# Section 1. HW 1

**Exersise 1.1** (Lemma 1.2) Let $z, w \in \mathbb{C}$.

(a) $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.

(b) $z + \overline{z} = 2\mathrm{Re}(z)$ and $z - \overline{z} = i2\mathrm{Im}(z)$.

(c) $|\overline{z}| = |z|$ and $|z \cdot w| = |z||w|$.

(d) $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$.

Proof) Let $z = z_1 + z_2 i$ and $w = w_1 + w_2 i$.

(a) $\overline{z+w} = \overline{(z_1+z_2i)+(w_1+w_2i)} = \overline{(z_1+w_1)+(z_2+w_2)i} = (z_1+w_1) - (z_2 + w_2)i = (z_1 - z_2i) + (w_1 - w_2i) = \overline{z} + \overline{w}$.
$\overline{z \cdot w} = \overline{(z_1+z_2i)\cdot(w_1+w_2i)} = \overline{(z_1w_1 - z_2w_2) + (z_1w_2 + z_2w_1)i} = (z_1w_1 - z_2w_2) - (z_1w_2 + z_2w_1)i = (z_1 - z_2i)\cdot(w_1 - w_2i) = \overline{z}\cdot\overline{w}$.

(b) $z + \overline{z} = (z_1 + z_2i) + (z_1 - z_2i) = 2z_1 = 2\mathrm{Re}(z)$
$z - \overline{z} = (z_1 + z_2i) - (z_1 - z_2i) = 2z_2i = 2i\mathrm{Im}(z)$.

(c) $|\overline{z}| = \sqrt{\overline{z}\cdot\overline{\overline{z}}} = \sqrt{\overline{z}\cdot z} = \sqrt{z\cdot\overline{z}} = |z|$.
$|z\cdot w|^2 = z\cdot w\cdot\overline{z}\cdot\overline{w} = z\cdot\overline{z}\cdot w\cdot\overline{w} = |z|^2|w|^2 \implies |z\cdot w| = |z||w|$.

(d) $|\mathrm{Re}(z)|^2 = z_1^2 \leq z_1^2 + z_2^2 = |z|^2 \implies |\mathrm{Re}(z)| \leq |z|$.
$|\mathrm{Im}(z)|^2 = z_2^2 \leq z_1^2 + z_2^2 = |z|^2 \implies |\mathrm{Im}(z)| \leq |z|$.

$\square$

**Exersise 1.2** (Lemma 1.5) If $f, g \in C(U)$, then $f+g, fg \in C(U)$.

Proof)

$(f+g)$ Given $\epsilon > 0$ and fixed $x_0 \in U$, choose open subsets $V_1, V_2 \subset U$ such that

$$x_0 \in V_1 \implies \sup_{x \in V_1} |f(x) - f(x_0)| < \frac{\epsilon}{2}$$

and

$$x_0 \in V_2 \implies \sup_{x \in V_2} |f(x) - f(x_0)| < \frac{\epsilon}{2}.$$

Then

$$\begin{aligned}
x_0 \in V = V_1 \cap V_2 \Rightarrow \sup_{x_0 \in V} &|(f+g)(x) - (f+g)(x_0)| \\
&\leq \sup_{x_0 \in V} |f(x) - f(x_0)| + \sup_{x_0 \in V} |g(x) - g(x_0)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

$(fg)$ Given $\epsilon > 0$ and fixed $x_0 \in U$, choose open subsets $V_1, V_2 \subset U$ such that

$$x_0 \in V_1 \implies \sup_{x_0 \in V_1} |f(x) - f(x_0)| < \frac{\epsilon}{4|g(x_0)|}$$

and

$$x_0 \in V_2 \implies \sup_{x_0 \in V_2} |g(x) - g(x_0)| < \frac{\epsilon}{2|f(x_0)|}$$

and

$$x_0 \in V_2 \implies \sup_{x_0 \in V_2} |g(x) - g(x_0)| < |g(x_0)|, \text{ i.e., } \sup_{x_0 \in V_2} |g(x)| < 2|g(x_0)|.$$

