Section 1. HW 1

Exercise 1.1 (Section 51 exercise 1) Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Proof) Let $H : X \times I \to Y, K : Y \times I \to Z$ are homotopies between h, h' and k, k' , respectively. Define $H' : X \times I \to Y$ by $H'(x,t) = (H(s,t), t)$. Since both H and t are continuous, H' is continuous. Let $F : X \times I \to Z(= K \circ H')$ be given by $F(x,t) = K(H(s,t), t)$. Then

(a) F is composition of continuous functions;

(b) $F(x, 0) = k(h(x))$;

(c) $F(x, 1) = k'(h'(x))$

Therefore, F is a homotopy between $k \circ h$ and $k' \circ h'$. □

Exercise 1.2 (Section 51 exercise 3) A space X is said to be ***contractible*** if the identity map $i_X : X \to X$ is nulhomotopic.

(a) Show that I and $\mathbb{R}$ are contractible.

(b) Show that a contractible space is path connected.

(c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.

(d) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

Proof)

(a) Define $F_X : X \times I \to I$ by $F_X(x,t) = xt$. If $X = I$, then F_I is continuous, $F_I(x, 0) = 0$, and $F_I(x, 1) = x$, for $x \in I$. If $X = \mathbb{R}$, $F_\mathbb{R}$ is continuous, then $F_\mathbb{R}(x, 0) = 0$, and $F_\mathbb{R}(x, 1) = x$, for $x \in \mathbb{R}$. Therefore, F_I and $F_\mathbb{R}$ are homotopies between i_I and 0 and between $i_\mathbb{R}$ and 0, respectively.

(b) Suppose a space X is contractible space. There is a point $p \in X$ such that a contant map e_p and i_X are homotopic. Let $a, b \in X$ be given. We want to show that there exists a curve $c : I \to X$ between a and b. To do this, let $H : X \times I \to X$ be a homotopy between e_p and i_X, where $H(x, 0) = i_X(x) = x$, $H(x, 1) = p$. We see that

  (a) $H(a,t)$: a path from a to p.

  (b) $H(b,t)$: a path from b to p.

If we define $c = H(a,t) * \overline{H(b,t)}$, then c is a path from a to b.

(c) Suppose $i_Y \simeq e_p$ for some $p \in Y$. Then $f = i_Y \circ f \simeq e_p \circ f = e_p \circ g \simeq i_Y \circ g = g$.

(d) Suppose $i_X \simeq e_p$ for some $p \in X$. Then $f = f \circ i_X \simeq f \circ e_p = e_{f(p)} \simeq e_{g(p)} \simeq g \circ i_X = g$. The homotopy equivalence between $e_{f(p)}$ and $e_{g(p)}$ is derived from the path-connectness of Y. Define $F : Y \times I \to Y$ by $F(s,t) = c(t)$ where c denote the curve from $f(p)$ to $g(p)$.

□

Exercise 1.3 (Section 52 exercise 4) Let $A \subset X$; suppose $r : X \to A$ is a continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a ***retraction*** of X onto A) If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective.

Proof) Define $r' : A \to X$ by $r'(x) = x$. Consider $r \circ r' : A \to A$. We easily see that $r \circ r'$ is well-defined and is a identity map. It implies that r' is a right inverse of r, which means r is surjective. □

Exercise 1.4 (Section 52 exercise 7) Let G be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

(a) Show that this operation makes the set $\Omega(G, x_0)$ into a group.

(b) Show that rhis operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

(c) Show that the two group operation $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same.[Hint: Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$]

(d) Show that $\pi_1(G, x_0)$ is abelian.

