Fitting mixed models

often: $G = \sigma_u^2 I_m$ $\quad R = \sigma_\varepsilon^2 I_N$

$$Y = X\beta + Zu + \varepsilon$$

$u \sim N(0, G)$ $\quad \varepsilon \sim N(0, R)$

$u \perp\!\!\!\perp \varepsilon$

X: design matrix for fixed effects

Z: design matrix for random effects

$$\underbrace{Zu} + \varepsilon \sim N(0, ZGZ^T + R)$$

$\mathrm{var}(Zu)$
$= Z\,\mathrm{var}(u) Z^T$

$$Y = X\beta + \delta \qquad \delta \sim N(0, \underbrace{ZGZ^T + R}_{V})$$

$$\Rightarrow \underbrace{V^{-\frac{1}{2}}Y}_{Y_w} = \underbrace{V^{-\frac{1}{2}}X}_{X_w}\beta + \underbrace{V^{-\frac{1}{2}}\delta}_{\substack{N(0,I) \\ \delta_w}}$$

$$V^{-\frac{1}{2}}\delta \sim N(0, I)$$

$\mathrm{Var}(V^{-\frac{1}{2}}\delta)$

$$Y_w = X_w\beta + \delta_w$$

$$\hat{\beta} = (X_w^T X_w)^{-1} X_w^T Y_w = (X^T V^{-1} X)^{-1} X^T V^{-1} Y$$

$\mathrm{var}(\hat{\beta}) =$
$(X^T V^{-1} X)^{-1}$

Ex:

$$Y_{ij} = \beta_0 + u_i + \beta_1 X_i + \varepsilon_{ij} \qquad X_i \in \{0,1\} \qquad n_i = \text{\# obs in group } i$$

$$X_1, \ldots, X_{m_0} = 0 \qquad X_{m_0+1}, \ldots, X_m = 1 \qquad \bar{Y}_{i\cdot} = \text{mean of } Y \text{ in group } i$$

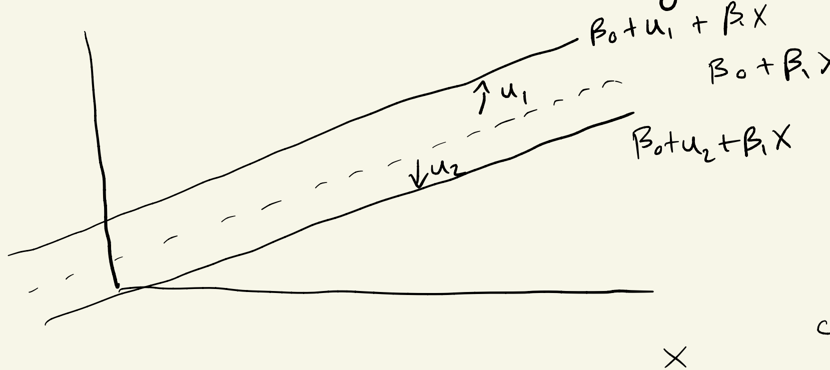
$$\hat{\beta}_1 = (X^T V^{-1} X)^{-1} X^T V^{-1} Y \qquad X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ \vdots & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} n_1 \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} n_2 \\ \vdots \\ \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} n_m \end{matrix}$$

$$\hat{\beta}_1 = \frac{\displaystyle\sum_{i=m_0+1}^{m} \frac{n_i \bar{Y}_{i\cdot}}{n_i \hat{\sigma}_u^2 + \hat{\sigma}_\varepsilon^2}}{\displaystyle\sum_{i=m_0+1}^{m} \frac{n_i}{n_i \hat{\sigma}_u^2 + \hat{\sigma}_\varepsilon^2}} - \frac{\displaystyle\sum_{i=1}^{m_0} \frac{n_i \bar{Y}_{i\cdot}}{n_i \hat{\sigma}_u^2 + \hat{\sigma}_\varepsilon^2}}{\displaystyle\sum_{i=1}^{m_0} \frac{n_i}{n_i \hat{\sigma}_u^2 + \hat{\sigma}_\varepsilon^2}}$$

$$\operatorname{Var}(\bar{Y}_{i\cdot}) = \sigma_u^2 + \frac{\sigma_\varepsilon^2}{n}$$