Then

$$\begin{aligned}
x_0 \in V = V_1 \cap V_2 \Rightarrow \sup_{x_0 \in V} &|fg(x) - fg(x_0)| \\
&= \sup_{x_0 \in V} |f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)| \\
&\leq \sup_{x_0 \in V} |g(x)(f(x) - f(x_0))| + \sup_{x_0 \in V} |f(x_0)(g(x) - g(x_0))| \\
&< 2|g(x_0)| \cdot \frac{\epsilon}{4|g(x_0)|} + |f(x_0)| \cdot + \frac{\epsilon}{2|f(x_0)|} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

$\square$

**Exersise 1.3** (Lemma 1.7)

(a) Let $f(z) = 1/z$. Then $f \in H(\mathbb{C} - \{0\})$ and $f'(z) = -1/z^2$.

(b) Let $f(z) = \overline{z}$. Then $f$ is nowhere differentiable.

Proof)

(a) Let $z_0 \in U \backslash \{0\}$ be given. Since the set $U = \mathbb{C}\backslash\{0\}$ is open, there is a neighborhood $V_0$ of $z_0$ contained in $U\backslash\{0\}$. Let $z_1 \in V_0$ be a point. Since $V_0$ is open, there is a neighborhood of $z_1$ does not contain 0. Therefore, we can apply definition 1.7 for every $z_1 \in V_0$:

$$\lim_{z \to z_1} \frac{f(z) - f(z_1)}{z - z_1} = \lim_{z \to z_1} \frac{\frac{1}{z} - \frac{1}{z_1}}{z - z_1} = \lim_{z \to z_1} \frac{-1}{zz_1}$$

Now we claim that

$$\lim_{z \to z_1} \frac{-1}{zz_1} = \frac{-1}{z_1^2}.$$

Let $\epsilon > 0$ be given, and assume $\delta \leq \frac{1}{2}|z_1|$. Then we have

$$\frac{1}{2}|z_1| < |z| < \frac{3}{2}|z_1|.$$

We observe that

$$\left|\frac{-1}{zz_1} - \frac{-1}{z_1^2}\right| = \left|\frac{-z_1 + z}{zz_1^2}\right| < \frac{\delta}{\frac{1}{2}|z_1|z_1^2}$$

But we want

$$\frac{\delta}{\frac{1}{2}|z_1|z_1^2} \leq \epsilon.$$

Consequently, it is sufficient to set $\delta = \min\left\{\frac{1}{2}|z_1|, \frac{1}{2}\epsilon|z_1|^3\right\}$.

(b) Let $z_0 \in \mathbb{C}$ be given, and suppose $z_0 = x_0 + y_0 i$ and $z = x + yi$ for $z \neq z_0$. Then

$$\lim_{z \to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0} = \lim_{z \to z_0} \frac{(x - x_0) + (-y + y_0)i}{(x - x_0) + (y - y_0)i}$$

If we fix $x = x_0$, then the value of limit is $-1$, or if we fix $y = y_0$, then the value of limit is 1, which implies that the limit does not converge.

□

**Exersise 1.4** (Corollary 1.9) If $f$ is differentiable at $c$, then $f$ is continuous at $c$.

Proof) By the theorem 1.8, $f$ can be expressed as a sum or product of continuous functions at $c$. By the lemma 1.5, it must also be continuous at $c$. □

**Exersise 1.5** (Lemma 1.10) If $f, g \in H(U)$, then $f+g, fg \in H(U)$ and $(f+g)' = f'+g'$ and $(fg)' = f'g + fg'$.

Proof) Let $z_0 \in U$ be given. There exists $r_f, r_g > 0$ such that $f, g$ are differentiable on $D(z_0, r_f), D(z_0, r_g)$ respectively. Set $r = \min(r_f, r_s)$. We claim that $f + g$ and $fg$ is differentiable on $D(z_0, r)$. To prove this, let $z_1 \in D(z_0, r)$ be given. For some $a_f, a_g \in \mathbb{C}$ and some $h_f, h_g : U \to \mathbb{C}$ which are continuous at $z_1$ and which are zero at $z_1$,

