Lab4: Fitting Bayesian Models in JAGS

STA721 Linear Models Duke University

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2 Block g-prior (Normal-Jeffreys)

Model in centered parameterization (review)

$$\begin{aligned} \mathbf{Y} &= \mathbf{1}\beta_0 + (\mathbf{I}_n - \mathbf{P}_1)\mathbf{X}_1\boldsymbol{\beta} + \boldsymbol{\epsilon} \\ p(\beta_0, \phi) &\propto 1/\phi \\ \boldsymbol{\beta} \mid \beta_0, \phi, g &\sim \mathrm{N}(\mathbf{0}, \frac{g}{\phi}(\mathbf{X}^T(\mathbf{I}_n - \mathbf{P_1})\mathbf{X})^{-1}) \end{aligned}$$

Log Likelihood (show)

$$\mathcal{L}(\beta_0, \beta_1, \phi) \propto \frac{n}{2}\log(\phi) - \frac{\phi}{2}\left(n(\beta_0 - \bar{y})^2 + (\mathbf{Y}_c - \mathbf{X}_c\boldsymbol{\beta})^T(\mathbf{Y}_c - \mathbf{X}_c)\right)$$

Since

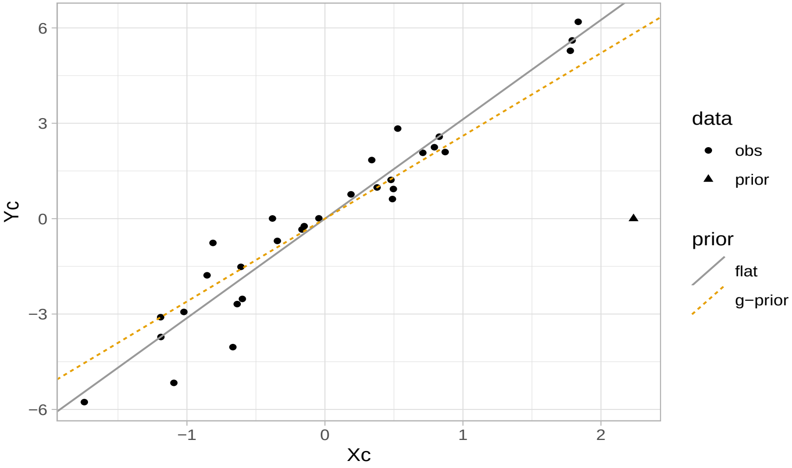
$$\mathbf{Y} = (\mathbf{I} - \mathbf{P_1})\mathbf{Y} + \mathbf{P_1}\mathbf{Y}$$

Bayesian Estimation with g prior (2 groups

$$
\begin{aligned}
\mathbf{Y} &= \mathbf{1}\beta_0 + \mathbf{I}_n - \mathbf{P}_1)\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \\
p(\beta_0, \phi) &\propto 1\phi \\
\boldsymbol{\beta} \mid \phi &\sim \mathsf{N}(\mathbf{0}, \frac{g}{\phi}(\mathbf{X}^T(\mathbf{I}_n - \mathbf{P}_{\mathbf{1}})\mathbf{X})^{-1})
\end{aligned}
$$

Example from class: g=5, n=30

In SLR g prior contribution is like adding an extra $Y_0 = 0$ at $\mathbf{X}_o = \sqrt{\frac{\mathrm{SS}_x}{g}}$:



Markov Chain Monte Carlo

- We know that $\beta_0, \boldsymbol{\beta}, \phi \mid \mathbf{Y}, g = 1/\tau$ has a Normal-Gamma distribution
- Derive full conditional for $\boldsymbol{\theta} = [\beta_0, \boldsymbol{\beta}, \phi]$ as a sequence of distributions β_0 and $\boldsymbol{\beta}$ given $\phi, \mathbf{Y}$ (are independent) and $\phi \mid \mathbf{Y}$ [Successive Substitution Sampling] (in HW)
- Let $\tau = 1/g$ have a Gamma distribution $G(1/b, n/2)$ We can show that $\tau \mid \beta_0, \boldsymbol{\beta}, \phi, \mathbf{Y}$ has a Gamma distribution (derive HW)

$$p(\tau \mid \boldsymbol{\beta}, \phi, \mathbf{Y}) \propto \mathcal{L}(\beta_0, \boldsymbol{\beta}, \phi)\tau^{p/2} e^{(-\tau \frac{\phi}{2} \boldsymbol{\beta}^T (\mathbf{X}^T \mathbf{X}) \boldsymbol{\beta})} \tau^{1/2-1} e^{-\tau n/2}$$

- alternate sampling from two blocks of full conditional distributions given current values of other parameters.
- implement in JAGS or STAN

