Contents

Section 1. Munkres Chapter 2

12. Topological space

1.1 Definition: Topology

A ***topology*** on a set X is a set of $\mathcal{T}$ of subsets of X having the following properties:

(a) $\varnothing, X \in \mathcal{T}$.

(b) The union of any subset of $\mathcal{T}$ is in $\mathcal{T}$.

(c) The intersection of any finite subset of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a ***topological space***. We say $U \subset X$ is an ***open set*** if $U \subset \mathcal{T}$.

Example 1.2 A topology $\mathcal{T} = \{\varnothing, X\}$ is called the ***indiscrete topology*** or ***trivial topology***. On the other hand, if $\mathcal{T} = P(X)$, then it is called the ***discrete topology***

Example 1.3 Let X ba a set, $\mathcal{T}_f$ be a set of all subset $U \subset X$ such that $X \backslash U$ either is finite or is all of X. Then

(a) $X \backslash X$ is finite and $X \backslash \varnothing$ is X.

(b) Suppose $\{U_\alpha\}$ is an indexed family of nonempty elements of $\mathcal{T}_f$. Then $X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$ is finite.

(c) $X - \bigcap_{i=1}^n = \bigcup_{i=1}^n (X - U_\alpha)$ is finite.

Therefore, $\mathcal{T}_f$ is a topology, called ***finite complement topology***. We can replace the condition 'finite' with 'countable', and the proposition still holds.

1.4 Definition

Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$, then we say that $\mathcal{T}'$ is ***finer(larger)*** than $\mathcal{T}$, and $\mathcal{T}$ is ***coarser(smaller)*** than $\mathcal{T}'$. Also, we say that $\mathcal{T}$ is ***comparable*** with $\mathcal{T}'$ if either $\mathcal{T} \supset \mathcal{T}'$ or $\mathcal{T}' \supset \mathcal{T}$.

13. Basis for a topology

1.5 Definition

If X is a set, a ***basis*** for a topology on X is a collection $\mathcal{B}$ of subsets of X(called a ***basis elements***) such that

(a) For each $x \in X$, there is at least one basis element B containing x.

(b) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$. We deine the ***topology*** $\mathcal{T}$ ***generated by*** $\mathcal{B}$ as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Example 1.6 Consider the set of all rectangular regions in $\mathbb{R}^2$ without border, where each rectangular have sides parallel to the coordinate axes.

1.7 Lemma

A topology is equal to the set of all union of its basis elements.

Munkres lemma 13.1 □

1.8 Lemma

Let X be a topological space. Suppose that $\mathcal{C}$ is a set of open sets of X such that for each $U \underset{\text{open}}{\subset} X$ and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Munkres lemma 13.2 □

1.9 Lemma

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. TFAE:

(a) $\mathcal{T}'$ is finer than $\mathcal{T}$.

(b) For each $x \in X$ and each $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Munkres lemma 13.3 $\square$

1.10 Definition

In $\mathbb{R}$,

(a) The set of all open interval is a basis of the ***standard topology*** on $\mathbb{R}$.

(b) The set of all half-open($[a, b)$) intervals is a basis of the ***lower limit topology*** on $\mathbb{R}$, denoted by $\mathbb{R}_l$.

(c) Let $K = \{1/n \mid n \in \mathbb{Z}\}$. The set of all open interval without the points in K is a basis of the ***K-topology*** on $\mathbb{R}$, denoted by $\mathbb{R}_K$.

1.11 Lemma

$\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

Munkres lemma 13.4 $\square$

1.12 Definition

A ***subbasis*** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The ***topology generated by the subbasis*** $\mathcal{S}$ is defined to be the set $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

14. The order topology

1.13 Definition

If X is a set with the relation $<$, the topology derived from the basis $\mathcal{B}$ of subset of X such that

(a) All open intervals (a, b) in X;

(b) all intervals of the form $[a_0, b)$ in X(if the smallest element a_0 exists);

(c) all intervals of the form $[a, b_0]$ in X(if the largest element b_0 exists),

is called the ***order topology***.

Example 1.14 The standard topology on $\mathbb{R}$ is a order topology.

