Time Series Analysis

Quiz I-Solutions

1. Answer is given below:

part (a) Assuming $e[k]$ to be white noise. Given the process

$$v[k] - 0.8v[k-1] + 0.15v[k-2] = e[k] \tag{1}$$

Taking Expectations both the sides

$$\sigma_{vv}[0] - 0.8\sigma_{vv}[1] + 0.15\sigma_{vv}[2] = \sigma_{ee}[0] \tag{2}$$

$$writing \quad \sigma_{vv}[k] \quad as \quad \sigma_k$$

$$\sigma_1 - 0.8\sigma_0 + 0.15\sigma_1 = 0 \tag{3}$$

$$\sigma_2 - 0.8\sigma_1 + 0.15\sigma_0 = 0 \tag{4}$$

$$\tag{5}$$

solving the above equation gives

$$\rho_1 = 0.696 \quad where \quad \rho_1 = \frac{\sigma_1}{\sigma_0}, \quad \text{ACF} \tag{6}$$

$\rho_2 = 0.406$, using these values we get $\sigma_0 = 1.985$

thus we get

$$\rho_0 = 1$$

$$\rho_1 = 0.696$$

$$\rho_2 = 0.406$$

similarly we get ACF for lags 3 and 4

$$\rho_3 = 0.221$$

$$\rho_4 = 0.116$$

2. Answer is given below:

Part A:

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & \text{otherwise} \end{cases}$$

Then

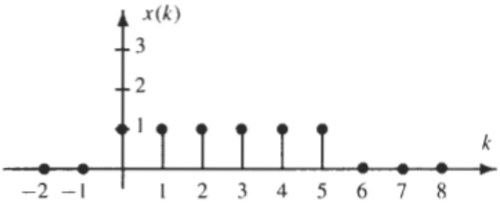
$$\mu_X(t) = \frac{A}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\, d\theta = 0 \tag{5.97}$$

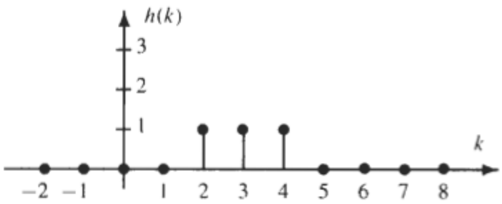
Setting $s = t + \tau$ in Eq. (5.7), we have

$$\begin{aligned} R_{XX}(t, t+\tau) &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\cos[\omega(t+\tau) + \theta]\, d\theta \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}[\cos \omega\tau + \cos(2\omega t + 2\theta + \omega\tau)]\, d\theta \\ &= \frac{A^2}{2}\cos \omega\tau \end{aligned} \tag{5.98}$$

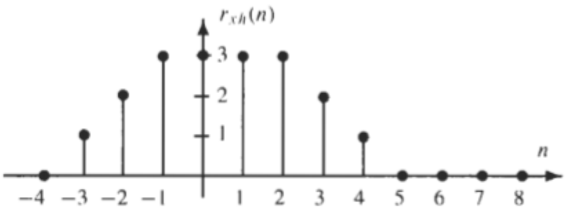
Since the mean of $X(t)$ is a constant and the autocorrelation of $X(t)$ is a function of time difference only, we conclude that $X(t)$ is WSS.

Part B (a)





Denoting the correlation by $r_{xh}(n)$, it is clear that for $n = 0$ the correlation is equal to 3. In fact, this will be the value of $r_{xh}(n)$ for $-1 \leq n \leq 2$. For $n = 3$, $x(k)$ and $h(3+k)$ only overlap at two points, and $r_{xh}(3) = 2$. Similarly, because $x(k)$ and $h(4+k)$ only overlap at one point, $r_{xh}(4) = 1$. Finally, $r_{xh}(n) = 0$ for $n > 4$. Proceeding in a similar fashion for $n < 0$, we find that $r_{xh}(-2) = 2$, and $r_{xh}(-3) = 1$. The correlation is shown in the figure below.



