1 a) $f(x) = \sin x - 1/2$

by taylor series

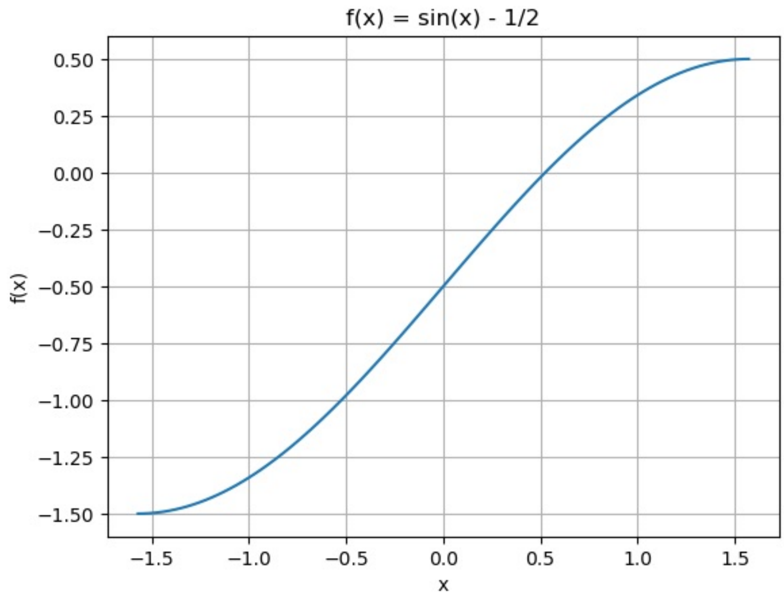
$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots$$

$f(0) = -1/2$; $f'(0) = 1$; $f''(0) = 0$

$f'''(0) = -1$

$$f(x) = \frac{-1}{2} + (x-a) - \frac{(x-a)^3}{3!} \dots$$

b)



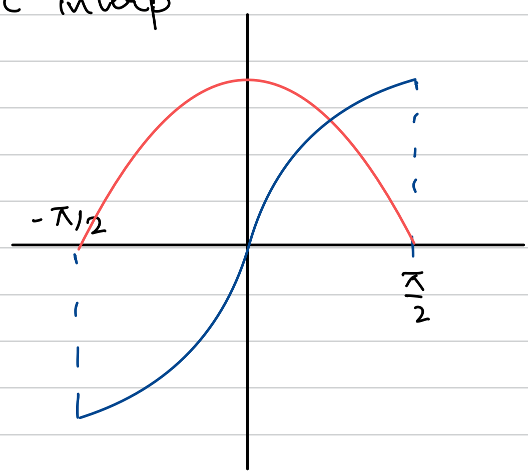
c) $f(x) = \sin x - 1/2$

Lets solve for $g(x) = \sin x$ and add $-1/2$

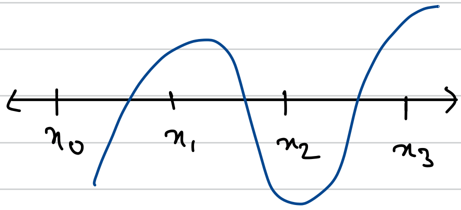
Since we have a quadratic interp^n

we need x_0, x_1, x_2, x_3

where $e(x_i)$ change signs



$h(x) = \sin x - (ax^2 + bx + c)$ has 3 roots in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$h'(x) = \cos x - 2ax + b$

since $h'(x)$ is non zero at x_0, x_3,
max error at the boundary

$x_0 = -\frac{\pi}{2}$, $x_3 = \frac{\pi}{2}$

What is x_1, x_2

$e(-\frac{\pi}{2}) = -1 - (a\frac{\pi^2}{4} - b\frac{\pi}{2} + c)$

$e(\frac{\pi}{2}) = 1 - (a\frac{\pi^2}{4} + b\frac{\pi}{2} + c)$

$e(-\frac{\pi}{2}) = -e(\frac{\pi}{2})$

$\Rightarrow \cancel{-1} - (a\frac{\pi^2}{4} - \cancel{b\frac{\pi}{2}} + c) = \cancel{-1} + (a\frac{\pi^2}{4} + \cancel{b\frac{\pi}{2}} + c)$