Example 1.15 The order topology on $\mathbb{Z}^+$ is the discrete topology.

Example 1.16 The basis elements for order topology of the set $\mathbb{R}\times\mathbb{R}$ in the dictionary order is of the form $(a \times b, c \times d)$ for $a < c$ or $a = c$ and $b < d$.

Example 1.17 The order topology of the set $X = \{1, 2\} \times \mathbb{Z}^+$ in the dictionary order is not the discrete topology. Consider the basis element containing $(2, 1)$.

1.18 Definition

Suppose X is an ordered set, and $a \in X$. There are four subsets of X:

(a) $(a, +\infty) = \{x \mid x > a\}$.

(b) $(-\infty, a) = \{x \mid x < a\}$.

(c) $[a, +\infty] = \{x \mid x \geq a\}$.

(d) $(-\infty, a) = \{x \mid x \leq a\}$.

They are called the ***rays*** determined by a. Sets of the first two types are called ***open rays***, and sets of the last two types are called ***closed rays***. The open rays form a subbasis for the order topology on X.

15. The product topology on $X \times Y$

1.19 Definition

Let X, Y be topological spaces. The ***product topology*** on $X \times Y$ is the topology having as basis the set $\mathcal{B}$ of all sets of the form $U \times V$, where $U \underset{\text{open}}{\subset} X$ and $V \underset{\text{open}}{\subset} Y$.

Remark $\mathcal{B}$ is not a topology on $X \times Y$.

1.20 Theorem

If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the set $\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$.

Munkres theorem 15.1 $\square$

1.21 Definition

Define $\pi_1 : X \times Y \to X$ by the equation $\pi_1(x, y) = x$, and define $\pi_2 : X \times Y \to Y$ by $\pi_2(x, y) = y$. The maps are called the ***projection*** of $X \times Y$ onto its first and second factors, respectively.

1.22 Theorem

$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \underset{\text{open}}{\subset} X\} \cup \{\pi_2^{-1}(V) \mid V \underset{\text{open}}{\subset} Y\}$ is a subbasis for the product topology on $X \times Y$.

Munkres theorem 15.2 $\square$

1.23 Definition

Let X be a topological space with a topology $\mathcal{T}$. If $Y \subset X$, $\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$ is a topology on Y, called the ***subspace topology***. With this topology, Y is called a subspace of X.

1.24 Lemma

If $\mathcal{B}$ is a basis for the topology on X, then the set $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.

Munkres lemma 16.1 □

We say $U \subset Y$ is ***open in(or open relative to)***Y if $U \subset \mathcal{T}_Y$.

1.25 Lemma

Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof) Munkres lemma 16.2 □

1.26 Theorem

Suppose A is a subspace of X and B is a subspace of Y. Then the product topology of $A \times B$ is same as the subspace topology of $A \times B$.

Proof) Munkres lemma 16.3 □

Example 1.27 Suppose $Y = [0, 1]$ is a subspace of $\mathbb{R}$. Then the subspace topology of Y and the order topology on $[0, 1]$ are same.

Example 1.28 Let $Y = [0, 1) \cup \{2\}$. If Y is a subspace of $\mathbb{R}$, then $\{2\}$ is open in the subspace topology on Y(since $(3/2, 5/2) \cap \{2\} = \{2\}$). But $\{2\}$ is not open in the order topology on Y(since $\{2\}$ must be in the set of the form $(a, 2]$).

Example 1.29 Let $I = [0, 1]$ and $I \times I$ be of the dictionary order on $\mathbb{R} \times \mathbb{R}$. Then its subspace topology and its order topology are not same. Consider the set $\{1/2\} \times (1/2, 1]$, which is open in $I \times I$ in the subspace topology, but not in order topology.

1.30 Theorem

Let X be an ordered set in the order topology, and let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

Munkres theorem 16.4 □

17. Closed sets and limit points

1.31 Definition

Let X be a topological space with topology $\mathcal{T}$. By a ***closed*** set, we mean a subset A of X that is $A = X - B$ for some $B \in \mathcal{T}$.

Example 1.32 In the discrete topology on the set X, every set is open and closed.

