Choice of Prior Distributions

STA721 Linear Models Duke University

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Bayesian Estimation

Model

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precision $\phi = 1/\sigma^2$

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\Phi_n &= \mathbf{X}^T\mathbf{X} + \Phi_0 \\
\mathbf{b}_n &= \Phi_n^{-1}(\mathbf{X}^T\mathbf{X}\hat{\beta} + \Phi_0\mathbf{b}_0) \\
\nu_n &= \nu_0 + n \\
\mathrm{SS}_n &= \mathrm{SSE} + \mathrm{SS}_0 + \hat{\beta}^T\mathbf{X}^T\mathbf{X}\hat{\beta} + \mathbf{b}_0^T\Phi_0\mathbf{b}_0 - \mathbf{b}_n^T\Phi_n\mathbf{b}_n \\
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\beta \mid \phi, \mathbf{Y} &\sim \mathrm{N}(\mathbf{b}_n, (\phi\Phi_n)^{-1}) \\
\phi \mid \mathbf{Y} &\sim \mathbf{G}\left(\frac{\nu_n}{2}, \frac{\nu_n\hat{\sigma}_n^2}{2}\right)
\end{aligned}$$

Marginal Distribution from Normal–Gamma

Theorem

Let $\boldsymbol{\theta} \mid \phi \sim N(m, \frac{1}{\phi}\Sigma)$ *and* $\phi \sim \mathbf{G}(\nu/2, \nu\hat{\sigma}^2/2)$. *Then* $\mathbf{t}$ $(p \times 1)$ *has a p dimensional multivariate t distribution*

$$\boldsymbol{\theta} \sim t_\nu(m, \hat{\sigma}^2\Sigma)$$

with density

$$p(\boldsymbol{\theta}) \propto \left[1 + \frac{1}{\nu}\frac{(\boldsymbol{\theta} - m)^T\Sigma^{-1}(\boldsymbol{\theta} - m)}{\hat{\sigma}^2}\right]^{-\frac{p+\nu}{2}}$$

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Marginal Posterior Distribution of $\boldsymbol{\beta}$

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Any linear combination $\lambda^T\boldsymbol{\beta}$

$$\lambda^T\boldsymbol{\beta} \mid \mathbf{Y} \sim t_{\nu_n}(\lambda^T\mathbf{b}_n, \hat{\sigma}^2\lambda^T\Phi_n^{-1}\lambda)$$

has a univariate t distribution with ν_n degrees of freedom

Predictive Distribution

Suppose $\mathbf{Y}^* \mid \boldsymbol{\beta}, \phi \sim \mathrm{N}(\mathbf{X}^* \boldsymbol{\beta}, \mathbf{I}/\phi)$ and is conditionally independent of $\mathbf{Y}$ given $\boldsymbol{\beta}$ and ϕ

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Choice of conjugate prior?

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Cannot represent real prior beliefs; double use of data but has the "right" behaviour.

Zellner's g-prior

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- $\frac{g}{1+g}$ weight given to the data

Zellner's g-prior

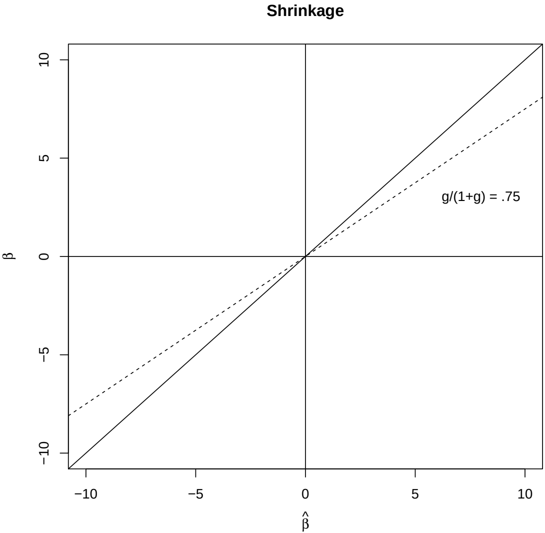
Zellner's g-prior(s) $\boldsymbol{\beta} \mid \phi \sim \mathsf{N}(\mathbf{b}_0, g(\mathbf{X}^T\mathbf{X})^{-1}/\phi)$

$$\boldsymbol{\beta} \mid \mathbf{Y}, \phi \sim \mathsf{N}\left(\frac{g}{1+g}\hat{\boldsymbol{\beta}} + \frac{1}{1+g}\mathbf{b}_0, \frac{g}{1+g}(\mathbf{X}^T\mathbf{X})^{-1}\phi^{-1}\right)$$

- Invariance: Require posterior of $\mathbf{X}\boldsymbol{\beta}$ equal the posterior of $\mathbf{X}\mathbf{H}\boldsymbol{\alpha}$ ($\mathbf{a}_0 = \mathbf{H}^{-1}\mathbf{b}_0$) (take $\mathbf{b}_0 = \mathbf{0}$)
- Choice of g?
- $\frac{g}{1+g}$ weight given to the data
- Fixed g effect does not vanish as $n \to \infty$
- Use $g = n$ or place a prior distribution on g

Shrinkage

Posterior mean under g-prior with $\mathbf{b}_0 = 0$ $\frac{g}{1+g}\hat{\beta}$



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Fisher Information Matrix

Log Likelihood

$$\log(\mathcal{L}(\boldsymbol{\beta}, \phi)) = \frac{n}{2}\log(\phi) - \frac{\phi}{2}\|(\mathbf{I} - \mathbf{P_x})\mathbf{Y}\|^2 - \frac{\phi}{2}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})$$

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Improper prior $\int\int p_J(\boldsymbol{\beta}, \phi)\, d\boldsymbol{\beta}\, d\phi$ not finite

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Bayesian Credible Sets $p(\boldsymbol{\beta} \in C_\alpha) = 1 - \alpha$ correspond to frequentist Confidence Regions

$$\frac{\boldsymbol{\lambda}^T\boldsymbol{\beta} - \boldsymbol{\lambda}\hat{\boldsymbol{\beta}}}{\sqrt{\hat{\sigma}^2\boldsymbol{\lambda}^T(\mathbf{X}^T\mathbf{X})^{-1}\boldsymbol{\lambda}}} \sim t_{n-p}$$

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