$\Rightarrow a\frac{\pi^2}{2} + 2c = 0$ (well this is not useful)

$c = -a\frac{\pi^2}{4}$

if $c = -a\frac{\pi^2}{4}$ can $h(x)$ have 3 roots

by symetry there are 2 sol^n since $\underline{\sin = -\sin(-x)}$

$ax^2 + bx + c$; $-ax^2 + bx - c$

Since $\sin(x)$ can have only one poly in Π_2

$ax^2 + bx + c = -ax^2 + bx - c$

$\Rightarrow ax^2 + c = 0$ iff $a = c = 0$

hence $g(x) = \underline{\underline{bx}}$ yay !!

$e(-\frac{\pi}{2}) = -1 + b\frac{\pi}{2}$

$e(\frac{\pi}{2}) = 1 - b\frac{\pi}{2}$

$e(x_1) = \sin(x_1) - bx_1$

$e(x_2) = \sin(x_2) - bx_2$

$\frac{de}{dx} = 0$ at x_1, x_2

$\frac{de}{dx} = \cos x - b \rightarrow x = \cos^{-1}(b)$

this has symmetric sol^n

$x_1 = -x_2$

$e(-\frac{\pi}{2}) = -e(x_1)$

$-1 + b\frac{\pi}{2} = -(\sin x - bx)$

and $b = \cos x$

$\sin x - x\cos x + \frac{\pi}{2}\cos x - 1 = 0$

On Solving for $x \in [-\frac{\pi}{2}, 0]$ we get

$x_1 = -0.7603$; $x_2 = 0.7603$

$x_0 = -\pi/2$; $x_3 = \frac{\pi}{2}$

$b = \cos(x_1) = 0.724$

Since we shifted by 0.5 ; we sift $g(x) \rightarrow g(x) - \frac{1}{2}$

hence

$$\sin x - \frac{1}{2} = 0.724\,x - \frac{1}{2}$$

$$L_\infty = e\left(\frac{\pi}{2}\right) = 0.138$$

$$L_2 = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sin x - \tfrac{1}{2} - (0.724x - \tfrac{1}{2})\right)^2 dx \right)^{1/2}$$

$$= 0.17$$

d) we solve for $Pc = q$

where $[\langle \phi_i, \phi_j \rangle]_{i,j} = P$

$c = [c_i]$ i.e coeff of ϕ_i

our polynomial is $\sum c_i \phi_i(x)$

we approximate $\langle \phi_i, \phi_j \rangle = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi_i(x)\,\phi_j(x)\,dx$

as a discrete sum on 10^3 pts

$$q = \left[\sum_i f(x_i)\,\phi_e(x_i) \right]_e$$

$$c = P^{-1} q$$

code provided

$c_0 = -0.5$; $c_1 = 0.7739$; $c_2 = 3.4 \times 10^{-16}$

$L_2 = 0.085$ $\quad L_\infty = 0.215$

2 $$p(x) = \sum_{i=0}^{\#poly} a_i x^i + \sum_{i=0}^{\#sin} b_i \sin(i\pi x) + \sum_{i=0}^{\#cos} c_i \cos(i\pi x)$$

$\underset{\alpha}{\#poly} = \underset{\beta}{\#sin} = \underset{\gamma}{\#cos} = 3$ (init)

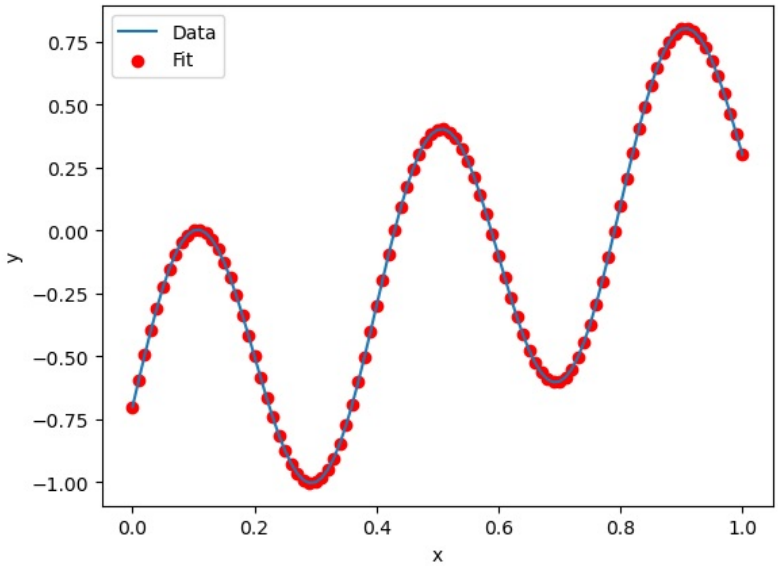
$$P = [\langle \phi_i , \phi_j \rangle]_{ij}$$