Proof)

(a) Let $I(= [0, 1])$ be the domain of loops in $\Omega(G, x_0)$. Let $f, g \in \Omega(G, x_0)$. Since, for each $s \in I$, both $f(s)$ and $g(s)$ are in G, $(f \otimes g)(s) \in G$. Thus the operation is closed in $\Omega(G, x_0)$. If $f, g, h \in \Omega(G, x_0)$, then $((f \otimes g) \otimes h)(s) = (f(s) \cdot g(s)) \cdot h(s) = f(s) \cdot (g(s) \cdot h(s)) = (f \otimes (g \otimes h))(s)$, for all $s \in I$, by the associativity of group operation. Furthermore, e_{x_0} is clearly the identity of $\otimes$. Suppose $f \in \Omega(G, x_0)$. Define $g : G \to G$ by $g(x) = x^{-1}$. By the definition of topological group, g is continuous, so is $g \circ f$. Then, $(g \circ f) \otimes f = f \otimes (g \circ f) = e_{x_0}$, which means that $(g \circ f)$ is the inverse element for f of $\otimes$.

(b) Given $[f], [g] \in \pi_1(G, x_0)$, define $[f] \otimes [g]$ by $[f \otimes g]$. We have to check whether the operation is well-defined. Suppose $h, h', k, k' \in \Omega(G, x_0)$, $h \simeq h'$, and $k \simeq k'$. Let H, K be the homotopy between f, f' and between g, g', respectively. Define $HK : I \times I \to G$ by $HK(s,t) = H(s,t) \cdot K(s,t)$. By the definition of topological group, HK is continuous. On top of that, $HK(s, 0) = h(s) \cdot k(s)$, $HK(s, 1) = h'(s) \cdot k'(s)$. Therefore, it is a homotopy between $h \otimes k$ and $h' \otimes k'$. Suppose $[f], [g], [h] \in \pi_1(G, x_0)$. Then $([f] \otimes [g]) \otimes [h] = [f] \otimes ([g] \otimes [h]) = [f \otimes g] \times [h] = [(f \otimes g) \otimes h] = [f \otimes (g \otimes h)] = [f] \otimes [g \otimes h] = [f] \otimes ([g] \otimes [h])$. The identity and inverse are followed by (a).

(c) Note that for each $f, g \in \Omega_1(G, x_0)$,

$$(f * e_{x_0}) \otimes (e_{x_0} * g) = \begin{cases} f(s) \cdot e_{x_0}(s) & \text{for } 0 \le s \le \frac{1}{2} \\ e_{x_0}(s) \cdot g(s) & \text{for } \frac{1}{2} \le s \le 1 \end{cases} = f * g.$$

Theremore, $[f] \otimes [g] = [f * e_{x_0}] \otimes [e_{x_0} * g] = [f * g] = [f] * [g]$.

(d) $[f] * [g] = [f * g] = [f * e_{x_0}] \otimes [e_{x_0} * g] = [e_{x_0} * f] \otimes [g * e_{x_0}] = [g * f] = [g] * [f]$, for each $[f], [g] \in \pi_1(G, x_0)$.

$\square$

Exercise 1.5 (Section 53 exercise 3) Let $p : E \to B$ be a covering map; let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$. In such a case, E is called a ***k-fold covering*** of B.

Proof) Let $B_0 := \{b \in B \mid |p^{-1}(b)| = k\}$. $B_1 := \{b \in B \mid |p^{-1}(b) \neq b\}$. By hypothesis, $|B_0| \neq 0$. Suppose, for contradiction, $|B_1| \neq 0$. Since p is a covering map, for each element of B_0, there exists a open subset U_α of B evenly covered by p. Likewise, for $b \in B_1$, there exists $V_\beta \subset B$ evenly covered by p, for some integer $n \neq k$. Let $U = \bigcup\{U_\alpha\}$, $V = \bigcup\{V_\beta\}$. Then

(a) Since U, V are union of open sets, they are open.

(b) Clearly $(U \cap B) \cup (V \cap B) = B$.

(c) $U \cap V = \varnothing$. Suppose $x \in U \cap V$, then $|p^{-1}(x)| = k = n$, leading to a contradiction.

That is, U, V seperate B, leading a contradiction. Thus, B_1 is an empty set. $\square$

Exercise 1.6 (Section 53 exercise 6) Let $p : E \to B$ be a covering map.

