Applied Time Series Analysis
Assignment 1

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Problem 1

Question

Consider a discrete-time signal $x[n] = \cos^2\left(\frac{2\pi}{3}n\right)$, which is obtained by sampling a continuous-time signal $x(t)$ at a sampling rate of 60 Hz. Determine the following:

1. What is the fundamental period of $x[n]$?
2. Identify the continuous-time signal $x(t)$.

Solution

Simplification Using Trigonometric Identities

We can simplify the $x[n]$ with trigonometry. So, we can use the formula $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$. Now our $x[n]$ becomes :

$$x[n] = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{4\pi}{3}n\right)$$

For a discrete-time signal $x[n]$ which is periodic, the following properties will be there:

1. $x[n] = x[n+N]$ for all n, where N is the period.
2. N must be a positive integer and $\frac{2\pi}{\omega_0}$ is a rational number.
3. N should be the *smallest positive integer* for which the periodicity condition holds, termed as the "fundamental period."
4. Periodicity may start from a specific initial point n_0, but this is a specific condition, not always applicable.
5. The signal is ideally assumed to exist for *all time n*.
6. The definition holds for both *real and complex signals.*

Determining the periodicity of this composite function

We can now consider $x[n_1] = \frac{1}{2}$ and $x[n_2] = \frac{1}{2}\cos\frac{4\pi}{3}n$

1. Periodicity of $x[n_1]$ is arbitrary N as it's a constant value. It can be any period as it stays always constant. So fundamental period is $N_0 = 1$
2. Periodicity of $x[n_2] = \frac{1}{2}\cos\frac{4\pi}{3}n$ can be calculated in the following manner:
   (a) Multiplying by a constant changes the amplitude but it won't impact the periodicity of the signal.
   (b) First we need to check if $\frac{1}{2}\cos\frac{4\pi}{3}n$ is periodic or not. Now, we can understand from the function that $\omega_0 = \frac{4\pi}{3}$. So, $\frac{2\pi}{\omega_0} = \frac{3}{2}$, which is a rational number. So periodicity exists. And we know period $\frac{N}{k} = \frac{2\pi}{\omega_0}$ where k is any positive integer. If I substitute the value of $\frac{2\pi}{\omega_0}$ then $N = \frac{3}{2} \times k$. So, for minimum $k = 2$, we get the fundamental period $N_0 = 3$.
3. To find the periodicity of the composite function $x[n_1]$ and $x[n_2]$ we will simply take the LCM of their fundamental periods. So, the LCM of 1 and 3 is 3.

Therefore, the fundamental period of $x[n]$ is 3. (Ans)

Problem 1 continued on next page...

Alternate solution

We can simply plot the graph and observe periodicity. [1]

Python code to plot the graph of the stem function :

```
import numpy as np
import matplotlib.pyplot as plt

# Define the sequence x[n] for n values ranging from 0 to 12
n_values = np.arange(0, 13)
x_n_values = np.cos(2*np.pi/3 * n_values)**2

# Plot the sequence x[n]
plt.figure(figsize=(10, 6))
plt.stem(n_values, x_n_values, basefmt=" ", linefmt='-r', markerfmt='oy')
plt.xlabel('$n$')
plt.ylabel('$x[n]$')
plt.title('Sequence $x[n] = \\cos^2\\left(\\frac{2\\pi}{3} n\\right)$')
plt.grid(True)
plt.show()
```

