Cauchy Priors: Mixtures of Normals & MCMC

STA721 Linear Models Duke University

Merlise Clyde

October 11, 2019

Bayesian Estimation with 2 Block g-prior (Normal-Jeffreys)

Model in centered parameterization

$$
\begin{aligned}
\mathbf{Y} &= \mathbf{1}\beta_0 + (\mathbf{I}_n - \mathbf{P}_1)\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \\
p(\beta_0, \phi) &\propto 1/\phi \\
\boldsymbol{\beta} \mid \beta_0, \phi &\sim \mathrm{N}(\mathbf{0}, \frac{g}{\phi}(\mathbf{X}^T(\mathbf{I}_n - \mathbf{P_1})\mathbf{X})^{-1})
\end{aligned}
$$

Log Likelihood

$$
\mathcal{L}(\beta_0, \boldsymbol{\beta}, \phi) \propto \frac{n}{2}\log(\phi) - \frac{\phi}{2}\left(n(\beta_0 - \bar{y})^2 + (\mathbf{Y}_c - \mathbf{X}_c\boldsymbol{\beta})^T(\mathbf{Y}_c - \mathbf{X}_c)\right)
$$

Since

$$
\mathbf{Y} = (\mathbf{I} - \mathbf{P_1})\mathbf{Y} + \mathbf{P_1}\mathbf{Y} \text{ and} \mathbf{X}_c \equiv (\mathbf{I} - \mathbf{P_1})\mathbf{X}
$$

Integrated Liklihood after integrating β_0

$$
\mathcal{L}(\boldsymbol{\beta}, \phi) \propto \frac{n-1}{2}\log(\phi) - \frac{\phi}{2}(\mathbf{Y}_c - \mathbf{X}_c\boldsymbol{\beta})^T(\mathbf{Y}_c - \mathbf{X}_c)
$$

Prior Data

Note

$$(\mathbf{X}^T(\mathbf{I}_n - \mathbf{P_1})\mathbf{X}) = (\mathbf{X}^T(\mathbf{I}_n - \mathbf{P_1})^T(\mathbf{I}_n - \mathbf{P_1})\mathbf{X}) = (\mathbf{X} - \mathbf{1}_n\bar{\mathbf{X}}^T)^T(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{X}})$$

Let $(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{X}}^T)^T(\mathbf{X} - \mathbf{1}\bar{\mathbf{X}}) = \mathsf{SS}_{\mathbf{X}} = \mathbf{U}^T\mathbf{U}$
Quadratic contribution to the log likelihood from prior after integrating out β_0

$$(\mathbf{Y}_c - \mathbf{X}_c\boldsymbol{\beta})^T(\mathbf{Y}_c - \mathbf{X}_c\boldsymbol{\beta}) + (\boldsymbol{\beta}^T\frac{\mathbf{U}^T\mathbf{U}}{g}\boldsymbol{\beta})$$

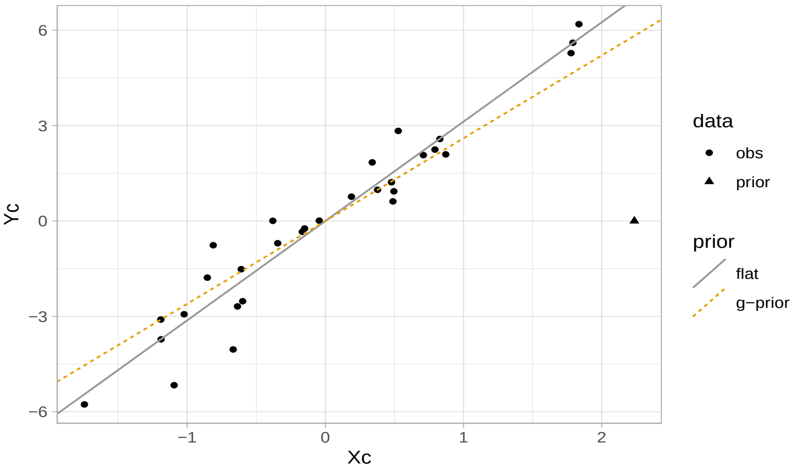
$$(\mathbf{Y}_c - \mathbf{X}_c\boldsymbol{\beta})^T(\mathbf{Y}_c - \mathbf{X}_c\boldsymbol{\beta}) + (\mathbf{0}_p - \frac{\mathbf{U}}{\sqrt{g}}\boldsymbol{\beta})^T(\mathbf{0}_p - \frac{\mathbf{U}}{\sqrt{g}}\boldsymbol{\beta})$$

Prior observations with $Yc = 0$.

