Multivariate Normal Theory

STA721 Linear Models Duke University

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Outline

- Multivariate Normal Distribution
- Linear Transformations
- Distribution of estimates under normality

Properties of MLE's Recap

- $\hat{\mathbf{Y}} = \hat{\boldsymbol{\mu}} = \mathbf{P_X Y}$ is an unbiased estimate of $\boldsymbol{\mu} = \mathbf{X}\beta$

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Is not an unbiased estimate of σ^2, but

$$\hat{\sigma}^2 \equiv \frac{\mathbf{e}^T\mathbf{e}}{n-p} = \frac{\mathbf{Y}^T(\mathbf{I} - \mathbf{P_X})\mathbf{Y}}{n-p}$$

where p equals the rank of $\mathbf{X}$ is an unbiased estimate.

Sampling Distributions

- Distribution of $\hat{\beta}$
- Distribution of $\mathbf{P_X Y}$
- Distribution of $\mathbf{e}$

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Density

If $\boldsymbol{\Sigma}$ is positive definite ($\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} > 0$ for any $\mathbf{x} \neq 0$ in $\mathbb{R}^d$) then $\mathbf{Y}$ has a density [1]

[1] with respect to Lebesgue measure on $\mathbb{R}^d$

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If $\mathbf{A}$ $(n \times n)$ *is a symmetric real matrix then there exists a* $\mathbf{U}$ $(n \times n)$ *such that* $\mathbf{U}^T\mathbf{U} = \mathbf{U}\mathbf{U}^T = \mathbf{I}_n$ *and a diagonal matrix* $\mathbf{\Lambda}$ *with elements* λ_i *such that* $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$

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- substitute $g(\mathbf{Y})$ for $\mathbf{Z}$ in density and multiply by Jacobian

$$
f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{Z}}(\mathbf{z}) J(\mathbf{Z} \to \mathbf{Y})
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$$\mathbf{Y} = \boldsymbol{\mu} + \mathbf{A}\mathbf{Z} \quad \text{for } \mathbf{Z} \sim \mathrm{N}(\mathbf{0}, \mathbf{I}_d) \tag{1}$$

Proof.

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- Jacobian of transformation $d\mathbf{Z} = |\mathbf{A}^{-1}|d\mathbf{Y}$
- Substitute and simplify algebra

$$f(\mathbf{Y}) = (2\pi)^{-d/2}|\boldsymbol{\Sigma}|^{-1/2}\exp(-\frac{1}{2}(\mathbf{Y} - \boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \boldsymbol{\mu}))$$

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see linked videos using characteristic functions:

$$Y \sim \textsf{N}(\mu, \sigma^2) \Leftrightarrow \varphi_Y(t) \equiv \textsf{E}[e^{itY}] = e^{it\mu - t^2\sigma^2/2}$$

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$$\mathbf{AY} \sim \mathsf{N}_m(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T)$$

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(Proof in book or linked video)

Distribution of $\hat{\mathbf{Y}}$ and $\mathbf{e}$ (marginally)

Equal in Distribution

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- $\mathbf{Z}_2 \sim \mathrm{N}(\mathbf{0}, \mathbf{I}_p)$, $\mathbf{Z}_2 \in \mathbb{R}^p$ and take $\mathbf{B}$ $d \times p$

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- Define $\mathbf{Y} = \boldsymbol{\mu} + \mathbf{A}\mathbf{Z}_1$

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Theorem

If $\mathbf{Y} = \boldsymbol{\mu} + \mathbf{A}\mathbf{Z}_1$ *and* $\mathbf{W} = \boldsymbol{\mu} + \mathbf{B}\mathbf{Z}_2$ *then* $\mathbf{Y} \stackrel{\mathrm{D}}{=} \mathbf{W}$ *if and only if* $\mathbf{A}\mathbf{A}^T = \mathbf{B}\mathbf{B}^T = \boldsymbol{\Sigma}$

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see linked video

Zero Correlation and Independence

Theorem

For a random vector $\mathbf{Y} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *partitioned as*

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} \right)$$

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then $Cov(\mathbf{Y}_1, \mathbf{Y}_2) = \boldsymbol{\Sigma}_{12} = \boldsymbol{\Sigma}_{21}^T = \mathbf{0}$ *if and only if* $\mathbf{Y}_1$ *and* $\mathbf{Y}_2$ *are independent.*

Independence Implies Zero Covariance

Proof.

$$\mathrm{Cov}(\mathbf{Y}_1, \mathbf{Y}_2) = \mathrm{E}[(\mathbf{Y}_1 - \boldsymbol{\mu}_1)(\mathbf{Y}_2 - \boldsymbol{\mu}_2)^T]$$

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therefore $\boldsymbol{\Sigma}_{12} = \mathbf{0}$

$\square$

Zero Covariance Implies Independence

Assume $\mathbf{\Sigma}_{12} = \mathbf{0}$

Proof

- Choose an

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix}$$

such that $\mathbf{A}_1 \mathbf{A}_1^T = \mathbf{\Sigma}_{11}$, $\mathbf{A}_2 \mathbf{A}_2^T = \mathbf{\Sigma}_{22}$

Zero Covariance Implies Independence

Assume $\mathbf{\Sigma}_{12} = \mathbf{0}$

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such that $\mathbf{A}_1\mathbf{A}_1^T = \mathbf{\Sigma}_{11}$, $\mathbf{A}_2\mathbf{A}_2^T = \mathbf{\Sigma}_{22}$
- Partition
$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix} \mathbf{0}_1 \\ \mathbf{0}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{I}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{bmatrix} \right) \text{ and } \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}$$

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- then $\mathbf{Y} \stackrel{\mathrm{D}}{=} \mathbf{AZ} + \boldsymbol{\mu} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

Continued

Proof.

- $$\begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} \stackrel{\mathrm{D}}{=} \begin{bmatrix} \mathbf{A}_1\mathbf{Z}_1 + \boldsymbol{\mu}_1 \\ \mathbf{A}_2\mathbf{Z}_2 + \boldsymbol{\mu}_2 \end{bmatrix}$$

Continued

Proof.

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- But $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are independent

Continued

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$\square$

For Multivariate Normal Zero Covariance implies independence

Corollary

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If $\mathbf{Y} \sim N(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_n)$ *and* $\mathbf{A}\mathbf{B}^T = \mathbf{0}$ *then* $\mathbf{AY}$ *and* $\mathbf{BY}$ *are independent.*

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- $$\begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} \mathbf{Y} = \begin{bmatrix} \mathbf{AY} \\ \mathbf{BY} \end{bmatrix}$$

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- $\mathrm{Cov}(\mathbf{W}_1, \mathbf{W}_2) = \mathrm{Cov}(\mathbf{AY}, \mathbf{BY}) = \sigma^2 \mathbf{A}\mathbf{B}^T$

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- $\mathrm{Cov}(\mathbf{W}_1, \mathbf{W}_2) = \mathrm{Cov}(\mathbf{AY}, \mathbf{BY}) = \sigma^2 \mathbf{A}\mathbf{B}^T$
- $\mathbf{AY}$ and $\mathbf{BY}$ are independent if $\mathbf{A}\mathbf{B}^T = \mathbf{0}$

$\square$

Joint Distribution of $\hat{\mathbf{Y}}$ and $\mathbf{e}$

More Distribution Theory

Distributions unconditional on σ^2

- χ^2 distributions ($\hat{\sigma}^2$)
- t distribution ($\hat{\mathbf{Y}}$, $\mathbf{e}$, $\hat{\boldsymbol{\beta}}$)