$$\phi_i = \begin{cases} x^i & 0 \le i < \alpha \\ \sin i\pi x & \alpha \le i < \alpha+\beta \\ \cos i\pi x & \alpha+\beta \le i < \alpha+\beta+\gamma \end{cases}$$

$$q = [\langle f, \phi_i \rangle]_i$$

Solve $Pc = q$ where c = coeff

we get the coeff and plot



given a good enough fit we filter all coeff below 10^{-2}

hence $p(x) \approx x + 0.6 \sin 5\pi x - 0.7$

3 a) $T_n(\cos(\theta)) = \cos n\theta$

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

$$T_0(x) = 1$$

$$T_1(\cos\theta) = \cos\theta \implies T_1(x) = x$$

$$T_2(x) = 2x\,T_1(x) - T_0(x)$$
$$= 2x^2 - 1$$

$$T_3(x) = 2x\,T_2(x) - T_1(x)$$
$$= 2x(2x^2-1) - x$$
$$= 4x^3 - 3x$$

$$T_4(x) = 2x\,T_3(x) - T_2(x)$$
$$= 2x(4x^3 - 3x) - (2x^2-1)$$
$$= 8x^4 - 6x^2 - 2x^2 + 1$$
$$= 8x^4 - 8x^2 + 1$$

$$T_5(x) = 2x\,T_4(x) - T_3(x)$$
$$= 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x)$$
$$= 16x^5 - 16x^3 + 2x - 4x^3 + 3x$$
$$= 16x^5 - 20x^3 + 5x$$

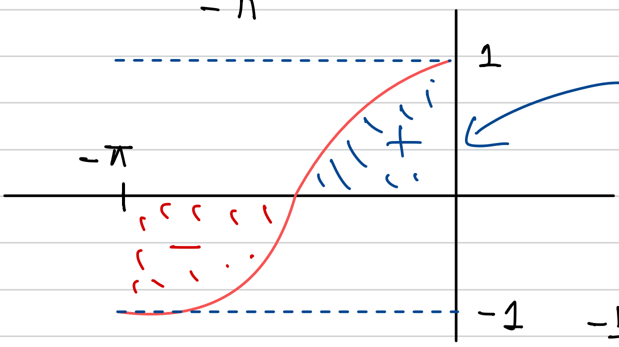
b)
$$\int_{-1}^{1} (1-x^2)^{-1/2} \, T_4(x) \, T_5(x) \, dx$$

$$x = \cos\theta \quad ; \quad dx = -\sin\theta \, d\theta$$

$$= \int_{-\pi}^{0} \frac{1}{\cancel{\sin\theta}} \, T_4(\cos\theta) \, T_5(\cos\theta) \, (-\cancel{\sin\theta} \, d\theta)$$

$$= \int_{-\pi}^{0} -\cos 4\theta \cdot \cos 5\theta \, d\theta$$

$$= -\frac{1}{2} \int_{-\pi}^{0} (\underbrace{\cos 9\theta}_{I_1} + \underbrace{\cos\theta}_{I_2}) \, d\theta$$



$$-\frac{1}{2} \int_{-\pi}^{0} \cos\theta \, d\theta = 0$$

$$I_2 = 0$$

$$\theta = t/9$$

$$\underbrace{-\frac{1}{2} \int_{-9\pi}^{0} \cos t \cdot \frac{1}{9} \, dt}_{I_1}$$

$$I_1 = -\frac{1}{18} \int_{-9\pi}^{0} \cos t \, dt = -\frac{1}{18} \int_{-4\cdot(2\pi)}^{0} \cancel{\cos t \, dt} \quad = 0 \quad -\frac{1}{18} \int_{-9\pi}^{-8\pi} \cos t \, dt$$

$$I_1 = -\frac{1}{18} \int_{-9\pi}^{-8\pi} \cos t \, dt \xrightarrow[\text{N. Time period preserves } \int]{\text{Sift by}} -\frac{1}{18} \int_{-\pi}^{0} \cos t \, dt = 0$$