Example: g=5, n=30

In SLR it is like an extra $Y_0 = 0$ at $\mathbf{X}_o = \sqrt{\frac{SS_x}{g}}$:

$$(\mathbf{Y}_c - \mathbf{X}_c\beta)^T(\mathbf{Y}_c - \mathbf{X}_c\beta) + (0 - \sqrt{\frac{SS_x}{g}}\beta)^T(0 - \sqrt{\frac{SS_X}{g}}\beta)$$



Disadvantages of Conjugate Priors

Disadvantages:

- Results may have be sensitive to prior "outliers" due to linear updating
- Problem potentially with all Normal priors, not just the g-prior.
- Cannot capture all possible prior beliefs
- Mixtures of Conjugate Priors

Mixtures of Conjugate Priors

Theorem (Diaconis & Ylivisaker 1985)

Given a sampling model $p(y \mid \boldsymbol{\theta})$ from an exponential family, any prior distribution can be expressed as a mixture of conjugate prior distributions

- Prior $p(\boldsymbol{\theta}) = \int p(\boldsymbol{\theta} \mid \omega) p(\omega)\, d\omega$
- Posterior

$$
\begin{aligned}
p(\boldsymbol{\theta} \mid \mathbf{Y}) &\propto \int p(\mathbf{Y} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta} \mid \omega) p(\omega)\, d\omega \\
&\propto \int \frac{p(\mathbf{Y} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta} \mid \omega)}{p(\mathbf{Y} \mid \omega)} p(\mathbf{Y} \mid \omega) p(\omega)\, d\omega \\
&\propto \int p(\boldsymbol{\theta} \mid \mathbf{Y}, \omega) p(\mathbf{Y} \mid \omega) p(\omega)\, d\omega \\
p(\boldsymbol{\theta} \mid \mathbf{Y}) &= \frac{\int p(\boldsymbol{\theta} \mid \mathbf{Y}, \omega) p(\mathbf{Y} \mid \omega) p(\omega)\, d\omega}{\int p(\mathbf{Y} \mid \omega) p(\omega)\, d\omega}
\end{aligned}
$$

Zellner-Siow prior (assume $\mathbf{X}$ is centered)

Zellner's g-prior $\boldsymbol{\beta} \mid \phi \sim \mathsf{N}(\mathbf{0}_p, g(\mathbf{X}_c^T\mathbf{X}_c)^{-1}/\phi)$

- Choice of g?
- $\frac{g}{1+g}$ weight given to the data
- Let $\tau = 1/g$ assign $\tau \sim G(1/2, n/2)$
- Marginal prior on $\boldsymbol{\beta} \sim C(0, \phi^{-1}(\mathbf{X}_c^T\mathbf{X}_c/n)^{-1})$
- Can express posterior as a mixture of g-priors

$$p(\tau \mid \mathbf{Y}) = \frac{p(\mathbf{Y} \mid \tau)p(\tau)}{\int p(\mathbf{Y} \mid \tau)p(\tau)\, d\tau}$$

- Problem: no analytic expression for integral
- Need 2 one dimensional integrals to obtain posterior.
- What about credible intervals?

