# Asymptotic properties of maximum likelihood estimators

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# Key results for the MLE

Let $Y_1, Y_2, \dots$ be iid from a distribution with probability function $f(y|\boldsymbol{\theta})$, where $\boldsymbol{\theta} \in \mathbb{R}^d$ is the parameter(s) we are trying to estimate. Let

$$\ell_n(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log f(Y_i|\boldsymbol{\theta})$$

$$\widehat{\boldsymbol{\theta}}_n = \text{argmax}_{\boldsymbol{\theta}} \ell_n(\boldsymbol{\theta})$$

**Theorem:** Under certain regularity conditions,

(a) $\widehat{\boldsymbol{\theta}}_n \xrightarrow{p} \boldsymbol{\theta}$

(b) $\sqrt{n}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \xrightarrow{d} N(\mathbf{0}, \mathcal{I}_1^{-1}(\boldsymbol{\theta}))$

# Application to regression models

Suppose that $(\mathbf{x}_1, Y_1), ..., (\mathbf{x}_n, Y_n)$ are iid from the linear regression model

$$Y_i|\mathbf{x}_i \sim N(\mu_i, \sigma^2)$$
$$\mu_i = \mathbf{x}_i^T \boldsymbol{\beta}$$

The MLE is $\widehat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$. Asymptotic normality of the MLE means that

$$\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{d} N(\mathbf{0}, \mathcal{I}_1^{-1}(\boldsymbol{\beta}))$$

# Fisher information for the linear regression model

$Y_i | \mathbf{x}_i \sim N(\mathbf{x}_i^T \beta, \sigma^2)$

**Score:** $U(\beta) = \frac{1}{\sigma^2} \mathbf{X}^T (\mathbf{y} - \mathbf{X}\beta)$

# Fisher information for the linear regression model

$Y_i | \mathbf{x}_i \sim N(\mathbf{x}_i^T \beta, \sigma^2)$

**Fisher information:** $\mathcal{I}_1(\beta) = \frac{1}{\sigma^2} \mathbb{E}[\mathbf{x}_i \mathbf{x}_i^T]$