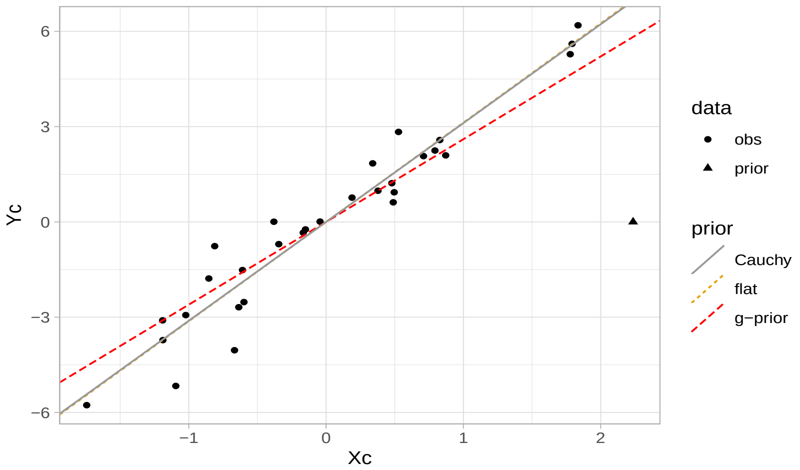
JAGS Code: library(R2jags)

```
model = function(){
  for (i in 1:n) {
      Y[i] ~ dnorm(beta0+ (X[i] -Xbar)*beta, phi)
  }
  beta0 ~ dnorm(0, .000001*phi) #precision is 2nd arg
  beta ~ dnorm(0, phi*tau*SSX)  #precision is 2nd arg
  phi ~ dgamma(.001, .001) # approximate independent Jeffr
  tau ~ dgamma(.5, .5*n)
  g <- 1/tau
  sigma <- pow(phi, -.5)
}
data = list(Y=Y, X=X, n =length(Y), SSX=sum(Xc^2),
            Xbar=mean(X))
ZSout = jags(data, inits=NULL,
             parameters.to.save=c("beta0","beta", "g",
                                  "sigma"),
             model=model, n.iter=10000)
```

HPD intervals

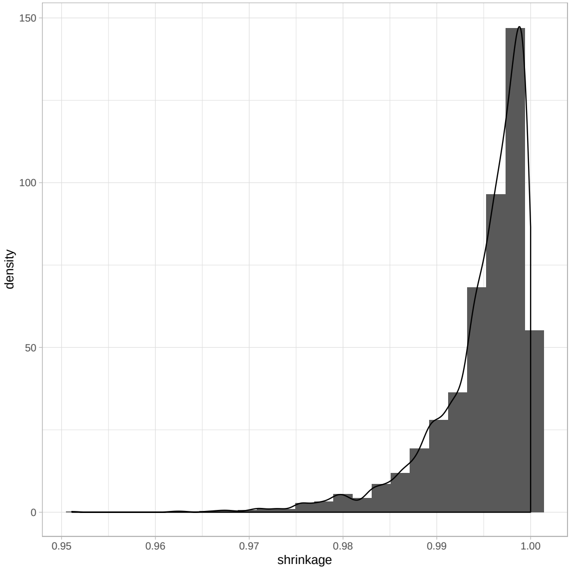
```
confint(lm(Y ~ Xc))

##                  2.5 %    97.5 %
## (Intercept) -0.3985359 0.2048303
## Xc           2.7945824 3.4555162

HPDinterval(as.mcmc(ZSout$BUGSoutput$sims.matrix))

##               lower        upper
## beta      2.7823047    3.4453690
## beta0    -0.3764027    0.2095465
## deviance 70.2043917   78.4813041
## g        19.4503373 3782.7134974
## sigma     0.6171029    1.0504892
## attr(,"Probability")
## [1] 0.95
```

Compare



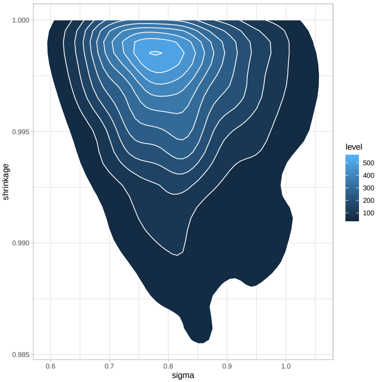
```
ZSout

## Inference for Bugs model at "/var/folders/n4/nj1122xj6bn5_xgbptv7bml40000gp/T//RtmptMfLcg/model1563918c
##  3 chains, each with 10000 iterations (first 5000 discarded), n.thin = 5
##  n.sims = 3000 iterations saved
##           mu.vect   sd.vect   2.5%     25%     50%     75%    97.5%  Rhat
## beta        3.112     0.170  2.782   2.997   3.115   3.225    3.445 1.001
## beta0      -0.099     0.152 -0.384  -0.204  -0.099   0.001    0.204 1.002
## g        2263.147 38967.029 48.273 146.129 282.298 697.063 9018.709 1.001
## sigma       0.827     0.114  0.636   0.747   0.816   0.896    1.079 1.001
## deviance   73.347     2.563 70.390  71.458  72.680  74.500   79.882 1.002
##          n.eff
## beta      3000
## beta0     1200
## g         3000
## sigma     3000
## deviance  1600
##
## For each parameter, n.eff is a crude measure of effective sample size,
## and Rhat is the potential scale reduction factor (at convergence, Rhat=1).
##
## DIC info (using the rule, pD = var(deviance)/2)
## pD = 3.3 and DIC = 76.6
## DIC is an estimate of expected predictive error (lower deviance is better).
```

Posterior Distribution of shrinkage



Joint Distribution of σ and $g/(1+g)$

```
ggplot(postdf, aes(x=sigma, y=shrinkage) ) +
 stat_density_2d(aes(fill = ..level..),
                 geom = "polygon", colour="white") +
  theme_light()
```



Cauchy Summary

- Cauchy rejects prior mean if it is an "outlier"
- robustness related to "bounded" influence (more later)
- requires numerical integration or Monte Carlo sampling (MCMC)