Applied Time Series Analysis
Assignment 2

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Problem 1

Question

Let X, Y, and Z be independent random variables with an identical distribution $N(0, 1)$ (Gaussian with zero mean and unit variance). Define the random variables $V = 2X + Y$ and $W = 3X - 2Z + 5$

1. Find the covariance between V and W.
2. Find the two parameters that completely specify the random variable $V + W$.

Solution

Let us discuss and prove some fundamental properties about variance and expectations which will come handy in the whole assignment.

Properties of Expectations:

1. $E[c] = c$ the expectation of a constant is also a constant, because the size of the sample can go to infinity, but still the value will be constant, and the probability is 1.
2. $E[aX] = aE[X]$, because $E[aX] = \sum aX \times P(X = x) = a \sum X \times P(X = x) = aE[X]$.
3. $E[aX+bY] = aE[X]+bE[Y]$, because $E[aX+bY] = \sum(aX+bY) \times P(X = x, Y = y) = a \sum X \times p(X = x) + b \sum Y \times P(Y = y) = aE[X] + bE[Y]$.
4. $E[XY] = E[X]E[Y] + Cov(X, Y)$, we will derive the covariance formula while doing the problem in the question.
5. $E[XY] = E[X]E[Y]$ when X and Y are independent variables. because covariance between them is 0.
6. $E[X^2] = Var(X) + |E[X]|^2$, this is coming from the formula of variance. But can be derived exactly similar to the covariance way.

Properties of Variance:

Variance can be defined as the spread of a random variable.

1. $Var(aX) = a^2 Var(X)$,we can prove from variance's formula. $Var(aX) = E[(aX - E[aX])^2] = a^2 E[(X - E[X])^2]$= $a^2 Var(X)$.
2. $Var(c) = 0$, because the dispersion of a constant is zero.
3. $Var(X) \geq 0$, as variance is a combination of squared terms, so it's always positive.
4. $Var(X+c) = Var(X)$, we can prove that, $Var(X+c) = E[(X+c-E[X+c])^2] = E[X+c-E[X]-c] = E[X - E[X]] = Var(X)$
5. $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$, this one can be proved like this - $Var(X + Y) = E[(X + Y - E[X + Y])^2] = E[((X - E[X]) + (Y - E[Y]))^2] = E[(X - E[X])^2 + 2(X - E[X])(Y - E[Y]) + (Y - E[Y])^2] = Var(X) + 2Cov(X, Y) + Var(Y)$.
6. If X and Y are independent then, $Var(X, Y) = Var(X) + Var(Y)$, because $Cov(X, Y) = 0$.

Problem 1 continued on next page. . .

Covariance between V and W (3 Ways) :

1. **Using direct formula of covariance**
   Let us start with our independent random variables X, Y and Z. They belong to the normal distribution $N(0, 1)$ which signifies the mean is $\mu = 0$ and the variance is $\sigma^2 = 1$ (where σ denotes the standard deviation).
   The mean represents the "central tendency" of the whole distribution. The Expectation of a random variable X, can be understood as a weighted average where each value x is weighted against it's probability of occurrence $p(x)$. We can write this as:

$$\mathbb{E}[X] = \sum_i p(x_i)x_i$$

   For a continuous random variable X with a probability density function $f(x)$, the expectation is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)\,dx$$

   And the Variance simply means the spread of the distribution, or how much it is far from the mean. So, We can understand now that from the question, $\mathbb{E}(X) = 0, \mathbb{E}(Y) = 0, \mathbb{E}(Z) = 0$ because all of them is coming from the Gaussian $N(0, 1)$.
   Now we will find out about covariance function. So, covariance is simply a kind of comparison between two random variables, for measuring if both of their trends match or not. If one goes up and the other also follows (vice versa) then it is called positive covariance, and if one goes down but other goes up (vice versa) then it's called the negative covariance. If there is fluctuations or no similarity of trends linearly then the covariance will be close to zero. If the covariance between two variables are zero, then they are linearly independent, but they might be non linearly dependent.
   So to calculate the covariance between two variables, we use this mathematical formula:

$$Cov(V, W) = \mathbb{E}\left[(V - \mathbb{E}[V]) \times (W - \mathbb{E}[W])\right]$$

   We first find out the deviations of variables from it's own expected values. then we multiply those because, if the sign of both of them are same (+ve or -ve) then the product will return us a positive value, otherwise a negative value. The sign is important to us. Now $(V - \mathbb{E}[V])$ and $(W - \mathbb{E}[W])$ is zero centred. We can prove it by taking the expectation of $(V - \mathbb{E}[V])$. $\mathbb{E}[(V - \mathbb{E}[V])] = \mathbb{E}[V] - \mathbb{E}[\mathbb{E}[V]]$. Now as Expectation is a linear transformation and Expectation of a constant is the constant itself. So, the formula will become $\mathbb{E}[V] - \mathbb{E}[V]$ which is 0. We are taking the Expectations of the product: $(V - \mathbb{E}[V]) \times (W - \mathbb{E}[W])$ because we want to quantify the outcome of two products and observe the combined trend (positive covariance or negative covariance) as expectation of each of the components are already 0 centred.
   So, the Covariance of V and W is :

$$
\begin{aligned}
Cov(V,W) &= \mathbb{E}\left[(V-\mathbb{E}[V]) \times (W-\mathbb{E}[W])\right] \\
&= \mathbb{E}\left[VW - V\mathbb{E}[W] - W\mathbb{E}[V] + \mathbb{E}[V]\mathbb{E}[W]\right] \\
&= \mathbb{E}[VW] - \mathbb{E}[V]\mathbb{E}[W] - \mathbb{E}[W]\mathbb{E}[V] + \mathbb{E}[V]\mathbb{E}[W] \quad \textit{(}\mathbb{E}[c]=c\textit{, where } c \textit{ is a constant)} \\
&= \mathbb{E}[VW] - \mathbb{E}[V]\mathbb{E}[W] \\
&= \mathbb{E}[(2X+Y)\times(3X-2Z+5)] - \mathbb{E}[(2X+Y)]\mathbb{E}[(3X-2Z+5)] \\
&= \mathbb{E}[(6X^2 - 4XZ + 10X + 3XY - 2YZ + 5Y)] - (\mathbb{E}[2X] + \mathbb{E}[Y]) \times (\mathbb{E}[3X] - \mathbb{E}[2Z] + \mathbb{E}[5]) \\
&= 6\mathbb{E}[X^2] - 4\mathbb{E}[X]\mathbb{E}[Z] + 10\mathbb{E}[X] + 3\mathbb{E}[X]\mathbb{E}[Y] - 2\mathbb{E}[Y]\mathbb{E}[Z] + 5\mathbb{E}[Y] \\
&\quad - (2\mathbb{E}[X] + \mathbb{E}[Y]) \times (3\mathbb{E}[X] - 2\mathbb{E}[Z] + 5) \quad \textit{(X and Y are independent, so } \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]\textit{)} \\
&= 6(Var[X] + |\mathbb{E}[X]|^2) - 0 + 0 + 0 - 0 + 0 - (0+0) \times (0 - 0 + 5) \\
&= 6 \times 1 \\
&= 6
\end{aligned}
$$