Example 1.33 Consider the set $Y = [0, 1] \cup (2, 3)$. Both interval $[0, 1]$ and $(2, 3)$ are open and closed.

1.34 Theorem

Let X be a topological space. Then

(a) $\varnothing$ and X are closed.

(b) Arbitrary intersections of closed sets are closed.

(c) Finite unions of closed sets are closed.

Proof) Clear by the definition of open set and DeMorgan's law. □

1.35 Theorem

Let Y be a subspace of X. Then a set A is ***closed in*** Y if and only if it equals the intersection of a closed set of X with Y.

Proof) Munkres theorem 17.2 □

1.36 Proposition

Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

1.37 Definition

Given a subset A of a topological space X, the ***interior***(denoted by Int$A$) of A is defined as the union of all open sets contained in A, and the ***closure***(denoted by ClA or $\overline{A}$) of A is defined as the intersection of all closed sets containing A.

1.38 Theorem

Let Y be a subspace of X, let A be a subset of Y, let $\overline{A}$ denote the closure of A in X. Then the closure of A in Y equals $\overline{A} \cap Y$.

Proof) Munkres theorem 17.5 □

A ***intersects*** B if $A \cap B \neq \varnothing$.

1.39 Theorem

Let A be a subset of the topological space X.

(a) $x \in \overline{A} \iff$ every open set U containing x intersects A.

(b) $x \in \overline{A} \iff$ every basis element B containing x intersects A.

Proof) Munkres p96 □

We can shorten the statement "U is an open set containing x" to the phrase "U is a ***neighborhood*** of x".

1.40 Definition

Let X be a topological space, and suppose $x \in A \subset X$. Then x is called a ***limit(or cluster or accumulation) point*** of A if every neighborhood of x intersects A in some points other than x. A' denote the set of all limit points of A.

1.41 Theorem

Let X be a topological space, and suppose $A \subset X$. Then $\overline{A} = A \cup A'$.

Proof) Munkres theorem 17.6 □

1.42 Corollary

Let X be a topological space, and suppose $A \subset X$. Then A is closed $\iff A' \subset A$.

Proof) Munkres corollary 17.7 □

1.43 Definition

If X is a topological space, a sequence of points of X is called ***converge*** to $x \in X$ if for each neighborhood U of x, there exists a positivie integer N such that $x_n \in U$ whenever $n \geq N$.

Remark In general topological space,

(a) A one point set in the space need not be closed.

(b) A sequence of points of the space can converge to more than one point.

1.44 Definition

A topological space X is called a ***Hausdorff space*** if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1, and U_2 of x_2, respectively, that are disjoint.

1.45 Theorem

Every finite point set in a Hausdorff space X is closed.

Proof) Munkres theorem 17.8 □

1.46 Theorem

Let X be a space satisfying the T_1 axiom(that is, finite set in X closed); let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Proof) Munkres theorem 17.9 □

1.47 Theorem

If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.

Proof) Munkres theorem 17.10 □

1.48 Definition

If a sequence x_n of points of the Hausdorff space X converges to the point x of X, we write $x_n \to x$, and we say that x is the ***limit*** of th sequence x_n.

1.49 Theorem

(a) Every simply ordered set is a Hausdorff set space in the order topology.

(b) The product of two Hausdorff spaces is a Hausdorff space.

(c) A subspace of a Hausdorff space is a Hausdorff space

Proof) Exercise. □

22. The quotient topology

1.50 Definition

Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. p is said to be a ***quotient map*** if

$$U \underset{\text{open}}{\subset} Y \iff p^{-1}(U) \underset{\text{open}}{\subset} X.$$

1.51 Definition

Let X and Y be topological spaces; let $p : X \to Y$ be surjective. A subset $C \subset X$ is ***saturated*** if
$$f^{-1}f(C) = C$$

1.52 Proposition

TFAE:

(a) $p : X \to Y$ is a quotient map.

(b) p is continuous and maps saturated open sets of X to open sets of Y.

(c) p is continuous and maps saturated closed sets of X to closed sets of Y.

1.53 Definition

A map f from a space to a space is ***open(closed) map*** if f maps each open(closed) sets in domain to open(closed) sets in codomain.