(b)

$$r_x(n) = x(n) \star x(n) = x(n) * x(-n) = \sum_{k=-\infty}^{\infty} x(k)x(n+k)$$

In addition observe that $r_x(n)$ is an even function of n:

$$r_x(-n) = \sum_{k=-\infty}^{\infty} x(k)x(-n+k) = \sum_{k'=-\infty}^{\infty} x(k'+n)x(k') = r_x(n)$$

Therefore, it is only necessary to find the values of $r_x(n)$ for $n \geq 0$. For $n \geq 0$, we have

$$r_x(n) = \sum_{k=-\infty}^{\infty} \alpha^k u(k)\alpha^{n+k}u(n+k) = \alpha^n \sum_{k=0}^{\infty} \alpha^{2k} = \frac{1}{1-\alpha^2}\alpha^n \qquad n \geq 0$$

Using the symmetry of $r_x(n)$, we have, for $n < 0$,

$$r_x(n) = \frac{1}{1-\alpha^2}\alpha^{-n} \qquad n \leq 0$$

Combining these two results together, we finally have

$$r_x(n) = \frac{1}{1-\alpha^2}\alpha^{|n|}$$

3) Be judicious. The answer is from lecture recordings and presentation.

4). Answer is given below:

Part A

A continuous-time sinusoid

$$x_a(t) = \cos(\Omega_0 t) = \cos(2\pi f_0 t)$$

that is sampled with a sampling frequency of f_s results in the discrete-time sequence

$$x(n) = x_a(nT_s) = \cos\left(2\pi \frac{f_0}{f_s} n\right)$$

However, note that for any integer k,

$$\cos\left(2\pi \frac{f_0}{f_s} n\right) = \cos\left(2\pi \frac{f_0 + kf_s}{f_s} n\right)$$

Therefore, any sinusoid with a frequency

$$f = f_0 + kf_s$$

will produce the same sequence when sampled with a sampling frequency f_s. With $x(n) = \cos(n\pi/8)$, we want

$$2\pi \frac{f_0}{f_s} = \frac{\pi}{8}$$

or

$$f_0 = \tfrac{1}{16} f_s = 625 \text{ Hz}$$

Therefore, two signals that produce the given sequence are

$$x_1(t) = \cos(1250\pi t)$$

and

$$x_2(t) = \cos(21250\pi t)$$

Part B

(*a*) The Nyquist rate is equal to twice the highest frequency in $x_a(t)$. If

$$y_a(t) = \frac{dx_a(t)}{dt}$$

then

$$Y_a(j\Omega) = j\Omega X_a(j\Omega)$$

Thus, if $X_a(j\Omega) = 0$ for $|\Omega| > \Omega_0$, the same will be true for $Y_a(j\Omega)$. Therefore, the Nyquist frequency is not changed by differentiation.

(*b*) The signal $y_a(t) = x_a(2t)$ is formed from $x_a(t)$ by *compressing* the time axis by a factor of 2. This results in an *expansion* of the frequency axis by a factor of 2. Specifically, note that

$$Y_a(j\Omega) = \int_{-\infty}^{\infty} y_a(t)e^{-j\Omega t}\,dt = \int_{-\infty}^{\infty} x_a(2t)e^{-j\Omega t}\,dt$$
$$= \int_{-\infty}^{\infty} \tfrac{1}{2}x_a(\tau)e^{-j\Omega\tau/2}\,d\tau = \tfrac{1}{2}X_a\left(\frac{j\Omega}{2}\right)$$

Consequently, if the Nyquist frequency for $x_a(t)$ is Ω_s, the Nyquist frequency for $y_a(t)$ will be $2\Omega_s$.

(*c*) When two signals are multiplied, their Fourier transforms are convolved. Therefore, if

$$y_a(t) = x_a^2(t)$$

then

$$Y_a(j\Omega) = \frac{1}{2\pi}X_a(j\Omega) * X_a(j\Omega)$$

Thus, the highest frequency in $y_a(t)$ will be twice that of $x_a(t)$, and the Nyquist frequency will be $2\Omega_s$.

(*d*) Modulating a signal by $\cos(\Omega_0 t)$ shifts the spectrum of $x_a(t)$ up and down by Ω_0. Therefore, the Nyquist frequency for $y_a(t) = \cos(\Omega_0 t)x_a(t)$ will be $\Omega_s + 2\Omega_0$.