So, covariance of V and W is 6. (Ans)

2. **Using linear properties of covariance**
   Let us see the properties of a covariance function:

   (a) **Homogenity of the arguments:**

   $$Cov(aX,Y) = aCov(X,Y), \quad Cov(X,aY) = aCov(X,Y)$$

   Let's prove one of them:

   $$
   \begin{aligned}
   Cov(aX,Y) &= E\left[(aX - E[aX])(Y - E[Y])\right] \\
   &= E\left[(aX - aE[X])(Y - E[Y])\right] \quad (\text{Since, } E[aX] = aE[X]) \\
   &= aE\left[(X - E[X])(Y - E[Y])\right] \quad (\text{Since, } E[aX] = aE[X]) \\
   &= aCov(X,Y)
   \end{aligned}
   $$

   Scaling one random variable by a constant a scales the covariance by the same constant. This holds irrespective of whether X and Y are independent or correlated.

   (b) **Additivity of the arguments:**

   $$Cov(X_1 + X_2, Y) = Cov(X_1, Y) + Cov(X_2, Y)$$

   $$Cov(X, Y_1 + Y_2) = Cov(X, Y_1) + Cov(X, Y_2)$$

   The covariance is additive for each of its arguments. This property holds even if X_1, X_2, Y_1, and Y_2 are not independent. Let us prove this:

   $$
   \begin{aligned}
   Cov(A+B,C) &= E\left[((A+B) - E[A+B])(C - E[C])\right] \\
   &= E\left[(A + B - E[A] - E[B])(C - E[C])\right] \quad (\text{Since, } E[A+B] = E[A] + E[B]) \\
   &= E\left[(A - E[A] + B - E[B])(C - E[C])\right] \\
   &= E\left[(A - E[A])(C - E[C]) + (B - E[B])(C - E[C])\right] \\
   &= E\left[(A - E[A])(C - E[C])\right] + E\left[(B - E[B])(C - E[C])\right] \\
   &= Cov(A,C) + Cov(B,C)
   \end{aligned}
   $$

(c) **Double Linearity:**

$$Cov(aX + bY, cZ + dW) = acCov(X, Z) + adCov(X, W) + bcCov(Y, Z) + bdCov(Y, W)$$

This general property combines both homogenity and additivity and shows that covariance is linear in both its arguments. This holds for any random variables X, Y, Z, W, whether they are independent or not.

(d) **Zero Covariance for Independent Variables:**

$$\text{if } X \text{ and } Y \text{ are independent then} \quad Cov(X, Y) = 0$$

If X and Y are independent variables, then their covariance is zero. However, the reverse is not necessarily true: zero covariance does not imply independence.

(e) **Covariance of a Random Variable with Itself:**

$$Cov(X, X) = Var(X)$$

The covariance of a random variable with itself is equal to its variance. This property holds for any random variable X, whether it is a part of a set of independent or dependent variables.

Finding the Covariance between V and W

We have X, Y, Z as independent random variables following a Gaussian distribution $N(0, 1)$. We have from question $V = 2X + Y$ and $W = 3X - 2Z + 5$.

We will now find $Cov(V, W)$. First, we use the double linearity property of covariance to express $Cov(V, W)$:

$$\begin{aligned} \text{Cov}(V, W) &= \text{Cov}(2X + Y, 3X - 2Z + 5) \\ &= 2 \times 3\,\text{Cov}(X, X) + 2 \times (-2)\,\text{Cov}(X, Z) + 1 \times 3\,\text{Cov}(Y, X) \\ &\quad + 1 \times (-2)\,\text{Cov}(Y, Z) + 2 \times 5\,\text{Cov}(X, 1) + 1 \times 5\,\text{Cov}(Y, 1) \end{aligned}$$

Covariance of two random independent variables are 0. So $Cov(X, Y), Cov(Y, Z), Cov(X, Z)$ will be zero as X,Y,Z are random independent variables. (Reverse always not true). Also, Covariance of X and a constant is also zero. Let us see why. $Cov(X, c) = E[(X - E[X])(c - E[c])] = E[(X - E[X])(c - c)] = E[(X - E[X]) \times 0] = 0$. We also know, Covariance of a variable with itself is it's variance. So, $Cov(X, X) = \sigma^2 = 1$. (As X belongs to $N(0, 1)$.

$$\begin{aligned} Cov(V, W) &= 6 \times Var(X) + 0 + 0 + 0 + 0 + 0 \\ &= 6 \times 1 \\ &= 6 \end{aligned}$$

So, our required covariance between V and W is 6. (Ans)

3. Finding (Approximating) Covariance using Monte Carlo simulation :

Derivation of Markov's Inequality

We can simply derive the markov's inequality which gives us the probability for extremities. Or we can say for the tail distribution.
Let X be any positive continuous random variable. We can write:

$$\begin{aligned} E[X] &= \int_0^\infty x f_X(x)\,dx \\ &= \int_0^a x f_X(x)\,dx + \int_a^\infty x f_X(x)\,dx \\ &\geq \int_a^\infty a f_X(x)\,dx \\ &= a\int_a^\infty f_X(x)\,dx \\ &= aP(X \geq a) \\ &\Rightarrow P(X \geq a) \leq \frac{E[X]}{a} \end{aligned}$$

Thus, we conclude that $P(X \geq a) \leq \frac{E[X]}{a}$, for any $a > 0$. This result is known as Markov's Inequality.

