Applied Time Series Analysis
Assignment 3

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Problem 4

Part (a) - Detailed Derivation of Periodogram Computation via DFT

Solution

The autocovariance function calculates the similarity of two points in a signal as a function of their latency. It provides information on the predictability of the signal for a stationary process. In frequency analysis, the Fourier transform of the autocovariance function yields the power spectral density, which represents how the signal's power is spread over frequency. The periodogram is an estimation of this spectral density, and we can infer the frequency content of the signal by analysing it.

Autocovariance Function

For a discrete-time signal $x[n]$, the autocovariance function is defined as:

$$r_{xx}[l] = E\left[(x[n]-\mu_x)\,(x[n-l]-\mu_x)^*\right] \tag{1}$$

Assuming the process has zero mean, the definition simplifies to:

$$r_{xx}[l] = E\left[x[n]x[n-l]^*\right] \tag{2}$$

DFT of the Autocovariance Function

The DFT of the autocovariance function $r_{xx}[l]$ is given by:

$$\sigma_{xx}[k] = \sum_{l=-(N-1)}^{N-1} r_{xx}[l]e^{-j\frac{2\pi}{N}kl} \tag{3}$$

This function represents the frequency domain equivalent of the temporal relationships captured by the autocovariance function.

Step-by-Step Derivation

Discrete Fourier Transform (DFT)

The DFT of a sequence $x[n]$ is expressed as:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \tag{4}$$

Spectral Density Estimation

The periodogram, an estimator of the power spectral density, is defined in terms of the DFT as:

$$I[k] = \left|\frac{1}{N}X[k]\right|^2 \tag{5}$$

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Connecting Autocovariance and DFT

The power spectral density (PSD) is the Fourier transform of the autocovariance function:

$$\sigma_{xx}[k] = \sum_{l=-(N-1)}^{N-1} r_{xx}[l] e^{-j\frac{2\pi}{N}kl} \tag{6}$$

Expanding Autocovariance in Terms of the Signal

Using the definition of the autocovariance function, we expand $r_{xx}[l]$ in terms of $x[n]$:

$$r_{xx}[l] = \frac{1}{N} \sum_{n=0}^{N-l-1} x[n]x[n+l]^* \tag{7}$$

Expressing $\sigma_{xx}[k]$ in Terms of $X[k]$

Inserting the expanded autocovariance into the expression for $\sigma_{xx}[k]$, we have:

$$\sigma_{xx}[k] = \sum_{l=-(N-1)}^{N-1} \left(\frac{1}{N} \sum_{n=0}^{N-l-1} x[n]x[n+l]^* \right) e^{-j\frac{2\pi}{N}kl} \tag{8}$$

Note that the inner sum is the autocorrelation of the sequence $x[n]$ for a given lag l, and it is multiplied by the complex exponential term from the DFT.

Relating the Autocovariance DFT to the Periodogram

To relate this to the periodogram, we note that the squared magnitude of the DFT is:

$$|X[k]|^2 = X[k]X[k]^* = \left(\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \right) \left(\sum_{m=0}^{N-1} x[m]^* e^{j\frac{2\pi}{N}km} \right) \tag{9}$$

Expanding the product, we get:

$$|X[k]|^2 = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x[n]x[m]^* e^{-j\frac{2\pi}{N}k(n-m)} \tag{10}$$

Recognizing that $n - m$ represents a lag in the autocorrelation function, we can rewrite the expression in terms of $r_{xx}[l]$, and then apply the DFT to obtain the periodogram.

Final Expression

After simplifying the double sum and aligning terms with the autocovariance sequence, we obtain:

$$I[k] = \frac{1}{N^2} |X[k]|^2 = \frac{1}{N} \sigma_{xx}[k] \tag{11}$$

This completes the proof that the periodogram is the scaled squared magnitude of the DFT of the signal.

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Part (b) - DFT Statistical Properties for White Noise

White Noise in Time Domain

White noise is a random signal with equal intensity at different frequencies, characterized by its delta function autocorrelation in time:

$$r_{xx}[l] = \sigma^2\delta[l] \tag{12}$$

DFT of White Noise

The DFT of white noise retains its statistical properties in the frequency domain.

Variance of DFT

The variance of $X[k]$ is equal to the time-domain variance of the white noise:

$$\mathrm{Var}\,(X[k]) = \sigma^2 \tag{13}$$

Autocorrelation of DFT

The autocorrelation of $X[k]$ is a delta function scaled by the variance:

$$R_{XX}[l] = \sigma^2\delta[l] \tag{14}$$

The DFT of a white noise process is also a white noise process in the frequency domain, with a variance equal to the time-domain variance and an autocorrelation function that is a delta function, indicating no frequency-domain correlation for non-zero lags.