Markov Chain Monte Carlo

- We know that $\beta_0, \boldsymbol{\beta}, \phi \mid \mathbf{Y}, g = 1/\tau$ has a Normal-Gamma distribution
- We can show that $\tau \mid \beta_0, \boldsymbol{\beta}, \phi, \mathbf{Y}$ has a Gamma distribution

$$p(\tau \mid \boldsymbol{\beta}, \phi, \mathbf{Y}) \propto \mathcal{L}(\beta_0, \boldsymbol{\beta}, \phi)\tau^{p/2} e^{(-\tau \frac{\phi}{2} \boldsymbol{\beta}^T (\mathbf{X}^T \mathbf{X}) \boldsymbol{\beta})} \tau^{1/2-1} e^{-\tau n/2}$$

- alternate sampling from full conditional distributions given current values of other parameters. (STA 601)
- JAGS or STAN

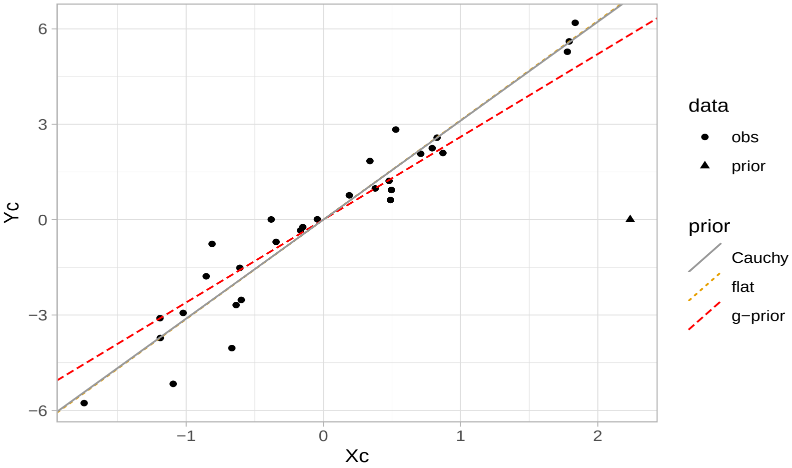
JAGS Code: library(R2jags)

```
model = function(){
  for (i in 1:n) {
      Y[i] ~ dnorm(beta0+ (X[i] -Xbar)*beta, phi)
  }
  beta0 ~ dnorm(0, .000001*phi) #precision is 2nd arg
  beta ~ dnorm(0, phi*tau*SSX)  #precision is 2nd arg
  phi ~ dgamma(.001, .001)
  tau ~ dgamma(.5, .5*n)
  g <- 1/tau
  sigma <- pow(phi, -.5)
}
data = list(Y=Y, X=X, n =length(Y), SSX=sum(Xc^2),
            Xbar=mean(X))
ZSout = jags(data, inits=NULL,
             parameters.to.save=c("beta0","beta", "g",
                                  "sigma"),
             model=model, n.iter=10000)
```

HPD intervals

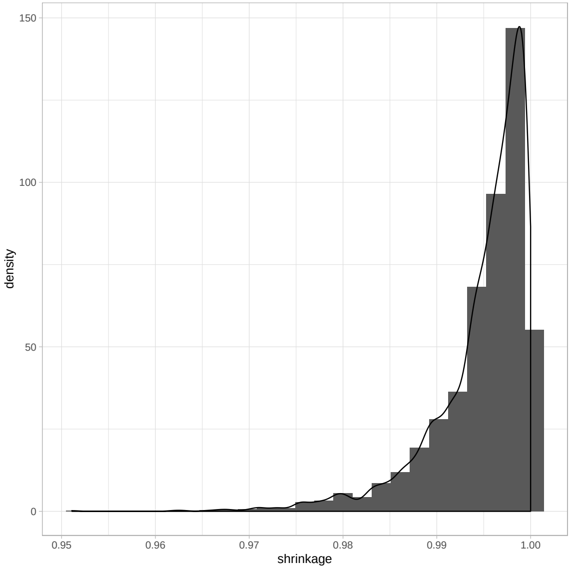
```
confint(lm(Y ~ Xc))

##                  2.5 %    97.5 %
## (Intercept) -0.3985359 0.2048303
## Xc           2.7945824 3.4555162

HPDinterval(as.mcmc(ZSout$BUGSoutput$sims.matrix))

##                lower        upper
## beta       2.7823047    3.4453690
## beta0     -0.3764027    0.2095465
## deviance  70.2043917   78.4813041
## g         19.4503373 3782.7134974
## sigma      0.6171029    1.0504892
## attr(,"Probability")
## [1] 0.95
```