Derivation of Chebyshev's Inequality Using Markov's Inequality

Let X be any random variable. Define $Y = (X - E[X])^2$. Y is a non-negative random variable, so we can apply Markov's Inequality to Y. For any positive real number b:

$$\begin{aligned} P(Y \geq b^2) &\leq \frac{E[Y]}{b^2} \\ E[Y] &= E[(X - E[X])^2] = \text{Var}(X) \\ P(Y \geq b^2) &= P((X - E[X])^2 \geq b^2) = P(|X - E[X]| \geq b) \end{aligned}$$

Thus, we conclude that $P(|X - E[X]| \geq b) \leq \frac{\text{Var}(X)}{b^2}$. This result is known as Chebyshev's Inequality. It's a much more robust framework than the markov's inequality because it also takes variance into consideration.

Derivation of WLLN using Chebyshev's Inequality

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with mean μ and variance σ^2. Consider the sample mean $S_n = \frac{1}{n}\sum_{i=1}^n X_i$.
We want to show that for any $\epsilon > 0$,

$$\lim_{n\to\infty} P\left(|S_n - \mu| \geq \epsilon\right) = 0$$

Using Chebyshev's Inequality,

$$\begin{aligned} P\left(|S_n - \mu| \geq \epsilon\right) &\leq \frac{\text{Var}(S_n)}{\epsilon^2} \\ &= \frac{\sigma^2}{n\epsilon^2} \end{aligned}$$

As $n \to \infty$, $\frac{\sigma^2}{n\epsilon^2} \to 0$, thus proving the weak law of large numbers which we will use in approximating in the next step.

Monte Carlo Approach using WLLN

To find the covariance between $V = 2X + Y$ and $W = 3X - 2Z + 5$ using Monte Carlo methods, one can take n samples (v_i, w_i) from the distributions of V and W. The sample covariance can be computed as:

$$\mathrm{Cov}(V, W) \approx \frac{1}{n}\sum_{i=1}^{n}(v_i - \bar{v})(w_i - \bar{w})$$

where $\bar{v}$ and $\bar{w}$ are the sample means of V and W respectively. According to WLLN, as $n \to \infty$, this approximation will converge to the actual covariance.

Python code

```
import numpy as np

# Number of samples
n = 100000

# Generate samples for X, Y, and Z
X = np.random.normal(0, 1, n)
Y = np.random.normal(0, 1, n)
Z = np.random.normal(0, 1, n)

# Compute V and W
V = 2 * X + Y
W = 3 * X - 2 * Z + 5

# Compute sample means
mean_V = np.mean(V)
mean_W = np.mean(W)

# Compute Cov(V, W)
cov_VW = np.mean((V - mean_V) * (W - mean_W))

cov_VW
```

The output of the Python code gives an approximate covariance value of ≈ 6.03, which aligns closely with the expected value of 6.

Finding the properties of V+W (second part of question) :

Definition and Gaussian PDF

The characteristic function $\phi_X(t)$ for a random variable X is defined as:

$$\phi_X(t) = E[e^{itX}]$$

For a Gaussian distribution $N(\mu, \sigma^2)$, the probability density function $f(x)$ is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Derivation of the Characteristic Function

Using the definition of expectation for continuous variables, we get:

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx$$

Substitute $f(x)$ into this integral:

$$\begin{aligned}
\phi_X(t) &= \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{itx-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2-2\mu x+\mu^2}{2\sigma^2}+itx} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu-i\sigma^2 t)^2}{2\sigma^2}} dx \\
&= e^{i\mu t-\frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= e^{i\mu t-\frac{\sigma^2 t^2}{2}} \quad \text{(Gaussian integral resolves to 1)}
\end{aligned}$$

Characteristic Function for Standard Gaussian Distribution $N(0,1)$

For a standard Gaussian distribution $N(0,1)$, the mean μ is 0 and the variance σ^2 is 1.
The general form of the characteristic function for a Gaussian distribution is:

$$\phi(t) = e^{i\mu t-\frac{\sigma^2 t^2}{2}}$$

Substituting $\mu = 0$ and $\sigma^2 = 1$ into the general formula, we obtain the characteristic function for $N(0,1)$ as:

$$\phi(t) = e^{i(0)t-\frac{1t^2}{2}} = e^{-\frac{t^2}{2}}$$

This is the characteristic function for a standard normal distribution $N(0,1)$.

Using Characteristic Function to Determine $U = V + W$

Characteristic Function for U

Let $U = V + W$, then $U = 5X + Y - 2Z + 5$. Given that X, Y, Z are standard normal random variables with characteristic function $e^{-\frac{t^2}{2}}$, the characteristic function of U, denoted as $\phi_U(t)$, can be derived as follows:

$$\begin{aligned}
\phi_U(t) &= E[e^{it(5X+Y-2Z+5)}] \\
&= e^{5it} \times E[e^{5itX}] \times E[e^{itY}] \times E[-2itZ] \\
&= e^{5it} \times e^{-\frac{25t^2}{2}} \times e^{-\frac{t^2}{2}} \times e^{-\frac{4t^2}{2}} \\
&= e^{5it} \times e^{-\frac{30t^2}{2}} \\
&= e^{5it} \times e^{-15t^2}
\end{aligned}$$

Problem 1 continued on next page. . .

Parameters of U

From the characteristic function $\phi_U(t) = e^{5it}e^{-15t^2}$, it's clear that U is Gaussian with:

- Mean $\mu_U = 5$
- Variance $\sigma_U^2 = 30$

Beyond the mean and variance, other statistical measures like skewness and kurtosis can also describe a distribution. However, for Gaussian distributions, these are standardized as follows:

- **Mean (μ_U)**: The mean is 5.
- **Variance (σ_U^2)**: The variance is 30.
- **Skewness**: The skewness of any Gaussian distribution is 0, indicating that the distribution is perfectly symmetrical. This is proven mathematically by:

  $$\gamma_1 = \frac{1}{\sigma^3}E[(X-\mu)^3] = 0$$

  The skewness is also known as the Fisher-Pearson coefficient of skewness. Adjustments for sample size are sometimes applied in the form of the adjusted Fisher-Pearson coefficient of skewness.
- **Kurtosis**: The kurtosis of any Gaussian distribution is 3, indicating the same level of "peakedness" as a standard normal distribution. This is proven mathematically by:

  $$\gamma_2 = \frac{1}{\sigma^4}E[(X-\mu)^4] - 3 = 3 - 3 = 0$$

  However, kurtosis is often reported as "excess kurtosis,". This form is used so that the standard normal distribution has an excess kurtosis of zero.

