Time Series Quiz 1

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1. Theoretical ACF of random process

$$v[k] - 0.8 v[k-1] + 0.15 v[k-2] = e[k]$$

for lags $l = 0, 1, 2$. Compute lags $l = 3, 4$

$e[k]$ is white noise, i.i.d.

→ **Answer**

This is an auto regressive equation of time series $v[k]$.

which can be generally written as

$$v[k] + d_1 v[k-1] + d_2 v[k-2] + \dots d_n v[k-n] = e[k]$$

where, $e[k] \in \mathcal{N}(0,1)$.

Now, to calculate the ACF values we can use the Luke-Walker (* forgot name exactly) equations, or least-square estimation or bayesian statistics or minimal likelihood estimation. If we use Luke-Walker then the general formula is:

$$ACF(n) = d_1 \cdot ACF(n-1) + d_2\, ACF(n-2) + \dots + d_{n-1}\, ACF(1) + d_n$$

$ACF(0) = 1$

$ACF(1) = d_1$

$ACF(2) = d_1 \cdot d_1 + d_2$

$ACF(3) = d_1 \cdot ACF(2) + d_2 \cdot d_1 + d_3$

$ACF(0)=1$

$ACF(1)=d_1$

$ACF(2)=d_1 \cdot d_1 + d_2$

$ACF(3)=d_1 \cdot ACF(2) + d_2 \cdot d_1 + d_3$

~~$ACF(1) = -0.8$~~

~~$ACF(2)$~~

$d_1 = -0.8$

$d_2 = +0.15$

$d_0 = 1$

$\therefore$ ACF of $V[K]$ at lag 3, 4 $\rightarrow$ we have to find.

$$\begin{array}{r} 0.64 \\ 0.15 \\ \hline 0.79 \end{array}$$

$ACF(0)=1$

$ACF(1)= -0.8$

$ACF(2)= (-0.8)^2 + 0.15 = 0.79$

$ACF(3) = (-0.8) \times 0.79 + 0.15 \times (-0.8) + 0$

[In the equation in question there is no d_3, so $d_3 = 0$]

$= -0.632 - 0.12$

$= -0.752$

$$\begin{array}{r} 0.79 \\ \times 0.8 \\ \hline .632 \end{array} \quad \begin{array}{r} 0.15 \\ \times 0.8 \\ \hline .120 \end{array} \quad \begin{array}{r} 0.632 \\ 0.120 \\ \hline 0.752 \end{array}$$

$ACF(4) = d_1 \cdot ACF(3) + d_2 \cdot ACF(2) + d_3 \cdot ACF(1) + d_4$

$= -0.8 \times (-0.752) + 0.15 \times 0.79 + 0 + 0$

[$\because$ d_3 & d_4 are not in equation so $d_3 = d_4 = 0$]

$= 0.6016 + 0.1185$

$= 0.7201$

$$\begin{array}{r} 0.752 \\ \times 0.8 \\ \hline .6016 \end{array} \quad \begin{array}{r} 0.79 \\ \times 0.15 \\ \hline .1185 \\ .6016 \\ \hline .7201 \end{array}$$

So, the ACF of $v[k]$ at lags $l = 3, 4$

are $ACF(3) = -0.752$
$ACF(4) = 0.7201$. (Ans)

Section-A

2) $X(t) = A\cos(\omega t + \theta)$ $\quad -\infty < t < \infty$

A, ω are constants, θ is r.v $[-\pi, \pi]$

Is this process stationary?

→ Answer:

We are assuming θ is uniformly distributed over $[-\pi, \pi]$. I already know that white noise ($\mathcal{N}[0,1]$) & sinusoidal functions like $A\sin(2\pi f t + \theta)$ are strictly stationary (SSS). But, for $X(t)$ lets see:

As mentioned in lecture, to prove stationarity proving WSS, or Wide Sense Stationary is enough. Let me try:

To prove if a function $X(t)$ is WSS we need to see 3 things:

i) $\mu(t)$ mean of $X(t)$ ~~or $E(X(t))$~~ must be constant
$E[X(t)]$

ii) $\sigma^2(t)$ variance of $X(t)$ must be constant
$E[(X(t) - \mu(t))^2]$

iii) Autocorrelation function $R(\tau, t)$

P.T.O

where, τ is time lag
this $R(\tau, t)$ must be independent of t.

so, $$x(t) = A\cos(\omega t+\theta)$$

$$E[x(t)] = E[A\cos(\omega t+\theta)]$$

$$= A.E[\cos(\omega t+\theta)]$$

$[\because E[aX] = aE[X]]$

we will use trigonometry:

$$\cos(\omega t+\theta)$$
$$= \cos\omega t\cos\theta - \sin\omega t\sin\theta$$

so, $$A.E[\cos(\omega t+\theta)]$$

$$= A.E[(\cos\omega t\cos\theta - \sin\omega t\sin\theta)]$$

$$= A.\{\cos\omega t\, E[\cos\theta] - \sin\omega t\, E[\sin\theta]\}$$

$[\because \omega$ is constant$]$

~~Now,~~ $\cancel{E[\cos\theta] = \int_{-\pi}^{\pi}\cos\theta . f(x)\,dx}$

we have assumed uniform distribution,

so, pdf of θ will be $\frac{1}{b-a} = \frac{1}{\pi-(-\pi)} = \frac{1}{2\pi}$

[a = lower bound
b = upper bound]

& $$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin\theta\,d\theta = \frac{1}{2\pi}[-\cos\pi + \cos(-\pi)]$$
$$= 0 \; [\because \cos\theta \text{ is even function}]$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\theta\,d\theta = \frac{1}{2\pi}[\sin\pi + \sin\pi]$$
$$= 0 \; [\because \sin\pi = 0]$$

So, $E[\sin\theta]$ & $E[\cos\theta]$ is 0.