1.54 Proposition

If $p : X \to Y$ is a surjective continuous map that is either open or closed, then p is a quotient map.

1.55 Definition

Let X be a space, A a set, and $p : X \to A$ surjective map. There exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map. It is called the ***quotient topology*** induced by p. $\mathcal{T} = \{U \in P(A) : p^{-1}(A) \underset{\text{open}}{\subset} X\}$.

1.56 Definition

Let

(a) X be a topological space;

(b) X^* be a partition of X into disjoint subsets whose uinon is X;

(c) $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it.

In the quotient topology induced by p, the space X^* is called a ***quotient space*** of X.

This can be viewed from a different perspective. For $U \subset X^*$, $p^{-1}(U)$ is a collection of equivalence classes whose union is open in X.

1.57 Theorem

Let $p : X \to Y$ be a quotient map; let A be a subspace of X that is saturated with respect to p; let $q : A \to p(A)$ be the map obtained by restricting p.

(a) If A is either open or closed in X, then q is quotient map.

(b) If p is either an open map or a closed map, then q is a quotient map.

Proof) Munkres p140 $\square$

Section 2. Munkres Chapter 3

24. Connected subspace of the real line

2.1 Theorem: Intermediate value theorem

Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If $f(a) < r < f(b)$, there exists a point $c \in X$ such that $f(c) = r$.

Proof) Munkres p154 □

25. Components and local connectedness

2.2 Definition

Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace(or path) X containing both x and y. The equivalence classes are called the ***components(or path components)*** of X.

2.3 Theorem

The (path) components of X are (path) connnected disjoint subspaces of X whose union is X such that each nonempty (path) connected subspace of X itersects only one of them.

Proof) Munkres p159,p160 □

2.4 Definition

A space X is said to be ***locally (path) connected at x*** if for every neighborhood U of x, there is a (path) connected neighborhood V of x contained in U.

2.5 Theorem

A space x is locally (path) connected if and only if for every open set U of X, each (path) components of U is open in X.

Proof) Munkres 161 □

2.6 Theorem

If X is a topological space, each path components of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.

Section 3. Munkres Chapter 9

51. Homotopy of paths

3.1 Definition

By the ***unit interval*** I, we mean the closed interval $[0,1]$.

3.2 Theorem

The unit interval has the following properties:

(a) I is a complete metric space. The metric on I is given by $d(x,y) = |x-y|$.

(b) I is compact, contractible, path connected, and locally path connected.

3.3 Definition: Homotopic

Let X, Y be topological spaces and $f, f' : X \to Y$ continuous. We say that f is ***homotopic*** to f' if there is a continuous map $F : X \times I \to Y$ such that

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = f'(x)$$

for each x. The map F is called a ***homotopy*** between f and f', denoted by $f \simeq f'$. In this case, if f' is constant, we say that f is ***nulhomotopic***.

3.4 Definition: Path

If a map $f : [0,1] \to X$ is continuous, $f(0) = x_0$, and $f(1) = x_1$, then we say f is a ***path*** in X from initial point x_0 to final point x_1.

3.5 Definition: Path homotopic

Two path $f, f' : [0,1] = I \to X$ are said to be ***path homotopic*** if thay have the same initial point x_0 and the same point x_1, and if there is a continuous map $F : I \times I \to x$ such that

$$F(s,0) = f(s) \quad \text{and} \quad F(s,1) = f'(s),$$

$$F(0,t) = x_0 \quad \text{and} \quad F(1,t) = x_1,$$

for each $s \in I$ and each $t \in I$. We call F a ***path homotopy*** between f and f', denoted by $f \simeq_p f'$.

3.6 Proposition

$\simeq$ and $\simeq_p$ are equivalence relations.

Proof)

(a) (reflexive) $F(x,t) = f(x)$.

(b) (symetric) $G(x,t) = F(x, 1-t)$.

(c) (transitive) $G(x,t) = \begin{cases} F(x,2t) & \text{for } t \in [0, \frac{1}{2}] \\ F'(x, 2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$. By pasting lemma, G is continuous.