The random variable $U = V + W$ is a Gaussian random variable, completely specified not only by its mean $\mu_U = 5$ and variance $\sigma_U^2 = 30$, but also by its skewness 0 and kurtosis 3.

Problem 2

Question

Calculate the first 10 values of impulse response of an LTI system described by the constant-coefficient difference equation:

$$y[n] - 4y[n-1] + 4y[n-2] = x[n] - x[n-1]$$

Is the LTI system stable?

Solution

Definitions and Concepts

Linear Time-Invariant (LTI) Systems: An Elaborate Discussion

A system is referred to as a Linear Time-Invariant (LTI) system if it possesses two fundamental properties: linearity and time-invariance.

Linearity

Linearity implies that the system follows the principle of superposition, which consists of two sub-properties: homogeneity and additivity.

- **Homogeneity**: If the input is scaled by a constant a, then the output will be scaled by the same constant. Mathematically, for an input $x[n]$ producing an output $y[n]$, a scaled input $a \cdot x[n]$ will produce $a \cdot y[n]$.
- **Additivity**: If $x_1[n]$ produces $y_1[n]$ and $x_2[n]$ produces $y_2[n]$, then $x_1[n] + x_2[n]$ will produce $y_1[n] + y_2[n]$.

Combining these, the linearity property can be expressed as:

$$T\{a_1x_1[n] + a_2x_2[n]\} = a_1T\{x_1[n]\} + a_2T\{x_2[n]\}$$

Here, $T\{\}$ represents the system's transformation. The equation states that a linear combination of inputs $a_1x_1[n] + a_2x_2[n]$ will produce the same linear combination of individual outputs $a_1T\{x_1[n]\} + a_2T\{x_2[n]\}$.

Time-Invariance

Time-invariance means that the behavior of the system remains unchanged over time. If $x[n]$ produces $y[n]$, then a time-shifted input $x[n-k]$ will result in a correspondingly time-shifted output $y[n-k]$. This can be mathematically expressed as:

$$T\{x[n-k]\} = y[n-k]$$

Here, k is the amount by which the signal is shifted. This equation signifies that shifting the input signal by k units will result in an identical shift in the output signal.

Impulse Response

The impulse response $h[n]$ is the output of the LTI system when the input is a unit impulse function $\delta[n]$.

Problem 2 continued on next page. . .

Difference Equation

The constant-coefficient difference equation for the LTI system is:

$$y[n] - 4y[n-1] + 4y[n-2] = x[n] - x[n-1]$$

Impulse Response Calculation

Solving through hand

The impulse response $h[n]$ is determined by solving the equation for different values of n. The initial conditions are $h[-1] = 0$ and $h[-2] = 0$.

$$\begin{aligned}
h[0] &= \delta[0] - \delta[-1] + 4 \cdot h[-1] - 4 \cdot h[-2] = 1 - 0 + 0 - 0 = 1 \\
h[1] &= \delta[1] - \delta[0] + 4 \cdot h[0] - 4 \cdot h[-1] = 0 - 1 + 4 \cdot 1 - 0 = 3 \\
h[2] &= \delta[2] - \delta[1] + 4 \cdot h[1] - 4 \cdot h[0] = 0 - 0 + 4 \cdot 3 - 4 \cdot 1 = 8 \\
h[3] &= \delta[3] - \delta[2] + 4 \cdot h[2] - 4 \cdot h[1] = 0 - 0 + 4 \cdot 8 - 4 \cdot 3 = 20 \\
h[4] &= \delta[4] - \delta[3] + 4 \cdot h[3] - 4 \cdot h[2] = 0 - 0 + 4 \cdot 20 - 4 \cdot 8 = 48 \\
h[5] &= \delta[5] - \delta[4] + 4 \cdot h[4] - 4 \cdot h[3] = 0 - 0 + 4 \cdot 48 - 4 \cdot 20 = 112 \\
h[6] &= \delta[6] - \delta[5] + 4 \cdot h[5] - 4 \cdot h[4] = 0 - 0 + 4 \cdot 112 - 4 \cdot 48 = 256 \\
h[7] &= \delta[7] - \delta[6] + 4 \cdot h[6] - 4 \cdot h[5] = 0 - 0 + 4 \cdot 256 - 4 \cdot 112 = 576 \\
h[8] &= \delta[8] - \delta[7] + 4 \cdot h[7] - 4 \cdot h[6] = 0 - 0 + 4 \cdot 576 - 4 \cdot 256 = 1280 \\
h[9] &= \delta[9] - \delta[8] + 4 \cdot h[8] - 4 \cdot h[7] = 0 - 0 + 4 \cdot 1280 - 4 \cdot 576 = 2816
\end{aligned}$$

Python Code

Below is the Python code used for calculating the impulse response.

```
# Import libraries
import numpy as np
import matplotlib.pyplot as plt

# Initialize constants
N = 10  # Number of steps
h = np.zeros(N)  # Impulse response
d = np.zeros(N)  # Unit impulse
d[0] = 1  # Set unit impulse at n=0

# Initial conditions (system initially at rest)
h[-2:0] = 0  # Set h[-2] and h[-1] to 0

# Compute Impulse Response
for n in range(N):
    h[n] = d[n] - d[n - 1] + 4 * h[n - 1] - 4 * h[n - 2]
```

Plot

The plot of the impulse response is shown below:

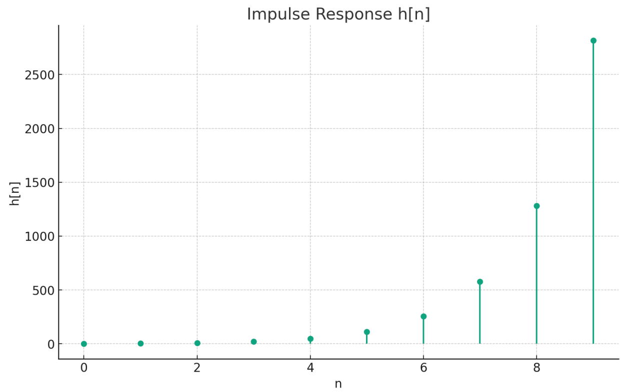

Figure 1: Impulse Response $h[n]$

Stability Analysis

In discrete-time Linear Time-Invariant (LTI) systems, one common criterion for assessing stability is the boundedness of the impulse response. Specifically, the system is considered stable if its impulse response is absolutely summable. Mathematically, this criterion can be expressed as follows:

$$\text{Stable if } \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

In the context of the given system, the impulse response $h[n]$ was found to demonstrate exponential growth, as observed in the calculated values for $h[0], h[1], \ldots, h[9]$. This indicates that the sum of the absolute values of the impulse response will not be finite. Consequently, the system fails the absolute summability test and is therefore unstable, expressed mathematically as:

$$\sum_{n=-\infty}^{\infty} |h[n]| \not< \infty$$

Implications

An unstable system implies that it will not reach a steady state under the influence of an impulse, and the output will grow without bounds. This is often undesirable in practical applications, such as signal processing and control systems, where stability is typically a prerequisite. Although absolute summability of the impulse response is a commonly used criterion for stability in discrete-time LTI systems, there are alternative methods for assessing stability. These include:

- Pole-Zero Analysis: The system is stable if all poles of its transfer function lie inside the unit circle in the z-plane.
- Lyapunov Stability: Utilizes Lyapunov functions to determine the stability of the system's equilibrium points.

However, it is worth noting that for linear systems, these alternative methods will yield the same conclusion regarding the system's stability.

Problem 2 continued on next page...

Solving through Z transforms

Types of Systems

Let us first see what kinds of systems are there.

Causal Systems

Causal systems are those where the output at any given time depends only on the present and past inputs. Mathematically, for a causal system, $y[n]$ depends on $x[n]$ for $n \leq N$ where N is the current time. Our problem is this one only.

Non-Causal or Acausal Systems

In non-causal systems, the output at any time can depend on future inputs. These are generally not practically implementable but are useful in theoretical contexts.

Anticausal Systems

Anticausal systems depend solely on future inputs. These are also largely theoretical constructs and are not practically implementable.

Bi-Directional or Two-Sided Systems

These systems depend on both past and future inputs. They are generally not implementable in real-time but can be used in offline data processing.

Z-Transform

The Z-transform provides a means of analyzing and representing discrete-time signals in the frequency domain. The Z-transform of a discrete-time signal $x[n]$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Poles and Stability

For the given system described by the constant-coefficient difference equation

$$y[n] - 4y[n-1] + 4y[n-2] = x[n] - x[n-1],$$

the transfer function $H(z)$ can be derived as follows:
Firstly, taking the Z-transform of both sides of the equation, we get

$$Y(z) - 4z^{-1}Y(z) + 4z^{-2}Y(z) = X(z) - z^{-1}X(z).$$

Isolating $Y(z)$ terms on one side and $X(z)$ terms on the other, we obtain

$$Y(z)(1 - 4z^{-1} + 4z^{-2}) = X(z)(1 - z^{-1}).$$

Simplifying, the transfer function $H(z) = \frac{Y(z)}{X(z)}$ becomes

$$H(z) = \frac{z-1}{z^2 - 4z + 4}.$$

Problem 2 continued on next page. . .

To evaluate stability, we examine the poles of the transfer function. The poles are the roots of the denominator of $H(z)$, which can be found by solving the equation $z^2 - 4z + 4 = 0$. This yields two identical poles at $z = 2$.
In the context of Discrete-Time Linear Time-Invariant (DT-LTI) systems, a system is considered stable if all its poles lie inside the unit circle in the z-plane. In this case, both poles are at $z = 2$, which is outside the unit circle. **Therefore, the system is unstable.** (Ans)

Problem 3

Question

1. **(Write a program)** Create 500 different realizations (of length 2048) of a zero-mean, unit variance Gaussian random process.
   (a) Compute the sample mean and variance for each realization. Plot the distribution of the resulting estimates. What do you observe?
   (b) Compute the average of the estimates and compare with the true value.
2. **Repeat the above steps for a chi-square distributed process (with 5 degrees of freedom)** and comment on the results.

Solution

Normal distribution

Let's talk about the properties of normal distribution before coding and plotting the graph.

Definition

The probability density function (PDF) of the Normal distribution is given by:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Where:

- μ is the mean of the distribution.
- σ is the standard deviation.
- σ^2 is the variance.

Parameters

- **Mean** (μ): The average value, indicating the center of the distribution.
- **Variance** (σ^2): Measures the dispersion or spread of the distribution around the mean.
- **Standard Deviation** (σ): The square root of the variance.

Problem 3 continued on next page...

Characteristic Function

The characteristic function $\phi(t)$ is given by:

$$\phi(t) = e^{i\mu t - \frac{\sigma^2 t^2}{2}}$$

Properties

1. **Symmetry**: The distribution is symmetric around its mean μ.
2. **Unimodal**: It is unimodal with a single peak at the mean.
3. **Asymptotic**: The tails extend to infinity but never touch the x-axis.
4. **Mean = Median = Mode**: All measures of central tendency are equal.

Special Cases

- **Standard Normal Distribution**: A Normal distribution with $\mu = 0$ and $\sigma = 1$.
- **Log-Normal Distribution**: If X is Normally distributed, then e^X has a Log-Normal distribution.