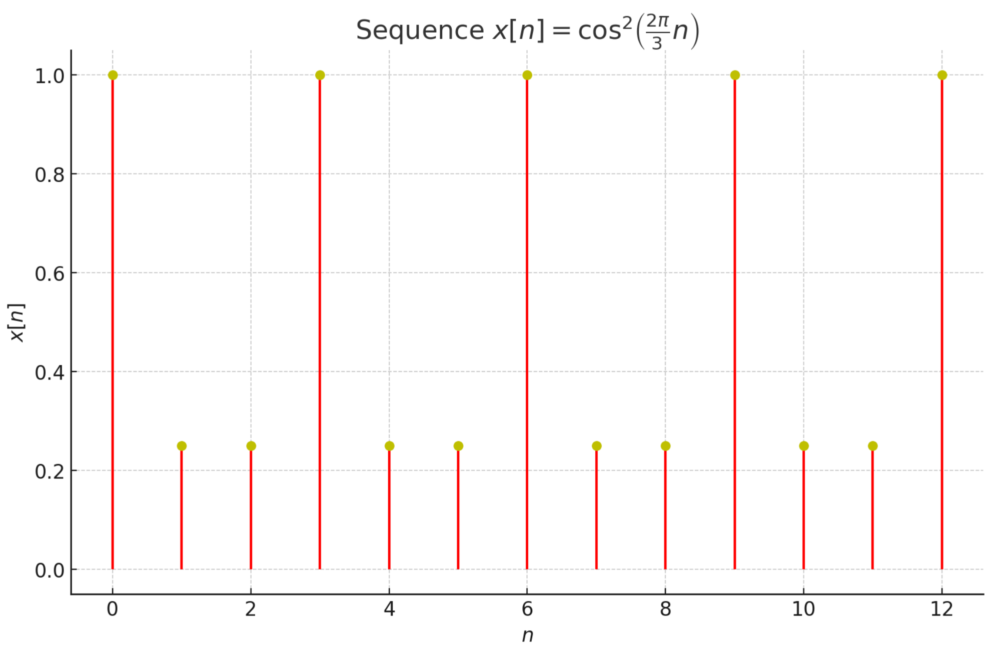


Figure 1: Sequence $x[n] = \cos^2\left(\frac{2\pi}{3} n\right)$

We can clearly, see from the plot that the function $x[n]$ is repeating after an interval of 3, so the fundamental time period N_0 is 3. (Ans)

[1] Another way I found while researching is that we can use the auto correlation function $R_{xx}(k)$ and plot it for all k, where k is a non-negative integer. The first non-zero ($k \neq 0$), peak which is observed, will be the fundamental period of x[n]. I haven't studied this part completely so, including it as a side note.

Finding the Continuous-Time Signal $x(t)$

The relationship between the discrete-time index n and the continuous-time variable t is given by:

$$n = f_s \cdot t$$

where f_s is the sampling rate. Given the discrete-time signal $x[n] = \cos^2\left(\frac{2\pi}{3}n\right)$ and a sampling rate $f_s = 60$ Hz, the continuous-time signal $x(t)$ is obtained as:

$$x(t) = \cos^2\left(\frac{2\pi}{3} \times \frac{t \times 60}{1}\right) = \cos^2(40\pi t)$$

$$x(t) = \cos^2(40\pi t) \quad \text{(Ans)}$$

Fundamental Frequency and Period (Extra)

The continuous-time signal $x(t) = \cos^2(40\pi t)$ has a fundamental angular frequency of 40π rad/s. The period of $\cos^2(\omega t)$ is $T = \frac{\pi}{\omega}$, hence for $x(t)$:

$$T = \frac{\pi}{40\pi} = \frac{1}{40} \text{ s}$$

Corresponding to a fundamental frequency of $f = \frac{1}{T} = 40$ Hz.

Nyquist Rate and Signal Reconstruction

The Nyquist-Shannon sampling theorem states that a continuous-time signal can be fully specified and reconstructed from its samples if it is sampled at a rate greater than twice the highest frequency component in the signal.[2] Mathematically, the Nyquist rate f_s is defined as:

$$f_s \geq 2 \times f_{\text{max}}$$

For the continuous-time signal $x(t) = \cos^2(40\pi t)$, the highest frequency component is $f_{\text{max}} = 40$ Hz. Therefore, the Nyquist rate is $f_s = 2 \times 40$ Hz $= 80$ Hz.

Given that the sampling rate is 60 Hz, which is less than the Nyquist rate of 80 Hz, the continuous-time signal $x(t) = \cos^2(40\pi t)$ cannot be perfectly reconstructed from its samples if sampled at 60 Hz. Aliasing will occur, leading to distortion in the reconstructed signal.