□

Example 3.7 Let f, g be any map of a space X into $\mathbb{R}^2$. The map $F(x,t) = (1-t)f(x) + tg(x)$ is called a ***straight-line homotopy***. More generally, let A be any convex subspace of $\mathbb{R}^n$. Then any two path f, g in A from x_0 to x_1 are path homotopic in A.

Example 3.8 Let X denote the ***punctured plane***, $\mathbb{R}^2\{0\}$, and let

$$f(s) = (\cos \phi s, \sin \phi s),$$

$$g(s) = (\cos \phi s, 2 \sin \phi s),$$

$$h(s) = (\cos \phi s, -\sin \phi s)$$

Then $f \simeq g$, but f, h are not path homotopic.

3.9 Definition

If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the ***product*** $f * g$ of f and g to be the path h given by the equations

$$h(s) = \begin{cases} f(2s) & \text{for } s \in [0, \frac{1}{2}] \\ g(2s-1) & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$$

3.10 Lemma

The product operation on path-homotopy classes is well-defined by the equation $[f] * [g] = [f * g]$.

Proof) Let F and G be the path homotopy between f, f' and g, g' respectively. Define $H(s,t) = \begin{cases} F(2s,t) & \text{for } s \in [0, \frac{1}{2}] \\ G(2s-1, t) & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$. Then H is well-defined; and continuous by the pasting lemma; that is, it is a path homotopy between $f * g$ and $f' * g'$. □

3.11 Lemma

Suppose

(a) $k : X \to Y$ is a continuous map;

(b) F is a path homotopy in X between the paths f and f'.

Then $k \circ F$ is a path homotopy in Y between the paths $k \circ f$ and $k \circ f'$.

Proof) $k \circ F$ is continuous, $k(F(s,0)) = k(f(s))$, and $k(F(s,1)) = k(f'(s))$. □

3.12 Lemma

Suppose

(a) $k : X \to Y$ is a continuous map;

(b) f and g are paths in X with $f(1) = g(0)$.

Then
$$k \circ (f * g) = (k \circ f) * (k \circ g).$$

The proof is trivial.

3.13 Lemma

If $[a, b]$ and $[c, d]$ are two intervals in $\mathbb{R}$, there are unique numbers $m, k \in \mathbb{R}$ that define the map $p : [a, b] \to [c, d]$, given by $p(x) = mx + k$. We call it the ***positive linear map*** of $[a, b]$ to $[c, d]$. This concept is closed under the inverse map and composition of maps.

$e_x : I \to X$ denote the constant path given by $e_x = x$.

$i : I \to I$ denote the identity map given by $i(s) = s$ for all $s \in I$.

3.14 Theorem

The operation $*$ has the following properties:

(a) (associativity) If $[f] * ([g] * [h])$ is defined, then so is $([f] * [g]) * [h]$, and they are equal.

(b) (right and left identities) Let $e_x : I \to X$ denote the constant path given by $e_x = x$. If f is a path from x_0 to x_1, then
$$[f] * [e_{x_1}] = [f] \quad \text{and} \quad [e_{x_0}] * [f] = [f].$$

(c) (inverse) Given the path f in X from x_0 to x_1, let $\overline{f}$ be the path defined by $\overline{f}(s) = f(1 - s)$. It is called the ***reverse*** of f. Then
$$[f] * [\overline{f}] = [e_{x_0}] \quad \text{and} \quad [\overline{f}] * [f] = [e_{x_1}].$$

Proof) To prove (1), define a map $k_{a,b} : I \to I \quad 0 < a < b < 1$ as follows:

(a) A positive linear map of $[0, a]$ to I followed by f;

(b) a positive linear map of $[a, b]$ to I followed by g;

(c) a positive linear map of $[b, 1]$ to I followed by h.

For $0 < c < d < 1$, $k_{c,d} \simeq_p k_{a,b}$. Define $p : I \to I$ as follows:

(a) A positive linear map of $[0, a]$ to $[0, c]$;

(b) a positive linear map of $[a, b]$ to $[c, d]$;

(c) a positive linear map of $[b, 1]$ to $[d, 1]$.