Graphical Representations

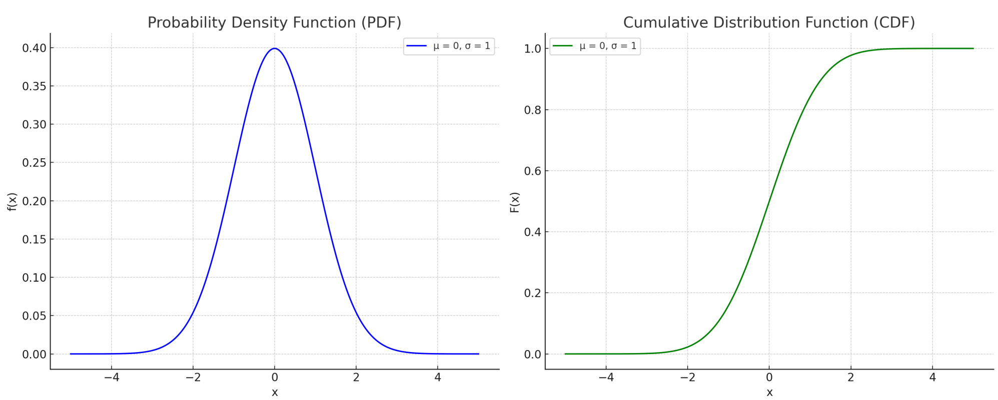

Figure 2: PDF and CDF of the Normal Distribution

Chi-Squared Distribution

The Chi-Squared distribution is a particular case of the gamma distribution and is one of the most widely used distributions in hypothesis testing and statistical inference.

Definition

The probability density function (PDF) of the Chi-Squared distribution with k degrees of freedom is given by:

Problem 3 continued on next page...

$$f(x;k) = \frac{x^{(k/2)-1}e^{-x/2}}{2^{k/2}\Gamma(k/2)}$$

Where:

- k are the degrees of freedom.
- Γ is the Gamma function.

About the Gamma Function

The Gamma function, denoted by $\Gamma(z)$, is an extension of the factorial function to complex numbers. It is defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

The Gamma function has the property that $\Gamma(n) = (n-1)!$ for positive integers n.

Chi-Squared Distribution Without Gamma Function

When $k = 2$, the Chi-Squared distribution simplifies to an Exponential distribution, and the Gamma function $\Gamma(1) = 1$. Therefore, the PDF becomes:

$$f(x;k=2) = \frac{1}{2}e^{-x/2}$$

This is the formula for an Exponential distribution with a rate parameter $\lambda = 1/2$, and we can here notice that the Gamma function term is absent.

Parameters

- **Degrees of Freedom** (k): A parameter that defines the shape of the distribution.

Characteristic Function

The characteristic function $\phi(t;k)$ is given by:

$$\phi(t;k) = (1-2it)^{-k/2}$$

Properties

1. **Skewness**: The distribution is skewed to the right.
2. **Unimodal**: It is unimodal with a single peak.
3. **Non-Negative**: All values are non-negative.
4. **Mean and Variance**: The mean is k and the variance is $2k$.

Problem 3 continued on next page. . .

Special Cases:

Chi Distribution

The Chi distribution is a special case of the Chi-Squared distribution when $k = 2$. More formally, if $Z_1, Z_2, \ldots, Z_k$ are independent standard normal random variables, then the square root of the sum of their squares:

$$\sqrt{Z_1^2 + Z_2^2 + \ldots + Z_k^2}$$

follows a Chi distribution with k degrees of freedom. Specifically, for $k = 2$, this square root follows a Chi distribution. In other words, the Chi distribution is the positive square root of a Chi-Squared distribution with $k = 2$.

Exponential Distribution

When $k = 2$, the Chi-Squared distribution simplifies to an Exponential distribution with rate parameter $\lambda = 1/2$. In this case, the Chi-Squared distribution is equivalent to:

$$f(x; k = 2) = \frac{1}{2} e^{-x/2}$$

This is the PDF of an Exponential distribution with $\lambda = 1/2$.

Graphical Representations

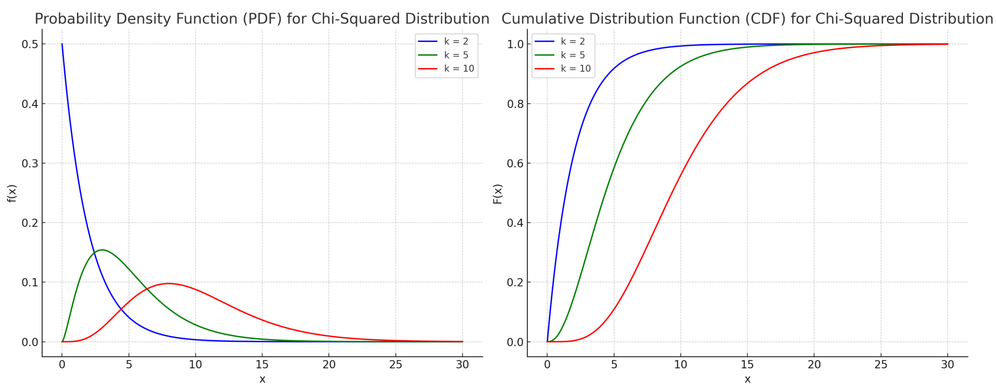

Figure 3: PDF and CDF of the Chi-Squared Distribution

Differences Between Normal and Chi-Squared Distributions

The Normal and Chi-Squared distributions are both foundational in statistics but are used in different contexts and have distinct properties. This subsection outlines the key differences between these two distributions.

1. **Definition and Parameters**:

   - The Normal distribution is defined by its mean μ and standard deviation σ.
   - The Chi-Squared distribution is defined by its degrees of freedom k.

2. **Symmetry**:
   - The Normal distribution is symmetric around its mean.
   - The Chi-Squared distribution is not symmetric and is skewed to the right.
3. **Domain**:
   - The Normal distribution has a domain of $-\infty$ to $+\infty$.
   - The Chi-Squared distribution is defined only for positive values.
4. **Applications**:
   - The Normal distribution is widely used in various fields for data that clusters around the mean.
   - The Chi-Squared distribution is often used in hypothesis testing, particularly for tests of independence.
5. **Derived from Normal Distribution**:
   - The Chi-Squared distribution is actually a special case of the Gamma distribution and can be derived from Normal distributions.
   - The Normal distribution is a fundamental distribution and is not generally derived from other distributions.
6. **Mathematical Complexity**:
   - The Normal distribution involves exponential functions in its PDF.
   - The Chi-Squared distribution involves the Gamma function in its PDF, making it more complex mathematically.

