Hypothesis Testing and Model Choice

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STA721 Linear Models

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Outline

Topics

- Climate Example
- t-tests
- Overall F-test
- Sequential F-tests
- Added Variable Plots (if time)
- Summary

Readings: Christensen Chapter 2 (section 7), Chapter 10, Appendix B

Climate Change ?

Scientists are interested in the Earth's temperature change since the last glacial maximum, about 20,000 years ago.

- The first study to estimate the temperature change was published in 1980
- Estimated a change of -1.5 degrees C, ± 1.2 degrees C in tropical sea surface temperatures.
- The negative value means that the Earth was colder then than now.
- Since 1980 there have been many other studies, which use different measurement techniques, or proxies.
- Some proxies can be used over land, others over water.

Proxies

The 8 proxies used are

1. "Mg/Ca" 1
2. "alkenone" 2
3. "Faunal" 3
4. "Sr/Ca" 4
5. "del 180" 5
6. "Ice Core" 6
7. "Pollen" 7
8. "Noble Gas" 8

Variables

```
climate =
read.table("http://www.stat.duke.edu/courses/Fall10/sta290/datasets/climate.dat",

header=T)
```

Each of the 53 studies reported

- `deltaT` an estimate of the temperature change
- `sdev` a standard deviation of that estimate
- `proxy` the proxy used (coded 1 to 8),
- `T.M` whether it was a terrestrial or marine study (T/M), which is coded as 0 for Terrestrial, 1 for Marine,
- `latitude` at which data were collected

Questions of Interest

1. Do estimates vary systematically by proxy?
2. Do terrestrial estimates differ systematically from marine estimates?
3. Do estimates vary systematically by latitude?
4. Can we combine the studies to get a better estimate of the overall temperature change?
5. Are temperatures changing?

Build a larger model or series of models to address these questions?

$$\mathrm{E}[\Delta T] = f(\text{ Proxy, latitude})$$

Model Building

George E. P. Box

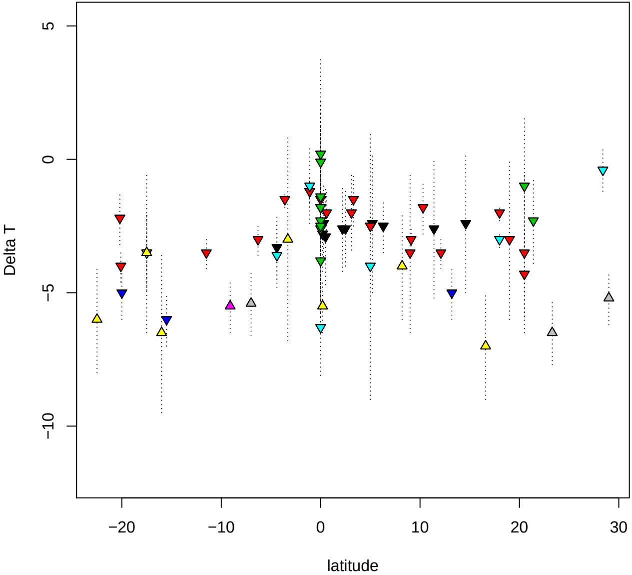
Essentially, all models are wrong, but some are useful. *Empirical Model-Building and Response Surfaces* (1987), co-authored with Norman R. Draper, p. 424

- "true" model may be a complicated function of latitude, proxy, as well as other (omitted) covariates
- Assume that for each proxy p, there is a nonlinear relationship between ΔT and latitude l and omitted variables o; $f(p, l, o)$
- Taylor's series expansion about some point l_0:

$$\begin{aligned} f(p, l, o) &= f(p, l_0, o) + (l - l_0) f'(p, l_0, o) + (l - l_0)^2 \frac{f''(p, l_0, o)}{2} \\ &\quad + R(l, p, o) \\ f(p, l) &\approx \beta_{p0} + \beta_{1p} l + \beta_{2p} l^2 \end{aligned}$$

- Ignore o and remainder term

A Picture is Worth a Thousand Words (sometimes)



duke.eps

R Model Formula

$$\texttt{DeltaT} \sim \texttt{proxy*poly(latitude, 2)}$$

- Expand out predictors as `proxy + poly(latitude, 2) + proxy:poly(latitude, 2)`
- `proxy` is a factor; default coding is to create 8 indicators of each proxy and then drop the column associated with the first level of the factor (MG/Ca in the example)
- `poly(latitude, 2)` creates an orthonormal basis for a second order polymomial in latitude $[1, l, l^2]$
- `proxy:poly(latitude, 2)` takes the product of each of the 7 dummy variables for `proxy` times the linear and quadratic terms for `latitude`
- Look at `model.matrix( ~ poly(latitude,2)*proxy, data=climate)`