Hence $T_4(x)$ and $T_5(x)$ are $\perp$

c) $\langle T_n, T_n \rangle = \int_{-1}^{1} (1-x^2)^{-1/2} T_n(x) T_n(x)\, dx$

using $x = \cos\theta$; $dx = -\sin\theta \cos\theta$

$$= \int_{-\pi}^{0} + \frac{1}{\cancel{\sin\theta}} (-\cancel{\sin\theta}) \cos^2 n\theta \, d\theta$$

$$= \frac{-1}{2} \int_{-\pi}^{0} (\cos 2n\theta - 1)\, d\theta$$

0

$$= \pi/2$$

d) $\langle T_i, T_j \rangle = h_{ij}$ if $i \neq j$ (assume $i > j$)

using part b), c)

$\langle T_i, T_j \rangle$ after $x \to \cos\theta$

and using $\cos A \cos B = +\frac{1}{2}(\cos A+B + \cos A-B)$

$$h_{ij} = -\frac{1}{2} \int_{-\pi}^{0} (\cos (i+j)\theta \, d\theta + -\frac{1}{2} \int_{-\pi}^{0} \cos (i-j)\theta \, d\theta$$

$(i+j)\theta \to m$ $\qquad$ $(i-j)\theta \to n$

$$h_{ij} = \frac{-1}{2(i+j)} \int_{-\pi(i+j)}^{0} \cos m \, dm \quad \frac{-1}{2(i-j)} \int \cos n \, dn$$

we know $\int_{-l\pi}^{0} \cos . x \, dx = 0 \quad l \in \mathbb{N}$

hence $h_{iz} = 0 + 0 = 0$

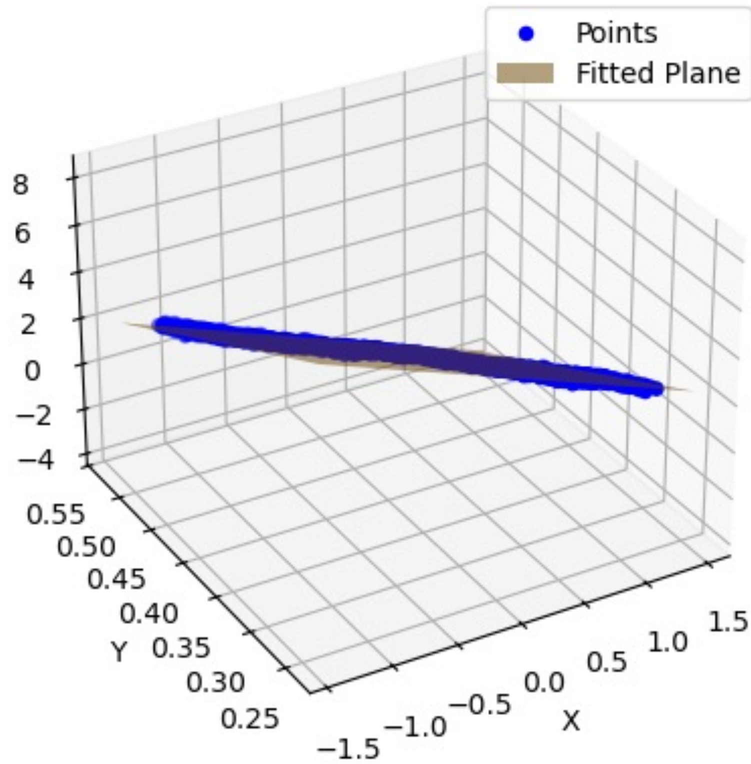
4 a) Given pts $[(x_i, y_i, z_i)]_i = A$