5. Answer is given below:

Part A:

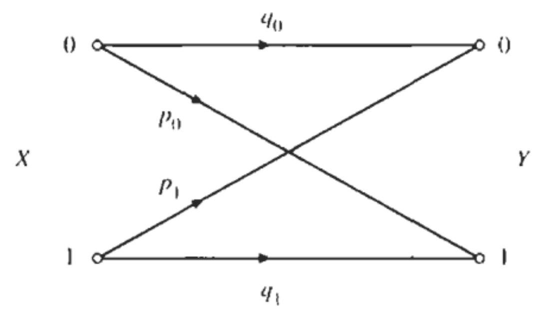


Fig. 1-15

(*a*) We note that

$$P(x_1) = 1 - P(x_0) = 1 - 0.5 = 0.5$$
$$P(y_0 \mid x_0) = q_0 = 1 - p_0 = 1 - 0.1 = 0.9$$
$$P(y_1 \mid x_1) = q_1 = 1 - p_1 = 1 - 0.2 = 0.8$$

$$P(y_0) = P(y_0 \mid x_0)P(x_0) + P(y_0 \mid x_1)P(x_1) = 0.9(0.5) + 0.2(0.5) = 0.55$$
$$P(y_1) = P(y_1 \mid x_0)P(x_0) + P(y_1 \mid x_1)P(x_1) = 0.1(0.5) + 0.8(0.5) = 0.45$$

(*b*) Using Bayes' rule (*1.42*), we have

$$P(x_0 \mid y_0) = \frac{P(x_0)P(y_0 \mid x_0)}{P(y_0)} = \frac{(0.5)(0.9)}{0.55} = 0.818$$

(*c*) Similarly,

$$P(x_1 \mid y_1) = \frac{P(x_1)P(y_1 \mid x_1)}{P(y_1)} = \frac{(0.5)(0.8)}{0.45} = 0.889$$

(*d*) The probability of error is

$$P_e = P(y_1 \mid x_0)P(x_0) + P(y_0 \mid x_1)P(x_1) = 0.1(0.5) + 0.2(0.5) = 0.15.$$

Part B:

(*a*) From the results of Prob. 1.52, we found that

$$P(X = 1) = 1 - P(X = 0) = 0.5$$
$$P(Y = 0 \mid X = 0) = 0.9 \qquad P(Y = 1 \mid X = 1) = 0.8$$

Then by Eq. (*1.41*), we obtain

$$P(X = 0,\ Y = 0) = P(Y = 0 \mid X = 0)P(X = 0) = 0.9(0.5) = 0.45$$
$$P(X = 0,\ Y = 1) = P(Y = 1 \mid X = 0)P(X = 0) = 0.1(0.5) = 0.05$$
$$P(X = 1,\ Y = 0) = P(Y = 0 \mid X = 1)P(X = 1) = 0.2(0.5) = 0.1$$
$$P(X = 1,\ Y = 1) = P(Y = 1 \mid X = 1)P(X = 1) = 0.8(0.5) = 0.4$$

Hence, the joint pmf's of $(X,\ Y)$ are

$$p_{XY}(0,\ 0) = 0.45 \qquad p_{XY}(0,\ 1) = 0.05$$
$$p_{XY}(1,\ 0) = 0.1 \qquad p_{XY}(1,\ 1) = 0.4$$

(*b*) By Eq. (*3.20*), the marginal pmf's of X are

$$p_X(0) = \sum_{y_j} p_{XY}(0,\ y_j) = 0.45 + 0.05 = 0.5$$
$$p_X(1) = \sum_{y_j} p_{XY}(1,\ y_j) = 0.1 + 0.4 = 0.5$$

By Eq. (*3.21*), the marginal pmf's of Y are

$$p_Y(0) = \sum_{x_i} p_{XY}(x_i,\ 0) = 0.45 + 0.1 = 0.55$$
$$p_Y(1) = \sum_{x_i} p_{XY}(x_i,\ 1) = 0.05 + 0.4 = 0.45$$

(*c*) Now

$$p_X(0)p_Y(0) = 0.5(0.55) = 0.275 \neq p_{XY}(0,\ 0) = 0.45$$

Hence X and Y are not independent.