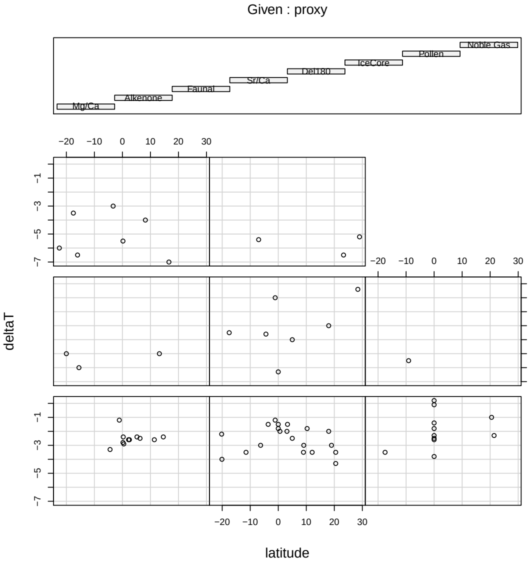
Estimates

```
> summary(climate.lm)
Coefficients: (2 not defined because of singularities)
                               Estimate Std. Error t value Pr(>|t|)
(Intercept)                     -2.7933     2.3189  -1.205    0.235
Alkenone                         0.4463     2.3234   0.192    0.849
Faunal                           0.7235     2.4525   0.295    0.769
Sr/Ca                           -2.9254     2.5318  -1.155    0.255
Del180                          -0.3037     2.4030  -0.126    0.900
IceCore                         -3.1407     2.8504  -1.102    0.277
Pollen                          -2.6751     2.4528  -1.091    0.282
Noble Gas                       -3.2520     2.5698  -1.265    0.213
poly(latitude, 2)1              -3.0092    10.5916  -0.284    0.778
poly(latitude, 2)2              -7.3654    26.6516  -0.276    0.784
Alkenone:poly(latitude, 2)1      3.5493    10.6675   0.333    0.741
Faunal:poly(latitude, 2)1        6.5637    11.7978   0.556    0.581
Sr/Ca:poly(latitude, 2)1        11.8701    15.6097   0.760    0.451
Del180:poly(latitude, 2)1        0.8912    11.7526   0.076    0.940
IceCore:poly(latitude, 2)1           NA         NA      NA       NA
Pollen:poly(latitude, 2)1       -4.0769    13.5600  -0.301    0.765
Noble Gas:poly(latitude, 2)1    -8.7078    17.9962  -0.484    0.631
Alkenone:poly(latitude, 2)2      3.0832    26.6984   0.115    0.909
Faunal:poly(latitude, 2)2        2.8690    27.4056   0.105    0.917
Sr/Ca:poly(latitude, 2)2        19.2753    31.4567   0.613    0.543
Del180:poly(latitude, 2)2       16.1802    26.9623   0.600    0.552
IceCore:poly(latitude, 2)2           NA         NA      NA       NA
Pollen:poly(latitude, 2)2        3.3119    27.6753   0.120    0.905
Noble Gas:poly(latitude, 2)2    18.6612    30.0579   0.621    0.538

Residual standard error: 2.112 on 41 degrees of freedom
Multiple R-squared: 0.682,^^IAdjusted R-squared: 0.5191
F-statistic: 4.187 on 21 and 41 DF,  p-value: 4.382e-05
```

Conditional Plot

`coplot(deltaT ~ latitude | proxy, data=climate)`



Estimates and t-statistics

- MLEs do not depend on the order of the variables in the model
- regression coefficients are adjusted for the other variables in the model
- t-statistics

  $$\frac{\lambda^T \boldsymbol{\beta} - \lambda^T b_0}{\hat{\sigma}\sqrt{\lambda^T(\mathbf{X}^T\mathbf{X})^{-1}\lambda}} \sim t(n-p, 0, 1)$$

  under the hypothis $\lambda^T b_0 = 0$
- t-values correspond to test statistic for testing hypothesis H$_o$: $\beta_j = 0$ versus H$_a$: $\beta_j \neq 0$ given the other variables are in the model
- all p-values greater than α does not mean that all coefficients are zero!
- redundancy
- with factors use ANOVA for simultaneous testing

