

Asymptotic properties of maximum likelihood estimators

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Big results so far

- ▶ **WLLN:** Under certain conditions, $\bar{X}_n \xrightarrow{P} \mu$
- ▶ **CLT:** Under certain conditions, $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0, 1)$

These are nice properties! But I don't just want to estimate the mean. Can we say something similar about maximum likelihood estimates in general?

Key results for the MLE

Definition: Let Y_1, Y_2, \dots, Y_n be a sample from some distribution with parameter $\theta \in \Theta$. Let $\hat{\theta}_n$ be an estimator constructed from the sample Y_1, \dots, Y_n . We say that $\hat{\theta}_n$ is a **consistent** estimator of θ if, for every $\varepsilon > 0$ and every $\theta \in \Theta$,

$$\lim_{n \rightarrow \infty} P_\theta(|\hat{\theta}_n - \theta| \geq \varepsilon) = 0$$

Key results for the MLE

Let Y_1, Y_2, \dots be iid from a distribution with probability function $f(y|\theta)$, where $\theta \in \mathbb{R}^d$ is the parameter(s) we are trying to estimate.
Let

$$\ell_n(\theta) = \sum_{i=1}^n \log f(Y_i|\theta)$$
$$\hat{\theta}_n = \operatorname{argmax}_{\theta} \ell_n(\theta)$$

Theorem: Under certain regularity conditions (to be discussed later),

- (a) $\hat{\theta}_n \xrightarrow{P} \theta$
- (b) $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(\mathbf{0}, v(\theta))$

(where we still need to determine what the variance $v(\theta)$ should be!)

Key results for the MLE

Theorem (consistency of the MLE): Let Y_1, Y_2, \dots be iid from a distribution with parameter θ , and let $\hat{\theta}_n$ be the MLE constructed from Y_1, \dots, Y_n . Under certain conditions (to be discussed later), $\hat{\theta}_n$ is a **consistent** estimator of θ .

Key results for the MLE

Theorem (asymptotic normality of the MLE): Let Y_1, Y_2, \dots be iid from a distribution with parameter θ , and let $\hat{\theta}_n$ be the MLE constructed from Y_1, \dots, Y_n . Under certain conditions (to be discussed later),

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(\mathbf{0}, v(\theta))$$

(where we still need to determine what the variance $v(\theta)$ should be!)

Class activity

Work on the class activity, then we will discuss as a group.

https://sta711-s26.github.io/class_activities/ca_12.html

Class activity

$$Y_1, \dots, Y_n \stackrel{iid}{\sim} Bernoulli(p)$$

$$\ell_n(p) = p \left(\sum_{i=1}^n Y_i \right) + (1-p) \left(n - \sum_{i=1}^n Y_i \right)$$

$$\ell'_n(p) = \frac{\sum_{i=1}^n Y_i}{p} - \frac{\left(n - \sum_{i=1}^n Y_i \right)}{1-p}$$

$$\ell''_n(p) = -\frac{\sum_{i=1}^n Y_i}{p^2} - \frac{\left(n - \sum_{i=1}^n Y_i \right)}{(1-p)^2}$$

$$\hat{p} = \frac{1}{n} \sum_{i=1}^n Y_i$$

The expected score

Let Y be a random variable with probability function $f(y|\theta)$.

Claim: Under regularity conditions,

$$\mathbb{E} \left[\frac{\partial}{\partial \theta} \log f(Y|\theta) \right] = \mathbf{0}$$

Fisher information

Let Y be a random variable with probability function $f(y|\theta)$.

Definition: The **Fisher information** for a single sample Y , denoted $\mathcal{I}_1(\theta)$, is defined as

$$\mathcal{I}_1(\theta) = \text{Var} \left(\frac{\partial}{\partial \theta} \log f(Y|\theta) \right)$$

Claim: Under regularity conditions,

$$\text{Var} \left(\frac{\partial}{\partial \theta} \log f(Y|\theta) \right) = -\mathbb{E} \left[\frac{\partial^2}{\partial \theta^2} \log f(Y|\theta) \right]$$

Fisher information

Claim: Under regularity conditions,

$$\text{Var} \left(\frac{\partial}{\partial \theta} \log f(Y|\theta) \right) = -\mathbb{E} \left[\frac{\partial^2}{\partial \theta^2} \log f(Y|\theta) \right]$$

Asymptotic normality: proof approach

We will sketch the proof in the case that $\theta \in \mathbb{R}$. Let

$$\ell'_n(\theta) = \frac{d}{d\theta} \ell_n(\theta), \quad \ell''_n(\theta) = \frac{d^2}{d\theta^2} \ell_n(\theta)$$

Using a Taylor expansion of ℓ'_n around θ :

$$\hat{\theta}_n - \theta \approx \frac{\ell'_n(\theta)}{-\ell''_n(\theta)}$$

Asymptotic normality: proof approach

$$\sqrt{n}(\hat{\theta}_n - \theta) \approx \frac{\frac{1}{\sqrt{n}}\ell'_n(\theta)}{-\frac{1}{n}\ell''_n(\theta)}$$