, X_2, X_3, \ldots$ be a sequence such that each $X_n \in \{0, n\}$, with $P(X_n = n) = \frac{1}{n}$ and $P(X_n = 0) = 1 - \frac{1}{n}$.

1. Show that $X_n \xrightarrow{p} 0$.

   Let $\varepsilon > 0$.

   For all $n > \varepsilon$, $P(|X_n - 0| \geq \varepsilon) = P(X_n = n) = \frac{1}{n} \to 0$

   $\Rightarrow$ $P(|X_n - 0| \geq \varepsilon) \to 0$

   so $X_n \xrightarrow{p} 0$.

2. Show that $\mathbb{E}[X_n] \not\to 0$.

   $\mathbb{E}[X_n] = n \cdot P(X_n = n) + 0 \cdot P(X_n = 0)$

   $= n\left(\frac{1}{n}\right) = 1$

   $\mathbb{E}[X_n] = 1$ for all n, so $\mathbb{E}[X_n] \not\to 0$.