(a) If B is Hausdorff, regular, completely regular, or locally compact Hausdorff, then so is E. [Hint: If $\{V_\alpha\}$ is a partition of $p^{-1}(U)$ into slices, and C is a closed set of B such that $C \subset U$, then $p^{-1}(C) \cap V_\alpha$ is a closed set of E.]

(b) If B is compact and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.

Proof) (b) Let an open cover $\{U_\alpha\}$ of E be given. For every $b \in B$, there exists V_b evenly covered by p. Each V_b has the covering space of finite disjoint open set W_{b_β}. Consider the intersection of each U_α and W_{b_β}. Since p is surjective and the set $\{V_b\}$ is an open cover of B, the union of every $U_\alpha \cap W_{b_\beta}$ contains E. It implies that the union of every $p(U_\alpha \cap W_{b_\beta})$ contains B. Therefore, there is a finite subcover $\{p(U_{\alpha_n} \cap W_{b_{\beta_m}})\}$ containing B. The preimage of each $p(U_{\alpha_n} \cap W_{b_{\beta_m}})$ is a finite-fold covering of $p(U_{\alpha_n} \cap W_{b_{\beta_m}})$, thus the number of every disjoint open set of preimage of $p(U_{\alpha_n} \cap W_{b_{\beta_n}})$ is also finite, and union of them contains E. Since $(U_{\alpha_n} \cap W_{b_{\beta_m}}) \subset U_{\alpha_n}$, E is contains in the union of $\{U_{\alpha_n}\}$. It is a finite subcover of E. $\square$

Exercise 1.7 (Section 54 exercise 5) Consider the covering map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ of Example 4 of setction 53. Consider the path $f(t) = (\cos 2\pi t, sin 2\pi t) \times (\cos 4\pi t, \sin 4\pi t)$ in $S^1 \times S^1$. Sketch what f looks like when $S^1 \times S^1$ is identified with the doughnut surface D. Find a lifting $\overline{f}$ of f to $\mathbb{R} \times \mathbb{R}$, and sketch it.

Proof) Define $\tilde{f} : I \to \mathbb{R} \times \mathbb{R}$ by

$$\tilde{f}(s) = (s, 2s).$$

Then $p \circ \tilde{f} = f$. To visualize this, let $g : S^1 \times S^1 \to \mathbb{R}^3$ given by

$$g((x_1, y_1) \times (x_2, y_2)) = ((1 + \frac{1}{3}x_1)x_2, (1 + \frac{1}{3}x_1)y_2, \frac{1}{3}y_1).$$

Then

$$\begin{aligned}(g \circ p)(u, v) &= g((\cos 2\pi u, \sin 2\pi u) \times (\cos 2\pi v, \sin 2\pi v)) \\ &= ((1 + \frac{1}{3}\cos 2\pi u)\cos 2\pi v, (1 + \frac{1}{3}\cos 2\pi u)\sin 2\pi v, \frac{1}{3}\sin 2\pi u)\end{aligned}$$

and

$$(g \circ f)(s) = g((\cos 2\pi s, \sin 2\pi s) \times (\cos 4\pi s, \sin 4\pi s)) \tag{1}$$

$$= ((1 + \frac{1}{3}\cos 2\pi s)\cos 4\pi s, (1 + \frac{1}{3}\cos 2\pi s)\sin 4\pi s, \sin 2\pi s) \tag{2}$$

If we render this using a graphics tool, the result looks like the following image. $\square$