If $i : I \to I$ is the identity map, then $p \simeq_p i$. Suppose P is a path-homotopy in I between p and i. Then $k_{c,d} \circ P$ is a path-homotopy in X between $k_{a,b}$ and $k_{c,d}$. Since $f * (g * h) = k_{a,b}$ where $a = 1/2$ and $b = 3/4$, and $(f * g) * h = k_{c,d}$ where $c = 1/4$ and $d = 1/2$, the associativity property holds.
To prove (2), since I is convex, we see that $[e_0 * i] = [i]$. By the provious lemma, $f \circ i$ and $f \circ (e_0 * i)$ are path-homotopic Consequently,
$$[f] = [f \circ i] = [f \circ e_0 * i] = [(f \circ e_0) * (f \circ i)] = [f \circ e_0] * [f \circ i] = [e_{x_0}] * [f].$$

A similar argument show that $[f] * [e_{x_1}] = [f]$.
To prove (3), note that $[i * \overline{i}] = [e_0]$. Therefore,
$$[e_{x_0}] = [f \circ e_0] = [f \circ (i * \overline{i})] = [f \circ i] * [f \circ \overline{i}] = [f] * [\overline{f}].$$

A similar argument show that $[\overline{f}] * [f] = [e_{x_1}]$. $\square$

3.15 Proposition: Exercise 1

If $h, h' : X \to Y$ are homotopic and k, k' are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

3.16 Proposition: Exercise 3

A space X is said to be ***contractible*** if the identity map $i_X : X \to X$ is nulhomotopic.

(a) Show that I and $\mathbb{R}$ are contractible.

(b) Show that a contractible space is path connected.

(c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.

(d) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

52. The fundamental group

monomorphism: homomorphism + injective. ***epimorphism***: homomorphism + surjective.

3.17 Definition

Let X be a space, $x_0 \in X$. A path in X that begins and ends at x_0 is called a ***loop*** based at x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the ***fundamental group*** of X relative to the ***base point*** x_0. It is denoted by $\pi_1(X, x_0)$, called the ***first homotopy group*** of X.

Example 3.18 The unit ball has trivial fundamental group.

3.19 Definition

Let α be a path in X from x_0 to x_1. We define a map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by $\hat{\alpha}([f]) = [\hat{\alpha}] * [f] * [\alpha]$.

3.20 Theorem

The map $\hat{\alpha}$ is a group isomorphism.

Proof) The proof for homomorphism is trivial. To prove bijectivity, consider $\hat{\hat{\alpha}}$. □

3.21 Corollary

If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

3.22 Definition

A space X is said to be ***simply connected*** if it is a path-connnected space and if $\pi_1(X, x_0)$ is trivial (one-element) group for some $x_0 \in X$, and hense for every $x_0 \in X$, write $\pi_1(X, x_0) = 0$.

3.23 Lemma

In a simply connected space X, any two paths having the same initial and final points are path homotopic.

Proof) Let α and β be two paths from x_0 to x_1. Then $[\alpha * \overline{\beta}] * [\beta] = [e_{x_0}] * [\beta]$. □

Suppose that $h : X \to Y$ is a continuous map that $h(x_0) = y_0$. Then we write $h : (X, x_0) \to (Y, y_0)$.

3.24 Definition

Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. Define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $h_*([f]) = [h \circ f]$. The map is called the ***homomorphism induced by h***, relative to the base point x_0.

Remark h_* depends not only on the map $h : X \to Y$ but also on the choice of the base point x_0.

3.25 Theorem

If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

Proof) The proof is trivial. □

3.26 Corollary

If h is a homeomorphism, then h_* is an isomorphism.

Proof) h_*^{-1} is the inverse of h_*. □

3.27 Proposition: Exercise 4

Let $A \subset X$; suppose $r : X \to A$ is a continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a ***retraction*** of X onto A) If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective.

3.28 Definition

G is called a ***topological group*** if it is both a topological space and a group such that the group operation map

$$G \times G \to G, \quad (x, y) \mapsto x \cdot y$$

and the inverse map

$$G \times G, \quad x \mapsto x^{-1}$$

are continuous maps with respect to the topology on G and the product topology on $G \times G$.