Statistical Analysis of Gaussian and Chi-Squared Distributions with 500 different realizations (of length 2048) of a zero-mean, unit variance

Gaussian Distribution

Python Code

```
import numpy as np
from scipy.stats import describe
import matplotlib.pyplot as plt

num_realizations = 500
length = 2048
true_mean = 0
true_variance = 1

gaussian_sample_means = []
gaussian_sample_variances = []

for _ in range(num_realizations):
```

```
    gaussian_realization = np.random.normal(loc=true_mean, scale=np.sqrt(
        true_variance), size=length)
    gaussian_stats = describe(gaussian_realization)
    gaussian_sample_means.append(gaussian_stats.mean)
    gaussian_sample_variances.append(gaussian_stats.variance)

avg_gaussian_sample_mean = np.mean(gaussian_sample_means)
avg_gaussian_sample_variance = np.mean(gaussian_sample_variances)
```

Graphs and Observations

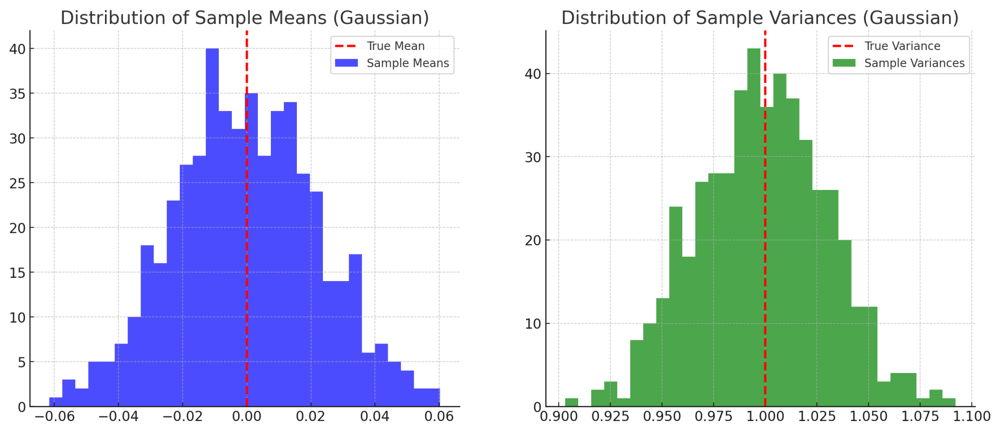

Figure 4: Distribution of Sample Means and Variances for Gaussian Random Process

The average of sample means is approximately -0.0002 and the average of sample variances is approximately 0.9985.

Chi-Squared Distribution

Python Code

```
chi_squared_sample_means = []
chi_squared_sample_variances = []

for _ in range(num_realizations):
    chi_squared_realization = np.random.chisquare(df=5, size=length)
    chi_squared_stats = describe(chi_squared_realization)
    chi_squared_sample_means.append(chi_squared_stats.mean)
    chi_squared_sample_variances.append(chi_squared_stats.variance)

avg_chi_squared_sample_mean = np.mean(chi_squared_sample_means)
avg_chi_squared_sample_variance = np.mean(chi_squared_sample_variances)
```

Graphs and Observations

The average of sample means is approximately 4.9996 and the average of sample variances is approximately 9.9897.

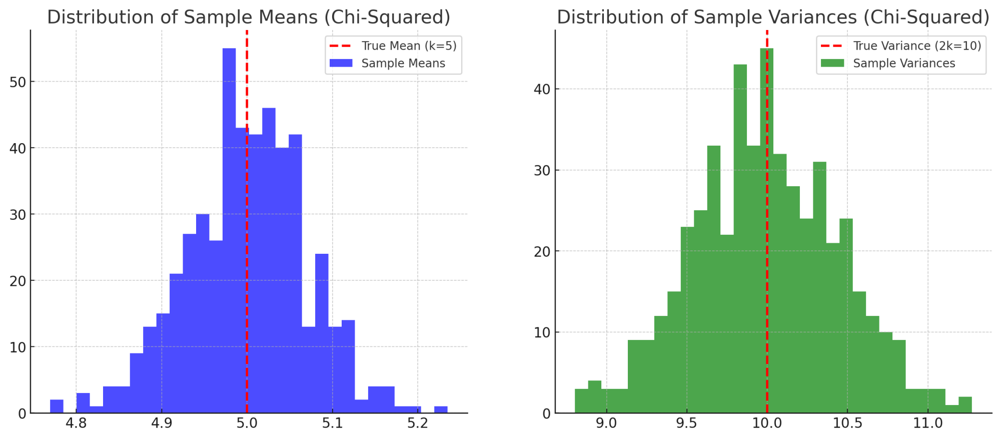

Figure 5: Distribution of Sample Means and Variances for Chi-Squared Random Process

Detailed Observations

Gaussian Distribution

For the Gaussian distribution, the sample means and variances are observed to be closely clustered around the true mean and variance, which are 0 and 1, respectively. This demonstrates the consistent and unbiased nature of the sample estimators for the Gaussian distribution. The distribution of sample means and variances is approximately normal, which is expected due to the Central Limit Theorem.

Chi-Squared Distribution

In the case of the Chi-Squared distribution with 5 degrees of freedom, the true mean and variance are 5 and 10, respectively. Similar to the Gaussian distribution, the sample means and variances are also closely clustered around these true values. This indicates that the sample estimators are reliable for estimating the population parameters of a Chi-Squared distribution.

Comparison of Average Estimates and True Values

Gaussian Distribution

The average of sample means is approximately -0.0002, and the average of sample variances is approximately 0.9985. Both of these estimates are very close to the true values of 0 and 1, thus validating the sample statistics as good estimators for the population parameters.

Chi-Squared Distribution

For the Chi-Squared distribution, the average of sample means is approximately 4.9996 and the average of sample variances is approximately 9.9897. These estimates are in close agreement with the true values of 5 and 10, further confirming the reliability of the sample estimators.

Problem 4

Question

Consider a continuous-time sinusoidal signal $x(t) = \sin(2\pi F_0 t)$. Suppose that $F_0 = 2$ kHz and that this signal is sampled at a sampling frequency of $F_s = 50$ kHz to produce $x[n]$. Then,

1. Plot the signal $x[n]$ (stem plot), $0 \leq n \leq 99$. What is the frequency f_0 of the signal $x[n]$?
2. Plot the signal $y[n]$ created by taking the even-numbered samples of $x[n]$. Is this a sinusoidal signal? Why? If so, what is its frequency?

