Frequentist Properties of Bayes Estimators

STA721 Linear Models Duke University

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October 14, 2019

Recap: Bayesian Estimation with 2 Block g-prior (Normal-Jeffreys)

- Model in centered parameterization $\mathbf{X}_c = (\mathbf{I}_n - \mathbf{P}_1)\mathbf{X}$

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\mathbf{Y} &= \mathbf{1}\beta_0 + (\mathbf{I}_n - \mathbf{P}_1)\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \\
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- Use Gibbs sampling or MCMC

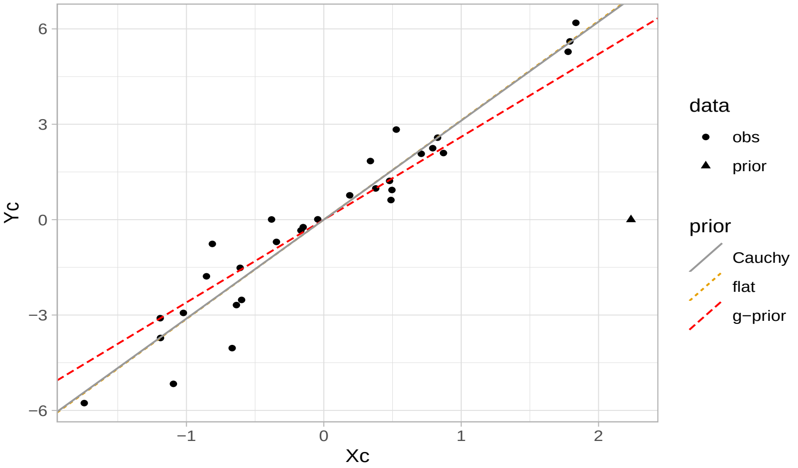
JAGS Code: library(R2jags)

```
model = function(){
  for (i in 1:n) {
      Y[i] ~ dnorm(beta0+ (X[i] -Xbar)*beta, phi)
  }
  beta0 ~ dnorm(0, .000001*phi) #precision is 2nd arg
  beta ~ dnorm(0, phi*tau*SSX)  #precision is 2nd arg
  phi ~ dgamma(.001, .001)
  tau ~ dgamma(.5, .5*n)
  g <- 1/tau
  sigma <- pow(phi, -.5)
}
data = list(Y=Y, X=X, n =length(Y), SSX=sum(Xc^2),
            Xbar=mean(X))
ZSout = jags(data, inits=NULL,
             parameters.to.save=c("beta0","beta", "g",
                                  "sigma"),
             model=model, n.iter=10000)
```

HPD intervals

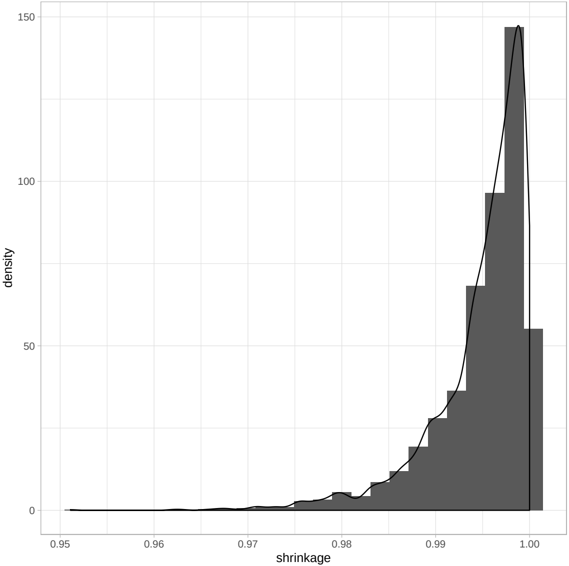
```r
confint(lm(Y ~ Xc))

##                  2.5 %    97.5 %
## (Intercept) -0.3985359 0.2048303
## Xc           2.7945824 3.4555162

HPDinterval(as.mcmc(ZSout$BUGSoutput$sims.matrix))

##                lower        upper
## beta       2.7823047    3.4453690
## beta0     -0.3764027    0.2095465
## deviance  70.2043917   78.4813041
## g         19.4503373 3782.7134974
## sigma      0.6171029    1.0504892
## attr(,"Probability")
## [1] 0.95
```