$A' = [(x_i, y_i, z_i, 1)]_i$

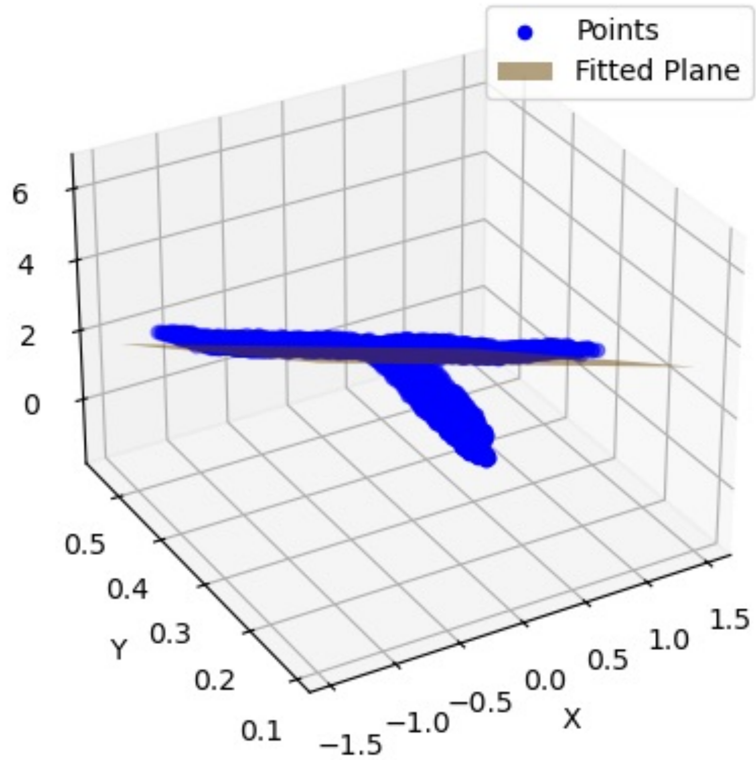
$AC = 0$ where $C = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

Such that $ax + by + cz + d = 0$

$AC = 0$; first 2 singular vector are in plane, 3^{rd} one is normal

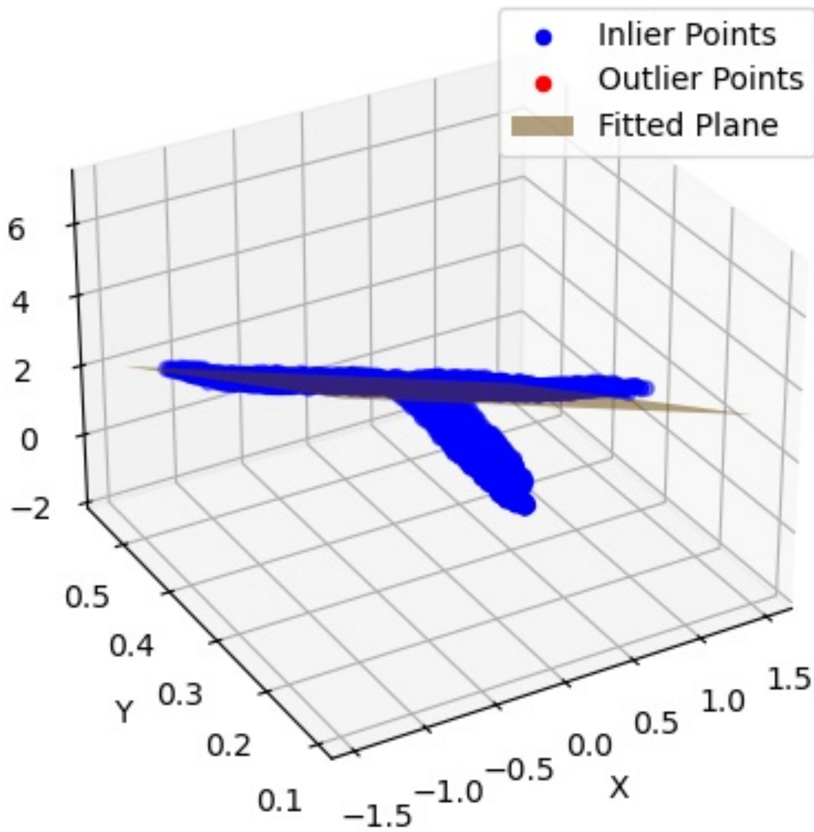


b)



