Generalized estimating equations

$$Y = X\beta + \delta \qquad \delta \sim N(0, V)$$

$$\mathbb{E}[Y] = X\beta \qquad \text{var}(Y) = V$$

$$\underbrace{V^{-\frac{1}{2}} Y}_{Y_w} = \underbrace{V^{-\frac{1}{2}}}_{X_w} X\beta + \underbrace{V^{-\frac{1}{2}}\delta}_{\delta_w} \implies Y_w = X_w \beta + \delta_w$$

$$\delta_w \sim N(0, \underline{I})$$

$\hat{\beta}$ minimizes $(Y_w - X_w\beta)^T (Y_w - X_w\beta)$

$\hat{\beta}$ solves $X_w^T (Y_w - X_w\beta) = 0$

$$\implies X^T V^{-1} (Y - X\beta) = 0 \qquad \hat{\beta} = (X^T V^{-1} X)^{-1} X^T V^{-1} Y$$

$$X^T V^{-1} (Y - X\beta) = 0$$

Suppose we have m groups, independence between groups

$$Y = \begin{bmatrix} Y_{1,1} \\ Y_{1,2} \\ \vdots \\ Y_{m n_m} \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{bmatrix}$$

$\leftarrow$ length n_1

$\leftarrow$ length n_2

$\vdots$

etc.

$$V = \begin{bmatrix} V_1 & & 0 \\ & V_2 & \\ 0 & & \ddots \; V_m \end{bmatrix}$$

V_i = covariance matrix for group i

$$V^{-1} = \begin{bmatrix} V_1^{-1} & & 0 \\ & V_2^{-1} & \\ 0 & & \ddots \; V_m^{-1} \end{bmatrix}$$

$$X^T V^{-1} (Y - X\beta) = \sum_{i=1}^{m} X_i^T V_i^{-1} (Y_i - X_i \beta) = 0$$

(sum over m independent groups)

Assumption: $E[Y] = X\beta$ $\quad$ m independent groups

V_i covariance matrix for group i

Estimate β by solving

$$u(\beta) = \sum_{i=1}^{m} X_i^T V_i^{-1} (Y_i - X_i\beta) = 0$$

Generalized estimating equation

Some common assumptions about V_i

Independence:

$$V_i = \underset{\text{scaling}}{\sigma^2} \underbrace{\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots & \\ & & & & 1 \end{bmatrix}}_{\text{correlation}}$$

Exchangeable:

$$V_i = \sigma^2 \begin{bmatrix} 1 & \rho & \rho & \rho & \cdots & \rho \\ \rho & 1 & & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ & & & & \ddots & \\ \rho & \rho & \cdots & & & 1 \end{bmatrix}$$

Auto-regressive (1) : $V_i = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 & \cdots \\ \rho & 1 & \rho & \rho^2 & \cdots \\ \rho^2 & \rho & & \ddots & \\ \vdots & \rho^2 & & & \\ \vdots & \vdots & & & 1 \end{bmatrix}$

Arbitrary : $V_i = \sigma^2 \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \cdots \\ \rho_{12} & 1 & & & \\ \rho_{13} & & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \end{bmatrix}$

Mixed effects model: $Y_{ij} = \beta_0 + u_i + \beta_1 X_{ij} + \varepsilon_{ij}$ $\quad u_i \sim N(0, \sigma_u^2)$

$\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$

(exchangeable correlation) $\quad V_i = \begin{bmatrix} \sigma_u^2 + \sigma_\varepsilon^2 & \sigma_u^2 & \sigma_u^2 & \cdots \\ & & & \\ & & & \end{bmatrix}$

$\sigma^2 = \sigma_u^2 + \sigma_\varepsilon^2 \qquad \rho = \dfrac{\sigma_u^2}{\sigma_u^2 + \sigma_\varepsilon^2}$

Estimation :

$$\hat{\beta} = (X^T V^{-1} X)^{-1} X^T V^{-1} Y$$

requires knowing V!

How do we estimate V?

Assume $V_i = \sigma^2 \begin{bmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & & & \\ \vdots & & & \ddots & \\ \rho & & & & 1 \end{bmatrix}$

need to estimate σ^2 and ρ

Let $e_{ij} = Y_{ij} - X_{ij}^T \hat{\beta}$ (Pearson residual)

N = total # obs

p = # parameters

$$\hat{\sigma}^2 = \frac{1}{N-p} \sum_{i=1}^{m} \sum_{j=1}^{n_i} e_{ij}^2$$

$$\hat{\sigma}^2 \hat{\rho} = \frac{1}{m} \sum_{i=1}^{m} \frac{1}{n_i(n_i-1)} \sum_{j \neq k} e_{ij} e_{ik}$$

Iterative GEE fitting procedure:

1) Start with initial $\hat{\beta}^0$

2) (a) Calculate e_{ij}

(b) Calculate $\hat{V}$

3) $$\hat{\beta}^{(r+1)} = \hat{\beta}^{(r)} + \left(\sum_{i=1}^{m} X_i^T \hat{V}_i^{-1} X_i \right)^{-1} \left(\sum_{i=1}^{m} X_i^T \hat{V}_i^{-1} (Y_i - X_i \hat{\beta}^{(r)}) \right)$$

4) iterate until convergence