Decomposition

Consider a series of nested models:

$$
\begin{aligned}
\mathcal{M}_0 : \mathbf{Y} &= \mathbf{1}_n\beta_0 + \boldsymbol{\epsilon} \\
\mathcal{M}_1 : \mathbf{Y} &= \mathbf{1}_n\beta_0 + \mathbf{X}_1\boldsymbol{\beta}_1 + \boldsymbol{\epsilon} \\
\mathcal{M}_2 : \mathbf{Y} &= \mathbf{1}_n\beta_0 + \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\epsilon} \\
\vdots \quad & \quad \vdots \\
\mathcal{M}_k : \mathbf{Y} &= \mathbf{1}_n\beta_0 + \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \ldots \mathbf{X}_k\boldsymbol{\beta}_k + \boldsymbol{\epsilon}
\end{aligned}
$$

Let $\mathbf{P}_j$ denote the projection on the column space in each of the models $\mathcal{M}_j$: $C(\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_j)$

$$
\begin{aligned}
\|\mathbf{Y}\|^2 = &\|\mathbf{P}_0\mathbf{Y}\|^2 + \|(\mathbf{P}_1 - \mathbf{P}_0)\mathbf{Y}\|^2 + \|(\mathbf{P}_2 - \mathbf{P}_1)\mathbf{Y}\|^2 + \ldots \|(\mathbf{P}_k - \mathbf{P}_{k-1})\mathbf{Y}\|^2 + \\
&\|(\mathbf{I}_n - \mathbf{P}_k)\mathbf{Y}\|^2
\end{aligned}
$$

Likelihood Ratio Tests

- Compare likelihoods under $\mathcal{M}_k$ to $\mathcal{M}_{k-1}$ where $\mathcal{M}_{k-1} \subset \mathcal{M}_k$.

$$\Lambda = \frac{\sup_{\theta_k \in \Theta_k} \mathcal{L}(\theta_k)}{\sup_{\theta_{k-1} \in \Theta_{k-1}} \mathcal{L}(\theta_{k-1})}$$

  where $\theta_j = (\beta_0, \ldots, \beta_j \sigma^2)$
- $\Lambda \geq 1$
- If Λ is "significantly" bigger than one, then should prefer the larger model (reject $\mathcal{M}_{k-1}$)
- Pick a constant such that if $\mathcal{M}_{k-1}$ is true, then $P(\Lambda > c \mid \mathcal{M}_{k-1}) = \alpha$, say 0.05
- Need distribution of Λ (or transformation of it)

Likelihood Ratio Tests

$$\Lambda = \frac{\sup_{\theta_k \in \Theta_k} \mathcal{L}(\theta_k)}{\sup_{\theta_{k-1} \in \Theta_{k-1}} \mathcal{L}(\theta_{k-1})}$$

F tests

The F statistic

$$F = \frac{\|(\mathbf{P}_k - \mathbf{P}_{k-1})\mathbf{Y}\|^2 / (r(\mathbf{P}_k) - r(\mathbf{P}_{k-1}))}{\hat{\sigma}^2} \sim F(r(\mathbf{P}_k) - r(\mathbf{P}_{k-1}), n-p)$$

under the null hypothesis.

- Numerator is a χ^2 over df
- Denominator is a χ^2 over df
- numerator and denominator are independent
- Nested models $C(M_k)$ contains $C(M_{k-1})$

Sequential F tests

Hypothesis*	SS	df	F
$\boldsymbol{\beta}_1 = 0$	$\Vert(\mathbf{P}_1 - \mathbf{P}_0)\mathbf{Y}\Vert^2$	$r(\mathbf{P}_1) - r(\mathbf{P}_0)$	$\frac{\frac{\Vert(\mathbf{P}_1 - \mathbf{P}_0)\mathbf{Y}\Vert^2}{r(\mathbf{P}_1) - r(\mathbf{P}_0)}}{\hat{\sigma}^2}$
$\boldsymbol{\beta}_2 = 0$	$\Vert(\mathbf{P}_2 - \mathbf{P}_1)\mathbf{Y}\Vert^2$	$r(\mathbf{P}_2) - r(\mathbf{P}_1)$	$\frac{\frac{\Vert(\mathbf{P}_2 - \mathbf{P}_1)\mathbf{Y}\Vert^2}{r(\mathbf{P}_2) - r(\mathbf{P}_1)}}{\hat{\sigma}^2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\boldsymbol{\beta}_k = 0$	$\Vert(\mathbf{P}_k - \mathbf{P}_{k-1})\mathbf{Y}\Vert^2$	$r(\mathbf{P}_k) - r(\mathbf{P}_{k-1})$	$\frac{\frac{\Vert(\mathbf{P}_k - \mathbf{P}_{k-1})\mathbf{Y}\Vert^2}{r(\mathbf{P}_k) - r(\mathbf{P}_{k-1})}}{\hat{\sigma}^2}$