So far: random intercepts model $Y_{ij} = \beta_0 + u_i + \beta_1 X_{ij} + \varepsilon_{ij}$

e.g. X is a continuous explanatory variable:

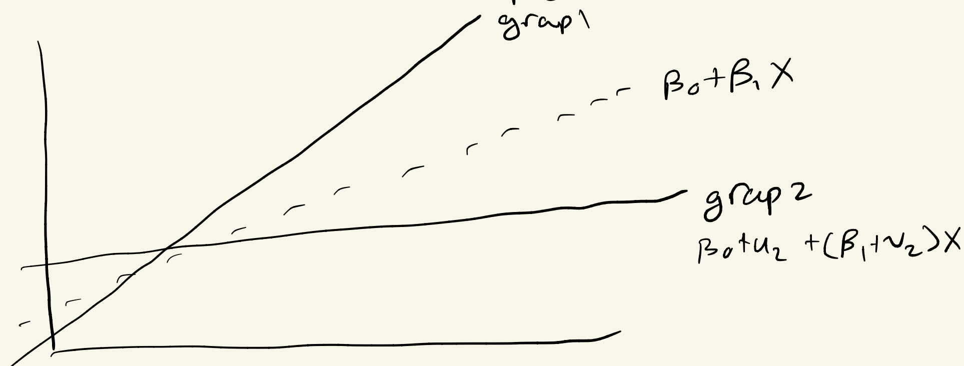


(average relationship across groups)

lines are <u>parallel</u>

(within a group, a unit change in X is associated w/ a change of β_1 in Y)

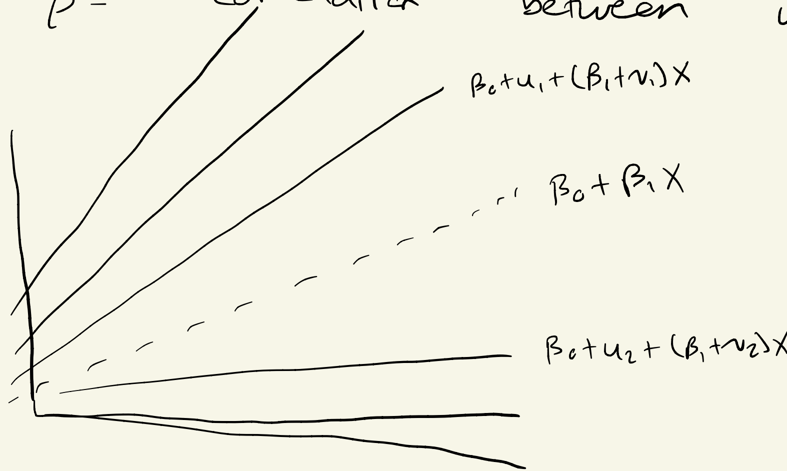
what if we want different slopes?



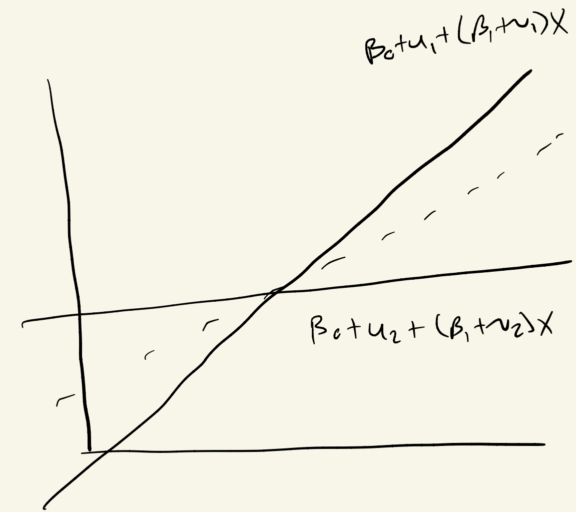
$$Y_{ij} = \beta_0 + u_i + (\beta_1 + v_i) X_{ij} + \varepsilon_{ij}$$

$$Y_{ij} = \beta_0 + u_i + (\beta_1 + v_i) X_{ij} + \varepsilon_{ij} \qquad \begin{bmatrix} u_i \\ v_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_u^2 & \rho\sigma_u\sigma_v \\ \rho\sigma_u\sigma_v & \sigma_v^2 \end{bmatrix} \right)$$

$\rho =$ correlation between u_i & v_i



$\rho > 0$



$\rho < 0$