[2] To prove it, we can consider the example of a sine function:

$$f(t) = \sin(2\pi f t)$$

Here, the time period after which the function repeats is $T = \frac{1}{f}$. We know the range of the sine function varies from $[-1, 1]$ and the domain varies from $[0, 2\pi]$. To capture the change of phase from positive to negative, we need at least two points. If the time period of a sine function is T, then we can capture points at $\frac{T}{4}$ and $\frac{3T}{4}$, which are two opposite points in the same cycle but different phases. Their difference is $\frac{T}{2}$ period. The sampling frequency needed to capture these two points is:

$$f_s = \frac{1}{\frac{T}{2}} = \frac{2}{T} = 2 \times f$$

This exact value is also called the Nyquist frequency.

Problem 2

Question

Let $X = \cos\theta$ and $Y = \sin\theta$ be two random variables. Let the mean value of each be zero. Let θ be a random variable in the interval $[0, 2\pi]$.

1. Find the value of $\sigma_{xy} = E(XY) - E(X)E(Y)$. (Hint: Do not get confused with the covariance function which is a function of τ. Use the formula for mean and $E(XY)$ to find the value).

2. Are X & Y variables independent? Support your answers. (Hint: No covariance between two random variables does not mean they are independent).

Solution

Before we start to solve the problems, let us analyze each component of the question very carefully. $X = \cos(\theta)$ and $Y = \sin\theta$. We are assuming that θ is uniformly distributed. The domain of X and Y is given to be $[0, 2\pi]$ and the range of $\sin\theta$ and $\cos\theta$, $X, Y \in [-1, 1]$. The mean value is given to be zero in the question. But from Figure 1 and Figure 2 below we can see the periodicity of the $\sin\theta$ and $\cos\theta$ functions. It is oscillating with a mean of zero. Let us first simplify XY:

$$\begin{aligned} XY &= \sin\theta\cos\theta \\ &= \frac{1}{2} \times 2 \times \sin\theta\cos\theta \\ &= \frac{1}{2}\sin 2\theta \end{aligned}$$

Visualisation

Figure 1: $\sin(\theta)$

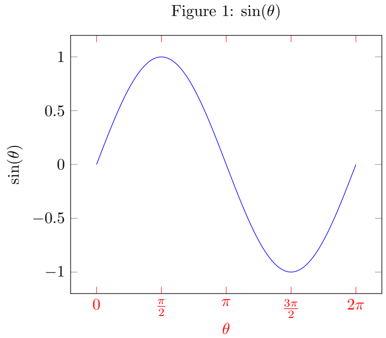


Figure 2: $\cos(\theta)$

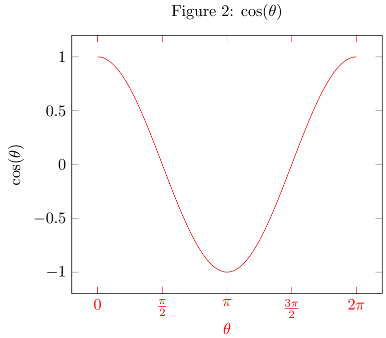
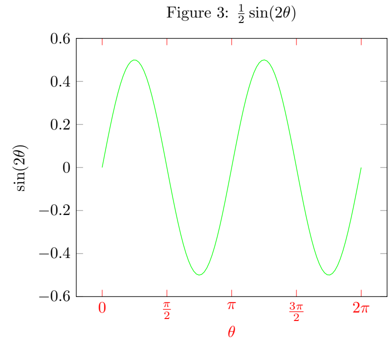


Figure 3: $\frac{1}{2}\sin(2\theta)$



Part: 1

The expectation of a continuous random variable gives us the "balance point" or the weighted average of the random variable. If we take enough observations of the random variable, their average will come very close to the Expected value. To calculate the Expectation we need the probability density function $f(x)$:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

So, we can calculate our pdf:

$$
\begin{aligned}
f(x) &= \frac{1}{b-a}, ([a,b] = [0, 2\pi]) \\
&= \frac{1}{2\pi - 0} \\
&= \frac{1}{2\pi}
\end{aligned}
$$

The expected value of a continuous random variable X with p.d.f. $f(x)$ with bounds $[a, b]$ is as follows:

$$E[X] = \int_a^b x \cdot f(x)\,dx$$

Finding out the σ_{xy}:

We already know from the question that the mean of X and Y is zero. So automatically, E(X) and E(Y) become zero. Next, let's find $E(XY)$:

$$
\begin{aligned}
E(XY) &= \frac{1}{2\pi}\int_0^{2\pi} \cos(\theta)\sin(\theta) d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} 2\cos(\theta)\sin(\theta) d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} \sin(2\theta) d\theta \\
&= \frac{1}{4\pi}\left[-\frac{1}{2}\cos(2\theta)\right]_0^{2\pi} \\
&= \frac{1}{4\pi}\left(-\frac{1}{2}\cos(4\pi) + \frac{1}{2}\cos(0)\right) \\
&= \frac{1}{4\pi}\left(-\frac{1}{2}(1) + \frac{1}{2}(1)\right) \\
&= 0
\end{aligned}
$$

Finally, we can find σ_{xy}:

$$
\begin{aligned}
\sigma_{xy} &= E(XY) - E(X)E(Y) \\
&= 0 - 0 \times 0 \\
&= 0
\end{aligned}
$$

So, the value of σ_{xy} is 0.

Part: 2

The covariance might be zero, but it doesn't guarantee independence from each other. Because covariance captures the linear relationship between two variables. If nonlinear relationships exist, then they can't capture the independence properly.

Problem 2 continued on next page. . .

Linear Independence

Let us try to prove the linear independence of $\sin\theta$ and $\cos\theta$. We know if two functions $f(x), g(x)$ have the relationship $af(x) + bg(x) = 0; a, b \in R$, then the functions are linearly independent only when a=0 **and** b=0. If either of those is 0 and the other one is non-zero, then it won't satisfy the condition. Let's see for our case, if any value of theta exists for which a and b can be zero To prove the linear independence of $\sin\theta$ and $\cos\theta$, we start by considering specific values of θ to determine the constants a and b in the equation $a\sin\theta + b\cos\theta = 0$. [On a side note: The domain of $a\sin\theta + b\cos\theta$ is $\theta \in [0, 2\pi]$ (given in question), and its range is $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$]

1. For $\theta = 0$:

$$\begin{aligned} a\sin(0) + b\cos(0) &= 0 \\ b &= 0, a \in C \end{aligned}$$

2. For $\theta = \frac{\pi}{2}$:

$$\begin{aligned} a\sin\left(\frac{\pi}{2}\right) + b\cos\left(\frac{\pi}{2}\right) &= 0 \\ a &= 0, b \in C \end{aligned}$$

3. For $\theta = \pi$:

$$\begin{aligned} a\sin(\pi) + b\cos(\pi) &= 0 \\ b &= 0, a \in C \end{aligned}$$

4. For $\theta = \frac{3\pi}{2}$:

$$\begin{aligned} a\sin\left(\frac{3\pi}{2}\right) + b\cos\left(\frac{3\pi}{2}\right) &= 0 \\ a &= 0, b \in C \end{aligned}$$

5. For $\theta = 2\pi$:

$$\begin{aligned} a\sin(2\pi) + b\cos(2\pi) &= 0 \\ b &= 0, a \in C \end{aligned}$$

No other values of π will make either of those functions 0. We can see that, no values of theta lead to being both a and b = 0. So, we can conclude that only a=0 and b=0 can make the whole function $a\sin\theta + b\cos\theta$ as 0. **And so we can conclude that these two functions are linearly independent**

Non-linear Dependence

To prove the non-linear dependence of $\sin\theta$ and $\cos\theta$, we can refer to the Pythagorean trigonometric identity:

$$\begin{aligned} \sin^2(\theta) + \cos^2(\theta) &= 1 \\ Y^2 + X^2 &= 1 \\ Y^2 &= 1 - = X^2 \\ Y &= \pm\sqrt{1 - X^2} \end{aligned}$$

We can clearly, see there is a nonlinear relationship between the two random variables and the two functions are not non-linearly independent although their covariance is 0.

Problem 3

Question

An electroencephalographic (EEG) signal has a maximum frequency of 300 Hz. The signal is sampled and quantized into a binary sequence by an A/D converter.

1. Determine the sampling rate if the signal is sampled at a rate 50% higher than the Nyquist rate.
2. The samples are quantized into 2,048 levels. How many binary bits are required for each sample?