- Sequential test $\boldsymbol{\beta}_j = 0$ includes variables from the previous model $\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{j-1}$ but $\boldsymbol{\beta}_i$ for $i > j$ are all set to 0
- All use estimate of $\hat{\sigma}^2 = \Vert(\mathbf{I}_n - \mathbf{P}_k)\mathbf{Y}\Vert^2/(n - r(\mathbf{P}_k))$ under largest model
- Unless $\mathbf{P}_j\mathbf{P}_i = \mathbf{0}$ for $i \neq j$, decomposition will depend on the order of $\mathbf{X}_j$ in the model
- If last $\mathbf{X}_k$ is $n \times 1$, then $t^2 = F$ for testing H_0: $\beta_k = 0$

Order 1: Sequential Sum of Squares

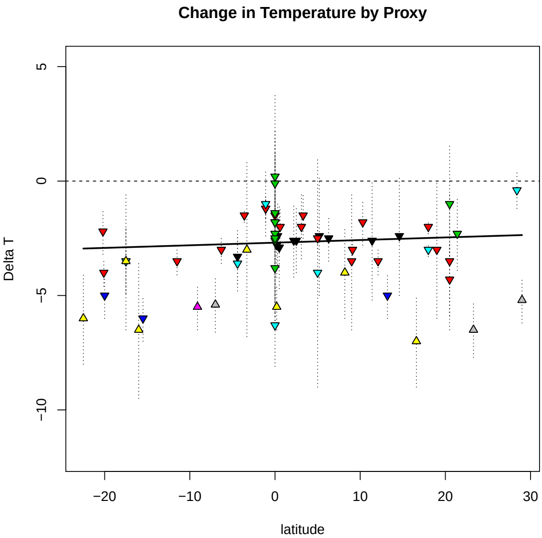
```
climate.lm = lm(deltaT ~ proxy *(poly(latitude,2)),
                weights=(1/sdev^2),
                data=climate)
anova(climate.lm)
Response: deltaT
                        Df  Sum Sq Mean Sq F value    Pr(>F)
proxy                    7 307.598  43.943  9.8541 3.848e-07 ***
poly(latitude, 2)        2  10.457   5.228  1.1725    0.3198
proxy:poly(latitude, 2) 12  74.065   6.172  1.3841    0.2126
Residuals               41 182.833   4.459
```

Order 2: Sequential Sum of Squares

```
>anova(lm(deltaT ~ (poly(latitude,2))* proxy, weights=1/sdev^2,
         data=climate))
 Analysis of Variance Table

Response: deltaT
                        Df  Sum Sq Mean Sq F value    Pr(>F)
poly(latitude, 2)        2  79.869  39.935  8.9553 0.0005931 ***
proxy                    7 238.185  34.026  7.6304  6.93e-06 ***
poly(latitude, 2):proxy 12  74.065   6.172  1.3841 0.2125512
Residuals               41 182.833   4.459
```

Prediction with Latitude



Multiple Model Objects and Anova in R

```
> anova(climate3.lm,climate2.lm,climate1.lm, climate.lm)
Analysis of Variance Table

Model 1: deltaT ~ T.M
Model 2: deltaT ~ poly(latitude, 2) + T.M
Model 3: deltaT ~ poly(latitude, 2) + proxy
Model 4: deltaT ~ proxy * (poly(latitude, 2))
  Res.Df    RSS Df Sum of Sq      F   Pr(>F)
1     61 385.66
2     59 347.11  2    38.542 4.3215 0.019814 *
3     53 256.90  6    90.215 3.3718 0.008552 **
4     41 182.83 12    74.065 1.3841 0.212551
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
```

Other order

```
> anova(climate3.lm,climate2.lm,climate1.lm, climate.lm)
Analysis of Variance Table

Model 1: deltaT ~ T.M
Model 2: deltaT ~ proxy
Model 3: deltaT ~ poly(latitude, 2) + proxy
Model 4: deltaT ~ proxy * (poly(latitude, 2))
  Res.Df    RSS Df Sum of Sq      F   Pr(>F)
1     61 385.66
2     55 267.35  6   118.301 4.4215 0.001555 **
3     53 256.90  2    10.457 1.1725 0.319767
4     41 182.83 12    74.065 1.3841 0.212551
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Terrestrial versus Marine

```
climate.final = lm(deltaT ~ T.M + proxy -1, weights=(1/sdev^2))

