MSE for estimating Normal variance

Suppose $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Consider two estimates of σ^2:

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 \qquad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Our goal is to find the MSE for these two estimators. We will use the following key facts:

- $\frac{1}{\sigma^2}\sum\limits_{i=1}^{n}(X_i - \overline{X})^2 \sim \chi^2_{n-1}$
- If $V \sim \chi^2_\nu$, then $\mathbb{E}[V] = \nu$ and $Var(V) = 2\nu$

Questions

1. Using the information above, show that $\mathbb{E}[s^2] = \sigma^2$ (that is, the sample variance s^2 is an unbiased estimator).

$$\mathbb{E}[s^2] = \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2\right] = \frac{\sigma^2}{n-1}\mathbb{E}\left[\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X})^2\right]$$

$$= \frac{\sigma^2}{n-1}(n-1) = \sigma^2 \quad //$$

2. Now calculate $\mathbb{E}[\widehat{\sigma}^2]$. Does $\widehat{\sigma}^2$ tend to overestimate or underestimate σ^2?

$$\hat{\sigma}^2 = \frac{n-1}{n} s^2$$

$$\Rightarrow \mathbb{E}[\hat{\sigma}^2] = \left(\frac{n-1}{n}\right)\mathbb{E}[s^2] = \left(\frac{n-1}{n}\right)\sigma^2$$

$$\text{Bias}(\hat{\sigma}^2) = -\frac{1}{n}\sigma^2$$

$\hat{\sigma}^2$ tends to slightly underestimate σ^2

3. Using the information above, compute $Var(s^2)$ and $Var(\widehat{\sigma}^2)$.

$$Var(s^2) = Var\left(\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \right)$$

$$= Var\left(\frac{\sigma^2}{\sigma^2(n-1)} \sum_{i=1}^{n} (X_i - \bar{X})^2 \right)$$

$$= \frac{\sigma^4}{(n-1)^2} Var\left(\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \right)$$

since $\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \sim \chi^2_{n-1}$

$$= \frac{\sigma^4}{(n-1)^2} \cdot 2(n-1) = \frac{2\sigma^4}{n-1}$$

$$\hat{\sigma}^2 = \left(\frac{n-1}{n}\right)s^2 \Rightarrow Var(\hat{\sigma}^2) = \left(\frac{n-1}{n}\right)^2 Var(s^2)$$

$$= \left(\frac{n-1}{n}\right)^2 \cdot \frac{2\sigma^4}{n-1} = \frac{2\sigma^4(n-1)}{n^2}$$

4. Which estimator has a lower MSE?

$$MSE(s^2) = Bias^2(s^2) + Var(s^2)$$

$$= 0 + \frac{2\sigma^4}{n-1}$$

$$= \frac{2\sigma^4}{n-1}$$

$$MSE(\hat{\sigma}^2) = Bias^2(\hat{\sigma}^2) + Var(\hat{\sigma}^2)$$

$$= \left(-\frac{1}{n}\sigma^2\right)^2 + \frac{2\sigma^4(n-1)}{n^2}$$

$$= \frac{\sigma^4}{n^2} + \frac{2\sigma^4(n-1)}{n^2}$$

$$= \frac{(2n-1)\sigma^4}{n^2} < \frac{2\sigma^4}{n-1}$$

$$\Rightarrow MSE(\hat{\sigma}^2) < MSE(s^2)$$

5. You are currently showing on homework that the Fisher information (for a single observation) for the variance of a normal distribution is $\mathcal{I}_1(\sigma^2) = \dfrac{1}{2\sigma^4}$. Does s^2 attain the CRLB?

$$\text{CRLB} = \frac{1}{n}\mathcal{I}_1^{-1}(\sigma^2)$$

$$= \frac{2\sigma^4}{n} < \frac{2\sigma^4}{n-1} = Var(S^2)$$

S^2 does <u>not</u> attain CRLB

6. Now **suppose we know the mean** μ, and consider the estimator $\widetilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$. Show that this estimator is unbiased for σ^2. (*Hint*: If $Z \sim N(0,1)$ then $Z^2 \sim \chi^2_1$)

$$X_i \sim N(\mu, \sigma^2) \Rightarrow \frac{X_i - \mu}{\sigma} \sim N(0,1)$$

$$\Rightarrow \left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_1$$

$$\Rightarrow \mathbb{E}\left[\left(\frac{X_i-\mu}{\sigma}\right)^2\right] = 1$$

$$\mathbb{E}[\tilde{\sigma}^2] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\right]$$

$$= \mathbb{E}\left[\frac{\sigma^2}{n}\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2\right]$$

$$= \frac{\sigma^2}{n}\sum_{i=1}^{n}\mathbb{E}\left[\left(\frac{X_i-\mu}{\sigma}\right)^2\right]$$

$$= \sigma^2 \qquad //$$

7. Does $\tilde{\sigma}^2$ from the previous question attain the CRLB?

$$\mathrm{Var}(\tilde{\sigma}^2) = \mathrm{Var}\left(\frac{\sigma^2}{n}\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2\right)$$

$$= \frac{\sigma^4}{n^2}\sum_{i=1}^{n}\mathrm{Var}\left(\left(\frac{X_i-\mu}{\sigma}\right)^2\right) \quad \text{(independence)}$$

$$= \frac{\sigma^4}{n^2}\cdot 2n = \frac{2\sigma^4}{n}$$

$\left(\text{since } \left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_1,\ \mathrm{Var}\left(\left(\frac{X_i-\mu}{\sigma}\right)^2\right) = 2\right)$

So, $\tilde{\sigma}^2$ <u>does</u> attain CRLB

<u>But</u> (important!) : $\tilde{\sigma}^2$ contains μ.

If μ is <u>unknown</u>, we can't calculate $\tilde{\sigma}^2$!

Turns out that if μ is unknown, there is no unbiased estimator of σ^2 which attains CRLB