Solution

Sampling Rate

We already know from the first problem that the Nyquist rate is defined as two times the maximum frequency of the signal (I proved it with the sin function in the footnote in the last question's solution) (the max frequency is denoted by f_{max}). For an EEG signal with a maximum frequency of 300 Hz(Given in question), the Nyquist rate (lets call it f_{Nyquist}) is:

$$f_{\text{Nyquist}} = 2 \times f_{\text{max}} = 2 \times 300\,\text{Hz} = 600\,\text{Hz}$$

The problem states that the signal is sampled at a rate 50% higher than the Nyquist rate. Therefore, the actual sampling rate (f_s) would be:

$$f_s = f_{\text{Nyquist}} \times \left(1 + \frac{50}{100}\right) = 600\,\text{Hz} \times 1.5 = 900\,\text{Hz}$$

Number of Binary Bits

The samples are quantized into 2,048 levels. The number of binary bits (n) required for each sample can be calculated using the formula:

$$n = \lceil \log_2(\text{Number of Levels}) \rceil = \lceil \log_2(2048) \rceil = \lceil 11 \rceil = 11$$

Where $\lceil . \rceil$ is the ceiling function. **Therefore,** 11 **binary bits are required for each sample, rounded to the nearest integer to quantize the signal at 2048 levels.** (Ans)

Problem 4

Question

We toss two fair coins simultaneously and independently. If the outcomes of the two coins are the same, we win; otherwise, we lose. Let A be the event that the first coin comes up heads, B be the event that the second coin comes up heads, and C be the event that we win. Which of the following statements is false?

1. Events A and B are independent.
2. Events A and C are not independent.
3. Events A and B are not conditionally independent given C.
4. The probability of winning is $\frac{1}{2}$.

Solution

We're dealing with 'fair' coins. A fair coin when tossed gives us one of the two outcomes only after it falls on a surface - either heads or tails. Coins can be both biased and unbiased, but fair coins mean the coin is unbiased. Out of the two outcomes heads or tails - a fair coin will have equal probability for both. One of the following:

$$P(H_{Heads}) = \frac{1}{2}$$
$$P(T_{Tails}) = \frac{1}{2}$$

The term 'simultaneously' here mentioned in the question means that both the coins are flipped at the same time exactly. The coins are mentioned to be also tossed "independently" which means, the two events are completely separate and there is no dependency on the outcome of one event with another event.

Now, if we consider the probability of an event occurring on these coins as $P(X)$, and the probability of another event occurring on these coins as $P(Y)$. Then the probability of both the events occurring together (**AND** condition) will be the $P(X \cap Y)$. According to the rule of joint probability, the probability of event X when event Y has already happened ($P(\frac{X}{Y})$) multiplied by the probability of event Y ($P(Y)$) gives us the probability of both X and Y happening together.

$$P(X \cap Y) = P(Y) \times P(\frac{X}{Y}) \tag{1}$$

But, as events X and Y are independent and there is no dependency of X upon Y, so :

$$P(X) = P(\frac{X}{Y}) \tag{2}$$

Now, we can simply substitute equation 2's values in equation 1 to get the following:

$$P(X \cap Y) = P(Y) \times P(X) \tag{3}$$

Calculating probabilities for events A, B, and C

Our sample space Ω can have four outcomes: {HH, HT, TH, TT}. If both coins have the same outcome then we will consider it as a win, otherwise a loss. We created a table below of the outcomes and, this closely resembles a XNOR logic table. In an XNOR table, like outcomes result in a 'true' or 'win' in our case, and unlike outcomes result in a 'false' or 'loss'.