                Estimate Std. Error t value Pr(>|t|)
T.MT             -5.6360     0.7132  -7.902 1.26e-10 ***
T.MM             -2.1145     0.4124  -5.127 3.93e-06 ***
proxyAlkenone    -0.1408     0.4381  -0.321    0.749
proxyFaunal      -0.1507     0.8971  -0.168    0.867
proxySr/Ca       -3.2188     0.7584  -4.244 8.49e-05 ***
proxyDel18O      -0.6378     0.5048  -1.263    0.212
proxyIceCore      0.1360     1.3130   0.104    0.918
proxyPollen       0.5283     1.0033   0.527    0.601
proxyNoble Gas        NA         NA      NA       NA

Multiple R-squared: 0.9115,^^IAdjusted R-squared: 0.8986

          Df  Sum Sq Mean Sq  F value  Pr(>F)
T.M        2 2635.27 1317.63 271.0625 < 2e-16 ***
proxy      6  118.30   19.72   4.0561 0.00195 **
Residuals 55  267.35    4.86
```

Even Simpler ?

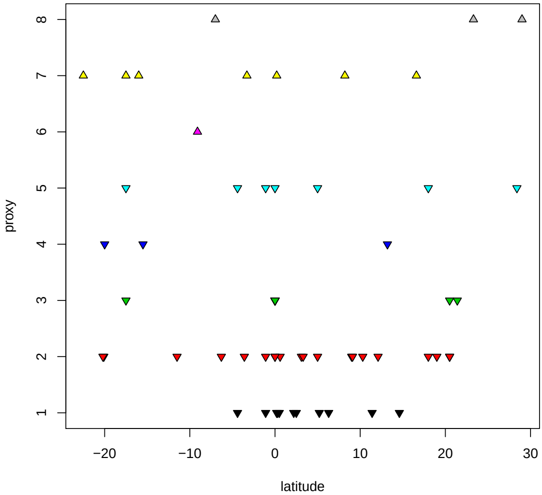
```
lm(formula = deltaT ~ T.M + I(proxy == "Sr/Ca"), weights = (1/sd

                          Estimate Std. Error t value Pr(>|t|)
(Intercept)               -5.3915     0.4486 -12.018  < 2e-16 ***
T.MM                       3.0585     0.4649   6.579 1.30e-08 ***
I(proxy == "Sr/Ca")TRUE  -3.0003     0.6371  -4.709 1.52e-05 ***

Residual standard error: 2.166 on 60 degrees of freedom
Multiple R-squared: 0.5103,^^IAdjusted R-squared: 0.4939

Model 1: deltaT ~ T.M + I(proxy == "Sr/Ca")
Model 2: deltaT ~ T.M + proxy - 1
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     60 281.58
2     55 267.36  5    14.228 0.5854  0.711
```

Design



Summary

- Ignoring proxies, there are systematic trends with latitude.
- Difference among proxies, even after adjusting for latitude
- Weak evidence of a latitude effect, after taking into account proxies (potential confounding)
- Terrestrial sites differ from Marine sites, however there are significant difference among proxies within the Marine group driven by the `Sr/Ca` proxy which indicates a significantly greater increases in temperatures
- Significant warming for Terrestrial ($5.4^\circ C$) with Marine sites significantly cooler ($3^\circ C$)
- `Sr/Ca` proxies are significantly cooler than other marine proxies by about $3^\circ C$

Uncertainty Measures? Normal Assumptions?

Added Variable Plots

1. Let $\mathbf{P}_{(-j)}$ denote the projection on the space spanned by $C(\mathbf{X}_0, \ldots, \mathbf{X}_{j-1}, \mathbf{X}_{j+1}, \ldots \mathbf{X}_k)$ (omit variable j)
2. Find residuals $\mathbf{e}_{\mathbf{Y}|\mathbf{X}_{(-j)}} = (\mathbf{I} - \mathbf{P}_{(-j)})\mathbf{Y}$ from regressing $\mathbf{Y}$ on all variables except $\mathbf{X}_j$
3. Remove the effect of other explanatory variables from $\mathbf{X}_j$ by taking residuals $\mathbf{e}_{\mathbf{X}_j|\mathbf{X}_{(-j)}} = (\mathbf{I} - \mathbf{P}_{(-j)})\mathbf{X}_j$
4. Plot $\mathbf{e}_{\mathbf{Y}|\mathbf{X}_{(-j)}}$ versus $\mathbf{e}_{\mathbf{X}_j|\mathbf{X}_{(-j)}}$
5. Slope is adjusted regression coefficient in full model $\boldsymbol{\mu} \in C(\mathbf{X}_0, \ldots, \mathbf{X}_{j-1}, \mathbf{X}_j, \mathbf{X}_{j+1}, \ldots \mathbf{X}_k)$
6. `library(car)`
7. `avPlots(climate1.lm, terms=~ .)`