STA 712 Homework 4

Due: Friday, October 6, 12:00pm (noon) on Canvas.

Instructions: Submit your work as a single PDF, created using LaTeX; see the course website for a homework template file and instructions on getting started with LaTeX and Overleaf. See the Overleaf guide on mathematical expressions to get started writing math in LaTeX.

Practice with exponential dispersion models

1. Suppose $Y \sim Gamma(\alpha, \beta)$, with shape $\alpha > 0$ and scale $\beta > 0$. The density function for Y is given by
$$f(y; \alpha, \beta) = \frac{y^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \exp\{-y/\beta\}.$$
   (a) Show that the gamma distribution is an EDM by identifying θ, $\kappa(\theta)$, and ϕ.
   (b) Find $\mu = \mathbb{E}[Y]$ as a function of α and β by using the fact that $\mu = \dfrac{\partial}{\partial \theta}\kappa(\theta)$.
   (c) Using (b), what is the canonical link function for the gamma distribution?
   (d) Using the fact that $Var(Y) = \phi \cdot \dfrac{\partial \mu}{\partial \theta} = \phi \cdot V(\mu)$, find $V(\mu)$ as a function of μ, and find $Var(Y)$ as a function of α and β.
   (e) Show that the unit deviance for the gamma distribution is
$$d(y, \mu) = 2\left(-\log\frac{y}{\mu} + \frac{y-\mu}{\mu}\right).$$
   (f) Write the density function for Y in dispersion model form.

2. A very cool property of EDMs is that the mean-variance relationship encoded by $V(\mu)$ uniquely determines the EDM. For example, a normal distribution is the only EDM for which $V(\mu) = 1$, a Poisson distribution is the only EDM for which $V(\mu) = \mu$, etc. This means that, given a desired mean-variance relationship, we can work backwards to figure out what the EDM should be!

   Suppose we are told that $Y \sim EDM(\mu, \phi)$, and we know that $V(\mu) = \mu^3$. In this problem, we will derive the corresponding EDM for Y.

   (a) Using the fact that $V(\mu) = \frac{\partial \mu}{\partial \theta}$, find θ as a function of μ. *Hint: recall from calculus that* $\dfrac{\partial \theta}{\partial \mu} = \dfrac{1}{\partial \mu / \partial \theta} = \mu^{-3}$.
   (b) Using the fact that $\mu = \frac{\partial \kappa(\theta)}{\partial \theta}$, show that $\kappa(\theta) = -\sqrt{-2\theta} = -\dfrac{1}{\mu}$.
   (c) Conclude that $f(y; \mu, \phi) = a(y, \phi)\exp\left\{\dfrac{-y/(2\mu^2) + 1/\mu}{\phi}\right\}$. Then rearrange to show that
$$f(y; \mu, \phi) = b(y, \phi)\exp\left\{-\frac{1}{2\phi}\frac{(y-\mu)^2}{y\mu^2}\right\}.$$

The density function of an inverse Gaussian distribution is given by

$$f(y; \mu, \phi) = (2\pi y^3 \phi)^{-1/2} \exp\left\{-\frac{1}{2\phi}\frac{(y-\mu)^2}{y\mu^2}\right\},$$

so we have shown that if $V(\mu) = \mu^3$, then Y must have an inverse Gaussian distribution!