Lecture 9

Recap: EDMs

EDM : $f(y; \theta, \phi) = \underbrace{a(y, \phi)}_{\text{normalizing function}} \exp\left\{ \frac{y\theta - \kappa(\theta)}{\phi} \right\}$

θ : canonical parameter

$\theta = g(\mu)$ $\quad \mu = \mathbb{E}[Y]$

↑ canonical link function

ϕ : dispersion parameter $\quad Var(Y) = \phi \cdot \underbrace{V(\mu)}_{\text{determines mean-variance relationship}}$

$\kappa(\theta)$: cumulant function

Recap: properties of EDMs

$$\mu = \mathbb{E}[Y] = \frac{\partial \kappa(\theta)}{\partial \theta}$$

$$\mathrm{Var}(Y) = \phi \cdot \frac{\partial^2}{\partial \theta^2} \kappa(\theta)$$

$$= \phi \cdot \underbrace{\frac{\partial}{\partial \theta} \mu(\theta)}_{V(\mu)}$$

So: $\theta = g(\mu) \Rightarrow \mu = g^{-1}(\theta)$

$$\mu = \frac{\partial \kappa(\theta)}{\partial \theta}$$

$$\mathrm{Var}(Y) = \phi \cdot V(\mu) = \phi \cdot \frac{\partial \mu}{\partial \theta}$$

Examples of EDMs

$Y \sim \text{Poisson}(\lambda)$

$\mu = \lambda \qquad Var(Y) = \overset{\phi}{1} \cdot \mu \leftarrow V(\mu)$

$\theta = \log \mu \qquad (= \log \lambda) \qquad$ log: canonical link function

$K(\theta) = \mu \qquad (= \lambda)$

$\mu = e^{\theta}$

$V(\mu) = \dfrac{d\mu}{d\theta} = \dfrac{d}{d\theta} e^{\theta} = e^{\theta} = \mu \qquad V(\mu) = \mu$ ✓

$K(\theta) = \mu = e^{\theta} \qquad \dfrac{dK(\theta)}{d\theta} = e^{\theta} = \mu$

$\Rightarrow E[Y] = \mu$

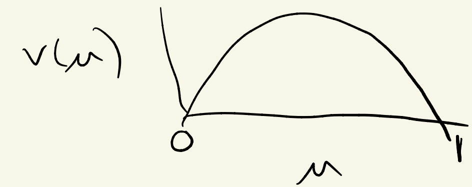
$Y \sim \text{Bernoulli}(\mu)$ $\qquad$ $E[Y] = \mu$ $\qquad$ $\text{Var}(Y) = \mu(1-\mu)$

$$= \phi \cdot V(\mu)$$

$$\phi = 1 \qquad V(\mu) = \mu(1-\mu)$$

$$\theta = \log\left(\frac{\mu}{1-\mu}\right)$$

$$\mu = \frac{e^{\theta}}{1+e^{\theta}}$$



verify that $\underset{\substack{\uparrow \\ \mu(1-\mu)}}{V(\mu)} = \dfrac{d\mu}{d\theta}$

$$\frac{d\mu}{d\theta} \quad \frac{e^{\theta}}{1+e^{\theta}} = \frac{(1+e^{\theta})e^{\theta} - (e^{\theta})^2}{(1+e^{\theta})^2} = \frac{e^{\theta}}{(1+e^{\theta})^2} = \mu(1-\mu) \checkmark$$

$$\kappa(\theta) = \log(1+e^{\theta})$$

verify: $\dfrac{d\kappa(\theta)}{d\theta} = \mu(\theta)$

$$\frac{d\kappa(\theta)}{d\theta} = \frac{1}{1+e^{\theta}} \cdot e^{\theta} = \mu(\theta) \checkmark$$

$Y \sim EDM(\mu, \phi)$ $\mu = \mathbb{E}[Y]$ $Var(Y) = \phi \cdot V(\mu)$

if $f(y; \theta, \phi) = a(y, \phi) \exp\left\{ \dfrac{y\theta - \kappa(\theta)}{\phi} \right\}$

where $\theta = g(\mu)$ g is monotonically increasing

$$\frac{\partial \kappa(\theta)}{\partial \theta} = \mu$$

$$V(\mu) = \frac{\partial \mu}{\partial \theta}$$

μ_i is a function of explanatory variables

<u>GLM</u> : $Y_i \sim EDM(\mu_i, \phi)$ <u>not</u> a function of X_i

$h(\mu_i) = \beta^T X_i$

h is some link function

if $h = g$, using the canonical link function

GLMs

Why abstraction is nice

$Y_i \sim EDM(\mu_i, \phi)$

$g(\mu_i) = \beta^T X_i$

canonical link

Want to estimate β

observe data $(X_1, Y_1), \ldots, (X_n, Y_n)$

$$L(\beta) = \prod_{i=1}^{n} a(Y_i, \phi) \exp\left\{ \frac{Y_i \theta_i - \kappa(\theta_i)}{\phi} \right\} \qquad \theta_i = g(\mu_i)$$

$$\Rightarrow \ell(\beta) = \sum_{i=1}^{n} \log(a(Y_i, \phi)) + \frac{1}{\phi} \sum_{i=1}^{n} (Y_i \theta_i - \kappa(\theta_i))$$

HW: Score: $$U(\beta) = \frac{1}{\phi} \sum_{i=1}^{n} (Y_i X_i - \mu_i X_i) = \frac{X^T (Y - \mu)}{\phi}$$

Information: $$\mathcal{I}(\beta) = \frac{1}{\phi^2} \sum_{i=1}^{n} Var(Y_i) X_i X_i^T = \frac{X^T W X}{\phi^2}$$

$W = diag(Var(Y_i))$

$V = diag(V(\mu_i))$

$$= \frac{X^T V X}{\phi}$$

$\hat{\beta}$ maximum likelihood estimates

$$\hat{\beta} \approx N(\beta, \mathcal{I}^{-1}(\beta))$$

Class activity

https://sta712-f23.github.io/class_activities/ca_lecture_9.html