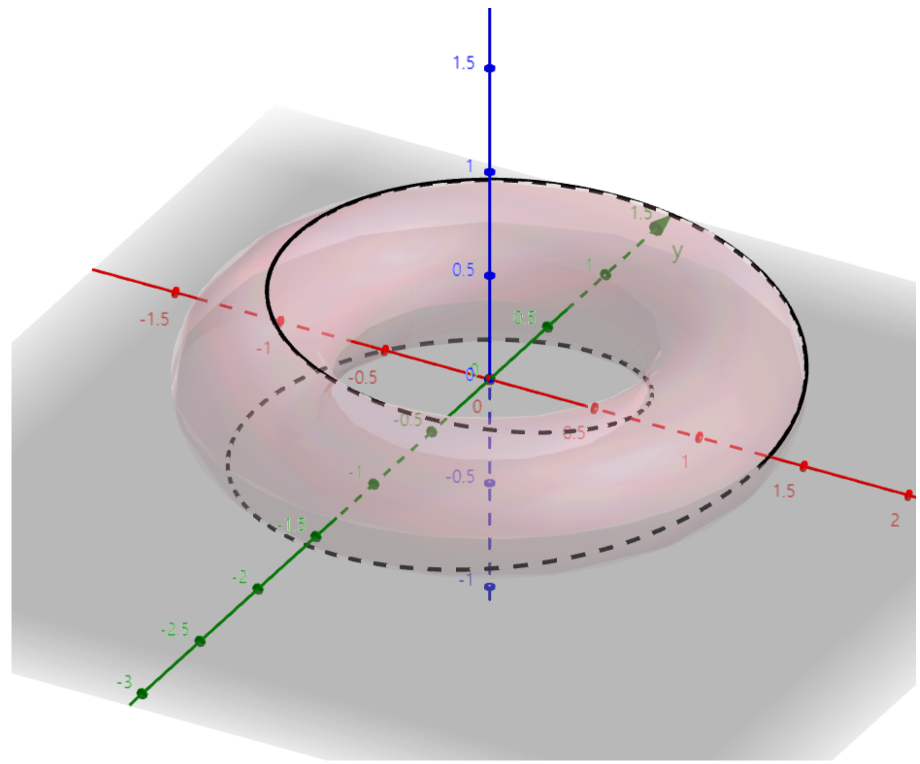

Figure 1: The path f on the torus p

Exercise 1.8 (Section 54 exersice 6) Consider the maps $g, h : S^1 \to S^1$ given $g(z) = z^n$ and $h(z) = 1/z^n$. (Here we represent S^1 as the set of complex numbers z of absolute value 1.) Compute the induced homomorphism g_*,h_* of the infinite cyclic group $\pi_1(S^1, b_0)$ into itself. [Hint: Recall the equation $(\cos\theta + i\sin\theta)^n = \cos n\theta + \sin n\theta$.]

Proof) Since S^1 is path-connected, $\pi_1(S^1, b_0) \cong \pi_1(S^1, (1,0))$. We may assume $b_0 = (1,0)$. Let $f : I \to S^1$ given by $f(t) = (\cos 2\pi t, \sin 2\pi t) = e^{2\pi i t}$. Then $f(0) = f(1) = (1,0)$, and the equivalence class $[f] \in \pi_1(S^1, (1,0)) \cong \mathbb{Z}$ corresponds to $1 \in \mathbb{Z}$, so it is a gererater of $\pi_1(S^1, (1,0))$. Define a isomorphism $\phi : \pi_1(S^1, (1,0)) \to \mathbb{Z}$ by $\phi([e^{2\pi n t}]) = n$. Then $(\phi \circ g_*)([f]) = \phi([g \circ f]) = \phi([e^{2\pi n t}]) = n$. Since $[f]$ is a generator of $\pi_1(S^1, (1,0))$, n is a generator of $(\phi \circ g_*)(\pi_1(S^1, (1,0)))$, $n\mathbb{Z}$. Consequently, $g_*(\pi_1(S^1, (1,0))) \cong (\phi \circ g_*)(\pi_1(S^1, (1,0))) = n\mathbb{Z}$. Similarly, $h_*(\pi_1(S^1, (1,0))) \cong (\phi \circ h_*)(\pi_1(S^1, (1,0))) = -n\mathbb{Z}$. $\square$

Section 2. HW2

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