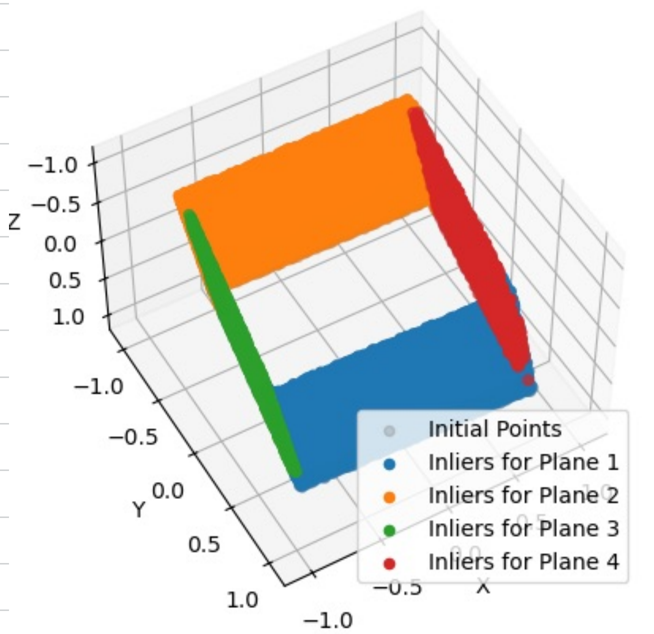
c) we apply RANSAC and find the plane where max pts are inliers

details in code:

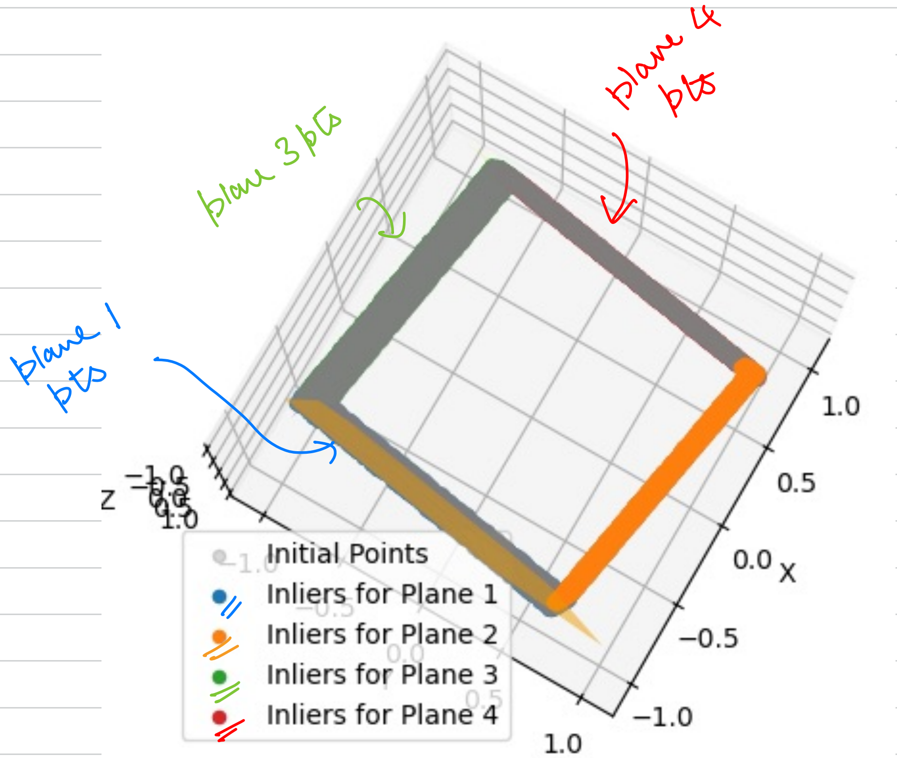


d) We apply RANSAC, and fit one plane. Then remove inliers and Redo RANSAC plane fit.

We do it a total 4 times



Note: I rescaled points to [-1, 1] and centered at origin for easier visualization.



e) the lower the singular value corr to the normal, tells us the roughness

The more variation along the $\perp$, higher will be $\Sigma_{3,3}$

-> so we apply RANSAC- plane fit

-> find $\Sigma_{3,3}$ corr to $\hat{n}$ of the plane

-> plot the plane w/ lowest $\Sigma_{3,3}$