So, $E[X(t)]$

$= A\{\cos\omega t . E[\cos\theta] - \sin\omega t\, E[\sin\theta]\}$

$= 0.$

$\mu(t)$ is 0 or it's constant.

$\sigma^2(t) = E[(X(t) - \mu(t))^2]$

$= E[(X(t))^2]$ $[\because \mu(t) = 0]$

$= E[\{A\cos(\omega t + \theta)\}^2]$

$= A^2 E[\cos^2(\omega t + \theta)]$ $2\cos^2\theta = 1 - \cancel{\sin 2\theta}$ $\cos 2\theta$

$= A^2 . E\left[\frac{(1 - \cos 2(\omega t + \theta))}{2}\right]$

$\cancel{= A^2}$ $\left[\because \cos^2\theta = \frac{1 - \cos 2\theta}{2}\right]$

(could be wrong, i don't remember)

$= \frac{A^2}{2} E(1) - \frac{A^2}{2} E[\cos(2\omega t + 2\theta)]$

$= \frac{A^2}{2} - \frac{A^2}{2} \left[\cos 2\omega t . E[\cos 2\theta] - \sin 2\omega t\, E[\sin 2\theta]\right]$

$= \frac{A^2}{2}$ $[\because E[\cos 2\theta]$ & $E[\sin 2\theta] = 0$ (don't have time to prove)$]$

So, $\sigma^2(t)$ is also constant.

iii) $R(\tau, t) = E[X(t) X(\tau + t)]$

$= E[A\cos(\omega t + \theta) . A\cos(\omega\tau + \omega t + \theta)]$

$= A^2 E[\cos(\omega t + \theta)\cos(\omega\tau + \omega t + \theta)]$

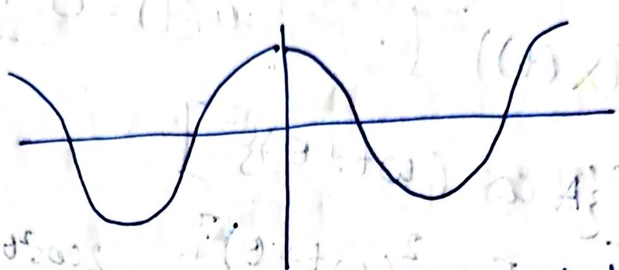
This will reduce to a statement with no t in it.

$$E\left[\cos(\omega t+\theta)\cos(\omega\tau+\omega t+\theta)\right].$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega t+\theta)\cos(\omega\tau+\omega t+\theta)$$

$\int \cos A \cos(A+B)$

(don't know how to solve)



whatever it is it must be symmetric with Y axis, because it's in the form $\cos\theta\cos(\theta+c)$ [where c is constant] ~~and it's~~ as $X(t)$ is a function, i can put $t=0$. & it becomes an even function. Probably, it will reduce to 0, & definitely it depends only on $\omega\tau$ (omega tao)..

So, $R(\tau, t)$ is independent of t & depends only on τ. So, all the 3 condition of WSS is fullfilled. So, it's stationary.

Bonus

There are 4 stationary type

1) Weak sense stationary
2) Strong " "
3) Double order "
4) Cyclostationary for periodic function,

X (t) in question is probably cyclostationary.

~~Section~~ B

3) Ensemble Mean.

A random variable X, may have multiple values $x_i \rightarrow (i \in 1,2,3 \ldots n)$

so the mean is represented as

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Now, ensemble means the mean of all realizations clumped together in one mean. Ensemble means collection.

Time average mean

It's an concept which means the ~~time as~~ average of time is used to calculate the mean in a time series.

$$\mu_t = \frac{1}{n}\sum_{t=1}^{n} x_t$$

Strict sense stationarity:

A random function $X(t)$ with values like $X(t_1), X(t_2) \ldots X(t_n)$ will be called strict stationary when after a time lag τ, $X(t_1+\tau), X(t_2+\tau) \ldots$ it won't change the function. We can ~~do~~ prove it through taking higher order moments of $X(t)$ & checking.

Wide sense Stationarity:

A function will be wide sense stationary when:

$\mu(t)$ is constant. (Mean)

$\sigma^2(t)$ " " (Variance)

$R(\tau, t)$ is independent of t.

→ Auto correlation function.

Ergodicity:

It's a special case of stationarity, seasonality, Trend & cycle should not be present.

4) $x(n) = \cos\left(\frac{n\pi}{8}\right)$

$x[n] = \cos(\pi/8 \; n)$

$f_s = 10\,Hz$

$x(t) = \cos(\pi/8 \times f_s \times t)$

$= \cos(\pi/8 \times 10 \times t)$

$= \cos(5\pi/4 \; t)$

$\frac{2\pi}{\omega} = \frac{5\pi}{4}$

$\Rightarrow \omega = \frac{8k}{5}$ $\quad k = 5, 10$

$x(t) = \cos(5\pi/4 \; t) \ldots$ (i)

$= \cos(10\pi/4 \; t) \ldots$ (ii)

Part B

a) $\frac{d x_a(t)}{d(t)}$ Nyquist rate will be still Ω_s

b) $x_a(2t)$ Nyquist rate will be $\frac{\Omega_s}{2}$

c) $x_a^2(t)$ Nyquist rate will be $\frac{\Omega_s}{2}$

d) $x_a(t)\cos(\Omega_0 t)$ Nyquist rate will be $\max(\Omega_s, \Omega_0)$