3.29 Proposition: Exercise 7

Let G be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

(a) Show that this operation makes the set $\Omega(G, x_0)$ into a group.

(b) Show that rhis operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

(c) Show that the two group operation $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same.[Hint: Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$]

(d) Show that $\pi_1(G, x_0)$ is abelian.

53. Covering spaces

3.30 Definition

Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be ***evenly covered*** by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into ***slices***.

3.31 Definition

Let $p : E \to B$ be continuous and surjective. If every point b of B has a neighborhood U that is evenly covered by p, then p is called a ***covering map***, and E is said to be a ***covering space*** of B.

3.32 Proposition

For each $b \in B$, the subspace $p^{-1}(b)$ has the discrete topology.

Proof) Munkres p336 □

3.33 Proposition

A covering map is open.

Example 3.34 For any space X, the identity map $i : X \to X$ is a covering map.

3.35 Theorem

The map $p : \mathbb{R} \to S^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map.

Proof) Let $U = \{(x, y) \mid x \in (0, 1], y \in (-1, 1)\}$. Then $p^{-1}(U)$ is the union of intervals of the form $V_n = (n - \frac{1}{4}, n + \frac{1}{4})$, for all $n \in \mathbb{Z}$. We easily see that $p|\overline{V_n}$ is bijective(by IVT), and is closed(since $\overline{V_n}$ is compact), that is, is a homeomorphism of $\overline{V}$ with $\overline{U}$. In particular, $p|V_n$ is a homeomorphism of V_n with U. Similar argument can be applied to the upper, left, down side of S^1. They cover S^1 and each of them is evenly covered by p. □

Example 3.36 Let $p : S^1 \to S^1$ be defined on the complex plane, given by $p(z) = z^2$. Then p is a covering map.

3.37 Definition

Let $p : E \to B$ is a ***local homeomorphism*** if for each $x \in E$, there is a neighborhood V of x such that $p|V$ is a homeomorphism

3.38 Proposition

A covering map is a local homeomorphism.

Example 3.39 $p : \mathbb{R}_+ \to S^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is surjective, and local homeomorphism, but is not a covering map. This example implies that the restriction of a covering map may not be a covering map.

3.40 Theorem

Let $p : E \to B$ be a covering map. If B_0 is a subspace of B, and if $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \to B_0$ obtained by restricting p is a covering map.

Proof) Munkres theorem 53.2 □

3.41 Theorem

If $p : E \to B$ and $p' : E' \to B'$ are covering maps, then $p \times p' : E \times E' \to B \times B'$ is a covering map.

Exercise 3.1 $T = S^1 \times S^1$ is called the ***torus***. The product map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ is a covering map of torus by the plane $\mathbb{R}^2$.

Section 54. The fundamental group of the circle

Lecture 0909

3.42 Theorem

$x \mapsto (\cos 2\pi x, \sin 2\pi x)$ is a covering map.

Proof)

$\square$

Example 3.43

$$exp : \mathbb{C} \to \mathbb{C}\backslash\{0\}$$
$$z \mapsto e^z$$

$\mathbb{C} \simeq \mathbb{R}^2$ and $\mathbb{C}\backslash\{0\} \simeq \mathbb{R}^+ \times S$ given by $(x, y) \mapsto (e^x, e^{iy})$. It is a product map of a homeomorphism $\mathbb{R} \to \mathbb{R}^+$ and a covering map $\mathbb{R} \to S$.

Lemma 54.1 $B = \bigcup_{j \in J} u_j$ where open sets $u_j \subset B$ are evenly covered by p. $I = \alpha^{-1}(u_{j_1}) \cup \cdots \cup \alpha^{-1}(u_{j_N})$ since compactness of I. $\exists 0 < s_0 < s_1 < \cdots < s_M = 1$ s.t. $[s_i, s_{i+1}] \subset \alpha^{-1}(u_{j_k})$ for some k by Lebesque's lemma. Assume that there exists a lifting of α $\tilde{\alpha} : [0, s_i] \to E$ with $\tilde{\alpha}(0) = e_0$. $\alpha([s_i, s_{i+1}]) \subset U \subset B$ for some evenly covered open set U. $\square$