Solution

Preliminary Definitions and Formulas

The continuous-time sinusoidal signal $x(t)$ is defined as $\sin(2\pi F_0 t)$, where $F_0 = 2$ kHz or 2000 Hz. The signal is sampled at a frequency $F_s = 50$ kHz or 50000 Hz to produce $x[n]$.
The relationship between continuous-time and discrete-time signals through sampling is given by:

$$x[n] = x(nT_s) = \sin(2\pi F_0 n T_s)$$

where $T_s = \frac{1}{F_s}$ is the sampling interval.

(a) Stem Plot of $x[n]$ and Frequency f_0

Firstly, let's find T_s and $x[n]$:

$$T_s = \frac{1}{F_s} = \frac{1}{50000} = 20 \times 10^{-6}\,\text{s}$$
$$x[n] = \sin(2\pi \times 2000 \times n \times 20 \times 10^{-6})$$
$$x[n] = \sin(0.08\pi n)$$

```
import numpy as np
import matplotlib.pyplot as plt

F_0 = 2000
F_s = 50000

T_s = 1 / F_s
n = np.arange(0, 100)

x_n = np.sin(2 * np.pi * F_0 * n * T_s)

plt.figure(figsize=(14, 6))
plt.stem(n, x_n, basefmt=" ", use_line_collection=True)
plt.title("Stem Plot of $x[n] = \\sin(0.08\\pi n)$")
plt.xlabel("Sample index (n)")
plt.ylabel("Amplitude")
plt.grid(True)
plt.show()
```

The frequency f_0 of the discrete-time signal $x[n]$ is $f_0 = 0.04$ kHz, equivalent to $\frac{1}{25}$ cycles per sample.

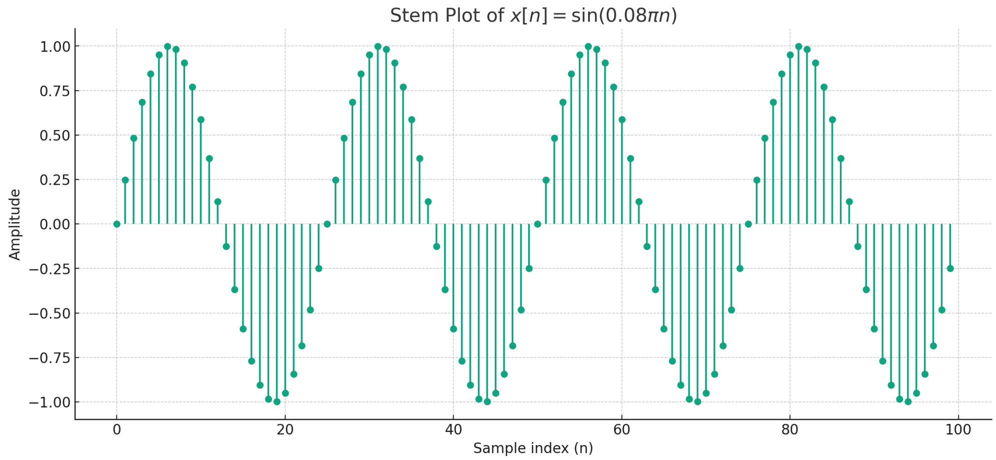

Figure 6: Stem plot of $x[n]$

(b) Stem Plot of $y[n]$ and Its Frequency

The signal $y[n]$ is created by taking the even-numbered samples of $x[n]$. Thus, $y[n] = \sin(0.16\pi n)$.

```
n_y = np.arange(0, 50)

y_n = np.sin(2 * np.pi * F_0 * 2 * n_y * T_s)

plt.figure(figsize=(14, 6))
plt.stem(n_y, y_n, basefmt=" ", use_line_collection=True)
plt.title("Stem Plot of $y[n] = \\sin(0.16\\pi n)$")
plt.xlabel("Sample index (n)")
plt.ylabel("Amplitude")
plt.grid(True)
plt.show()
```

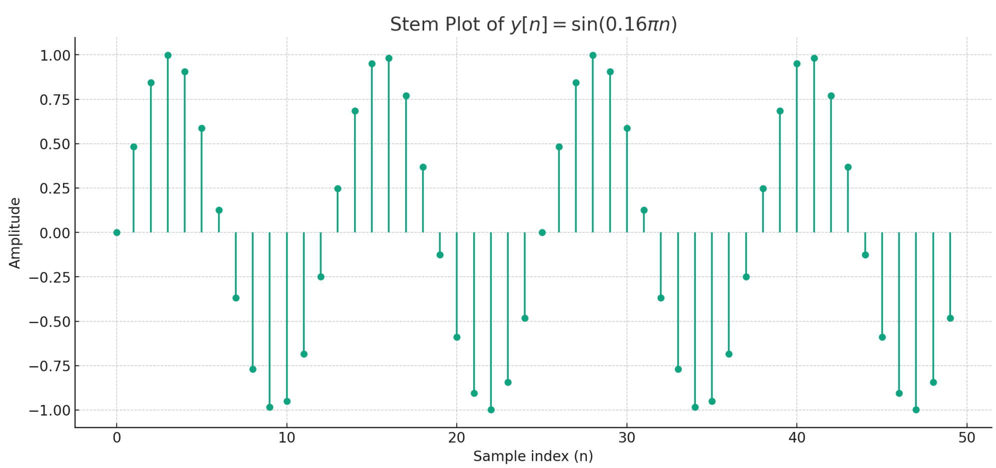

Figure 7: Stem plot of $y[n]$

$y[n]$ is also a sinusoidal signal with a frequency of $f_1 = 0.08$ kHz. $y[n]$ is also a sinusoidal signal with a frequency of $f_1 = 0.08$ kHz, equivalent to $\frac{1}{12.5}$ cycles per sample.

Acknowledgments

- I have extensively used browsing, YouTube, Khan Academy, Coursera, books, and lecture notes to do this assignment. I have tried to prove concepts as I have written this assignment, deriving formulas in process, just for my understanding. I tried to make it as elaborate and detailed as possible just for my learning. I can use it as my revision notes before exam,
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