Compare



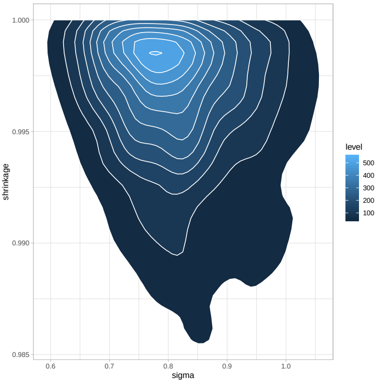
```
ZSout

## Inference for Bugs model at "/var/folders/n4/nj1122xj6bn5_xgbptv7bml40000gp/T//RtmpABkjXF/model185e51fc
##  3 chains, each with 10000 iterations (first 5000 discarded), n.thin = 5
##  n.sims = 3000 iterations saved
##           mu.vect   sd.vect   2.5%     25%     50%     75%    97.5%  Rhat
## beta        3.112     0.170  2.782   2.997   3.115   3.225    3.445 1.001
## beta0      -0.099     0.152 -0.384  -0.204  -0.099   0.001    0.204 1.002
## g        2263.147 38967.029 48.273 146.129 282.298 697.063 9018.709 1.001
## sigma       0.827     0.114  0.636   0.747   0.816   0.896    1.079 1.001
## deviance   73.347     2.563 70.390  71.458  72.680  74.500   79.882 1.002
##          n.eff
## beta      3000
## beta0     1200
## g         3000
## sigma     3000
## deviance  1600
##
## For each parameter, n.eff is a crude measure of effective sample size,
## and Rhat is the potential scale reduction factor (at convergence, Rhat=1).
##
## DIC info (using the rule, pD = var(deviance)/2)
## pD = 3.3 and DIC = 76.6
## DIC is an estimate of expected predictive error (lower deviance is better).
```

Posterior Distribution of shrinkage



Joint Distribution of σ and $g/(1+g)$

```
ggplot(postdf, aes(x=sigma, y=shrinkage) ) +
 stat_density_2d(aes(fill = ..level..),
                 geom = "polygon", colour="white") +
  theme_light()
```



Cauchy Summary

- Cauchy rejects prior mean if it is an "outlier"
- robustness related to "bounded" influence (more later)
- requires numerical integration or Monte Carlo sampling (MCMC)

How Good are these Estimators?

Quadratic loss for estimating $\boldsymbol{\beta}$ using estimator $\mathbf{a}$

$$L(\boldsymbol{\beta}, \mathbf{a}) = (\boldsymbol{\beta} - \mathbf{a})^T(\boldsymbol{\beta} - \mathbf{a})$$

- Consider our expected loss (before we see the data) of taking an "action" $\mathbf{a}$
- Under OLS or the Reference prior the Expected Mean Square Error

$$\begin{aligned} \mathsf{E}_{\mathbf{Y}}[(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^T(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}) &= \sigma^2 \mathrm{tr}[(\mathbf{X}^T\mathbf{X})^{-1}] \\ &= \sigma^2 \sum_{j=1}^{p} \lambda_j^{-1} \end{aligned}$$

  where λ_j are eigenvalues of $\mathbf{X}^T\mathbf{X}$.
- If smallest $\lambda_j \to 0$ then MSE $\to \infty$
- Note: estimate is unbiased!

Is the g-prior better?

Explore Frequentist properties of using a Bayesian estimator

$$\mathsf{E}_{\mathbf{Y}}[(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_g)^T(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_g)$$

but now $\hat{\boldsymbol{\beta}}_g = g/(1+g)\hat{\boldsymbol{\beta}}$

when is the g prior better than the Reference prior or OLS? Is it always better?

Estimator Properties

- Bias
- Variability
- MSE $=$ Bias^2 $+$ Variance (multivariate analogs)
- Problems with OLS, g-priors & mixtures of g-priors with collinearity
- Solutions:
  - removal of terms
  - other shrinkage estimators