Compare



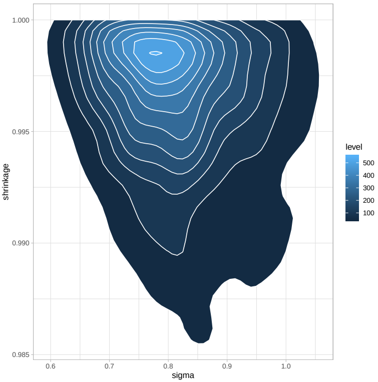
```
ZSout

## Inference for Bugs model at "/var/folders/n4/nj1122xj6bn5_xgbptv7bml40000gp/T//RtmpwDmY3Y/model17fcc743
##  3 chains, each with 10000 iterations (first 5000 discarded), n.thin = 5
##  n.sims = 3000 iterations saved
##           mu.vect   sd.vect   2.5%     25%     50%     75%    97.5%  Rhat
## beta        3.112     0.170  2.782   2.997   3.115   3.225    3.445 1.001
## beta0      -0.099     0.152 -0.384  -0.204  -0.099   0.001    0.204 1.002
## g        2263.147 38967.029 48.273 146.129 282.298 697.063 9018.709 1.001
## sigma       0.827     0.114  0.636   0.747   0.816   0.896    1.079 1.001
## deviance   73.347     2.563 70.390  71.458  72.680  74.500   79.882 1.002
##          n.eff
## beta      3000
## beta0     1200
## g         3000
## sigma     3000
## deviance  1600
##
## For each parameter, n.eff is a crude measure of effective sample size,
## and Rhat is the potential scale reduction factor (at convergence, Rhat=1).
##
## DIC info (using the rule, pD = var(deviance)/2)
## pD = 3.3 and DIC = 76.6
## DIC is an estimate of expected predictive error (lower deviance is better).
```

Posterior Distribution of shrinkage



Joint Distribution of σ and $g/(1+g)$

```
ggplot(postdf, aes(x=sigma, y=shrinkage) ) +
 stat_density_2d(aes(fill = ..level..),
                geom = "polygon", colour="white") +
  theme_light()
```



Cauchy Summary

- Cauchy rejects prior mean if it is an "outlier"
- robustness related to "bounded" influence (more later)
- requires numerical integration or Monte Carlo sampling (MCMC)

How Good are Bayes Estimators?

Quadratic loss for estimating $\boldsymbol{\beta}$ using estimator $\mathbf{a}$

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$$\mathsf{E}_{\mathbf{Y}}[(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^T(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}) \quad = \quad \sigma^2 \mathrm{tr}[(\mathbf{X}^T\mathbf{X})^{-1}]$$

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  where λ_j are eigenvalues of $\mathbf{X}^T\mathbf{X}$.
- If smallest $\lambda_j \to 0$ then MSE $\to \infty$
- Note: estimate is unbiased!

Is the g-prior better?

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- Explore Frequentist properties of using a Bayesian estimator

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- HW: show that there is a value of g prior such that the g-prior is always better than the Reference prior/OLS
- Potential problem: MSE also blows up if smallest eigenvalue goes to zero!

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  - other shrinkage estimators

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  where $\mathbf{U}_0 = \mathbf{1}/\sqrt{n}$
- $\mathbf{U}_0^T \mathbf{U}_p = 0$, $\mathbf{U}_0^T \mathbf{U}_{n-p-1} = 0$ and $\mathbf{U}_p^T \mathbf{U}_{n-p-1} = 0$ (orthogonal columns)

Orthogonal Regression

Rotate by multiplyting by $\mathbf{U}^T$:

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- $\text{Var}(\hat{\gamma}_i) = \sigma^2/l_i^2$

Directions in $\mathbf{X}$ space $\mathbf{U}_j$ with small eigenvectors l_i have the largest variances. Unstable directions.

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(Another) Normal Conjugate Prior Distribution on γ:

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$$\tilde{\gamma} = (L^T L + k\mathbf{I})^{-1} L^T \mathbf{U}_p^T \mathbf{Y} = (L^T L + k\mathbf{I})^{-1} L^T L \hat{\gamma}$$

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- When $\lambda_i \to 0$ then $\tilde{\gamma}_i \to 0$
- When $k \to 0$ we get OLS back but if k gets too big posterior mean goes to zero.

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- importance of standardizing
- Is there a value of k for which ridge is better in terms of Expected MSE than OLS?

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- Transform back $\tilde{\boldsymbol{\beta}} = \mathbf{V}\tilde{\boldsymbol{\gamma}}$

$$\tilde{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X} + k\mathbf{I})^{-1}\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}$$

- importance of standardizing
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- Choice of k?

MSE

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Since $k = 0$ is OLS, this means that is a value of k that will always be better than OLS

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