First Coin	Second Coin	Outcome
H	H	Win
H	T	Lose
T	H	Lose
T	T	Win

For Event A, the first coin will come up as head. So the Event space will be {HH, HT} Probability of Event A is $P(A) = \frac{2}{4} = \frac{1}{2}, P(A) > 0$
For Event B, the second coin comes up as head, So the Event space will be {HH, TH} Probability of Event B is $P(B) = \frac{2}{4} = \frac{1}{2}, P(B) > 0$
For Event C, we are only considering the win conditions. So the Event space will be {HH, TT} Probability of Event C is $P(C) = \frac{2}{4} = \frac{1}{2}, P(C) > 0$

Part (i): Let us check if Events A and B are independent : The probability of Event A and Event B happening together can be represented by $P(A \cap B)$. Which is nothing but, the probability of the first coin having heads **AND** second coin having heads. So, there is only one such outcome out of our 4 possibilities ({HH}). So $P(A \cap B) = \frac{1}{4}$. We need to prove $P(A \cap B) = P(A) \times P(B)$.

$$\begin{aligned} P(A) &= \frac{1}{2} \\ P(B) &= \frac{1}{2} \\ P(A) \times P(B) &= \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{4} \end{aligned}$$

Which is equal to $P(A \cap B)$. So, we can conclude **the Events A and B are independent and the statement is true.**

Part (ii): Let us check if Events A and C are independent : The probability of Event A and Event C happening together can be represented by $P(A \cap C)$. Which is nothing but, the probability of the first coin having heads **AND** having a winning condition according to the question. So, there is only one such outcome out of our 4 possibilities ({HH}) which is both winning and the first coin is heading. So $P(A \cap C)$ $= \frac{1}{4}$. We need to prove $P(A \cap C) = P(A) \times P(C)$.

$$\begin{aligned} P(A) &= \frac{1}{2} \\ P(C) &= \frac{1}{2} \\ P(A) \times P(C) &= \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{4} \end{aligned}$$

Which is equal to $P(A \cap C)$. So, we can conclude **the Events A and C are independent and the statement in the question is false.**

Part (iii): We have already determined the probability of event A **and** event B is $P(A \cap B) = \frac{1}{4}$. Now we have to find the conditional probability of $P(A \cap B)$ when C has already given. Logically, we can say that if the outcome of Event C is given then there are only 2 possibilities {HH, TT}. If the outcome of C is {HH} then it's easy to conclude that - if A is {H} then B also will be {H} and vice versa. The dependence is visible. If the outcome of C is {TT} then we can conclude neither A nor B has a head. So, the conditional probability is not independent. Let's see this mathematically. Given that we win, there are two possibilities {HH, TT}, so both Event A and Event B happen only at { HH}. So $P(\frac{A \cap B}{C}) = \frac{1}{2}$. We know, the formula for conditional independence as:

$$P(\frac{A \cap B}{C}) = P(A/C) \times P(B/C)$$

We already know, the events A and C are independent, so $P(A/C) = P(A) = \frac{1}{2}$. Let's check if event B and event C are independent or not.

The probability of Event B and Event C happening together can be represented by $P(B \cap C)$. Which is nothing but, the probability of the second coin having heads **AND** having a winning condition according to the question. So, there is only one such outcome out of our 4 possibilities ({HH}) which is both winning and the second coin is heading. So $P(B \cap C) = \frac{1}{4}$. We need to prove $P(B \cap C) = P(B) \times P(C)$.

$$\begin{aligned}
P(B) &= \frac{1}{2} \\
P(C) &= \frac{1}{2} \\
P(B) \times P(C) &= \frac{1}{2} \times \frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

Which is equal to $P(B \cap C)$. So, we can conclude **the Events B and C are independent** and so $P(B/C) = P(B) = \frac{1}{2}$

$$P(A/C) \times P(B/C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

which is not equal to $P(\frac{A \cap B}{C})$.
So the events A and B are not conditionally independent given C. This statement is true.

Part (iv): The probability of winning will be 2 conditions among the four conditions. All outcomes are {HH, HT, TH, TT} which we are denoting by ω and the winning outcomes are {HH, TT}. So, the probability is $\frac{2}{4}$ or $\frac{1}{2}$. **So the probability of winning is $\frac{1}{2}$. This statement is true.**

From our analysis, we found that Part (ii) statement which was "Events A and C are not independent" is only false.

Acknowledgments

- I have extensively used browsing, YouTube, Khan Academy, Coursera, books, and lecture notes to do this assignment. I have tried to prove concepts as I have written this assignment, just for my understanding. I tried to make it as elaborate and detailed as possible just for my learning.
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