$$f(z) = f(z_1) + a_f(z - z_1) + h_f(z)(z - z_1)$$

$$g(z) = g(z_1) + a_g(z - z_1) + h_g(z)(z - z_1)$$

$a_f, a_g$ denote the derivative of $f$ and $g$ at $x_0$, respectively. Then

$$(f + g)(z) = (f + g)(z_1) + (a_f + a_g)(z - z_1) + (h_f + h_z)(z)(z - z_1).$$

If we set $a_{f+g} = a_f + a_g$ and $h_{f+g} = h_f + h_z$, then it satisfies the conditions for differentiability($\because$ $h_{f+g}(z) = 0$ at $z_1$ and continuous at $z_1$ by the lemma 1.5). On the other hand,

$$\begin{aligned}(fg)(z) =&(f(z_1) + a_f(z - z_1) + h_f(z)(z - z_1))(g(z_1) + a_g(z - z_1) + h_g(z)(z - z_1))\\ =&fg(z_1) + f(z_1)a_g(z - z_1) + f(z_1)h_g(z)(z - z_1)\\ &+ a_f g(z_1)(z - z_1) + a_f a_g(z - z_1)^2 + a_f h_g(z)(z - z_1)^2\\ &+ h_f(z)g(z_1)(z - z_1) + h_f(z)a_g(z - z_1)^2 + h_f h_g(z)(z - z_1)^2\\ =&fg(z_1) + (f(z_1)a_g + a_f g(z_1))(z - z_1)\\ &+ (f(z_1)h_g(z) + a_f a_g(z - z_1) + a_f h_g(z)(z - z_1)\\ &\quad + h_f(z)g_(z_1) + h_f(z)a_g(z - z_1) + h_f h_g(z)(z - z_1))(z - z_1)\end{aligned}$$

If we set $a_{fg} = f(z_1)a_g + a_f g(z_1)$ and $h_{fg}$ equal to the third term on the right-hand side of the above equation, then, since $h_f, h_g, h_g h_f, (z - z_1)$ are all continuous at $z_1$ and have a value of zero at $z_1$, by the lemma 1.5, $h_{fg}$ satisfies the condition for differentiability.

□

**Exersise 1.6** (Lemma 1.11) If $f : U \to \mathbb{C}$ is differentiable at $c$ and $g : f(U) \to \mathbb{C}$ is differentiable at $f(c)$, then $g \circ f$ is differentiable at $c$ and $(g \circ f)'(c) = g'(f(c))f'(c)$.

Proof) To prove this, we should first prove a lemma regarding the continuity of composition of continuous functions.

(Lemma 1.12) Let $A, B, C$ be sets, $f : A \to B$ continuous, and $g : f(A) \to C$ continuous. Then $g \circ f$ is also continuous.

By the definition of the continuity of functions, for every open set $U \in C$, $g^{-1}(U)$ is open in $B$, and $f^{-1}(g^{-1}(U))$ is open in $A$. Therefore, the lemma holds. Let

$$f(x) = f(c) + a_f(x - c) + h_f(x)(x - c)$$

$$g(y) = g(f(c)) + a_g(y - f(c)) + h_g(y)(y - f(c))$$

Here, $a_f$ and $a_g$ denote the derivatives of $f$ and $g$ at $c$ and $f(c)$, respectively, and $h_f$ and $h_g$ denote continuous functions which are zero at $c$ and $f(c)$, respectively. Then,

$$\begin{aligned}(g \circ f)(x) =&g(f(c)) + a_g(f(x) - f(c)) + h_g(f(x))(f(x) - f(c))\\ =&g(f(c)) + a_g(f(c) + a_f(x - c) + h_f(x)(x - c) - f(c))\\ &+ h_g(f(x))(f(c) + a_f(x - c) + h_f(x)(x - c) - f(c))\\ =&g(f(c)) + a_g a_f(x - c) + (a_g h_f(x) + h_g(f(x))a_f + h_g(f(x))h_f(x))(x - c)\end{aligned}$$

If we set $a_{g\circ f} = a_g a_f$ and $h_{g\circ f} = a_g h_f(x) + h_g(f(x))a_f + h_g(f(x))h_f(x)$, then $h_{g\circ f}$ is continuous by the lemma 1.5 and $h_{g\circ f}(c) = 0$. Then, by the theorem 1.8, $g \circ f$ is differentiable at $c$. □