Contents

Section 1. BASIC TOPOLOGY

Set theory

Definition 1.1 (pma 2.1, 2.2, 2.3) For $f : A \to B$

(1) f: ***function*** from A to B(or ***mapping*** of A ***into*** B)

(2) A: ***domain*** of f

(3) $f(x \in A)$: ***value*** of f

(4) $f[A]$: ***range*** of f

(5) $f[E \subset A]$: ***image*** of E under f

(6) If $f[A] = B$, we say that f maps A ***onto*** B.

(7) $f^{-1}[E]$: ***inverse*** image of E under f

(8) If for $y \in B$ $f^{-1}(y)$ consists of at most one element of A, we say that f is a ***one-to-one*** mapping of A into B.

(9) If there exists a one-to-one mapping A onto B, we say that A and B can be put into ***one-to-one correspondence***, have the same ***cardinal number***, or are ***equivalent***(written as $A \sim B$), which has the following properties:

  (a) reflexive: $A \sim A$.

  (b) symmetric: If $A \sim B$, then $B \sim A$.

  (c) transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$.

  Any ***relation*** with these properties is called an ***equivalence*** relation.

Definition 1.2 (pma 2.4) Let $J_n := \{1, 2, \dots, n\}$, $J := \{1, 2, \dots\}$, A be an any set. Then

(1) A is ***finite*** if $A \sim J_n$ for some n.

(2) A is ***infinite*** if A is not finite.

(3) A is ***countable*** if $A \sim J$(or ***enumerable*** or ***denumerable***).

(4) A is ***uncountable*** if A is neither finite nor countable.

Definition 1.3 (pma 2.7) For $f : J(:= \mathbb{N}) \to A(:= \{x_1, x_2, \dots,\})$ given by $f(n) = x_n$,

(1) f: ***sequence***, denoted by $\{x_n\}$ or $x_1, x_2, \dots$. Also $\{x_n\}$ is called a sequence in A.

(2) x_n: A ***term*** of the sequence.

Remark (pma 2.7) Every countable set is the range of a sequence of distinct terms.

Theorem 1.4 (pma 2.8) Every infinite subset of a countable set is countable.

Proof) Let $E \subset A$. Arrange A in a sequence $\{x_n\}$. Define n_k as follows:

(1) n_1 is the smallest positive integer where $x_{n_1} \in E$.

(2) n_{k+1} is the smallest integer where $x_{n_{k+1}} \in E$ greater than $x_{n_1}, x_{n_2}, \dots, x_{n_k}$.

Then $\{x_{n_k}\}$ is an one-to-one correspondence between E and J. □

Remark (pma 2.9) Let A and Ω be sets, suppose that for each $\alpha \in A$, there is a corresponding subset of Ω which is denoted by E_α. Then $\{E_\alpha\}$ means a set of sets.

Theorem 1.5 (pma 2.12) The countable union of countable sets is countable.

Theorem 1.6 (pma 2.13) Let A be a countable set, and let B_n be the set of n-tuples where each term is in A. Then B_n is countable.

Corollary (pma 2.13) The set of all rational numbers is countable.

Theorem 1.7 (pma 2.14) Let A be a set of all sequence whose elements are the digits 0 and 1. This set A is uncountable.

Proof) By diagonal construction, we can see that every countable subset of A is a proper subset of A, i.e., A is uncountable. □

Metric spaces

Definition 1.8 (pma 2.15) Let X be a set and let d be a function with the following properties for any $p, q \in X$:

(1) $d(p, q) \geq 0$, and the inequality is equality ***if and only if*** $p = q$.

(2) $d(p, q) = d(q, p)$.

(3) $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.

Then d is called a **distance** or **metric**, and X is called a **metric space**.

Definition 1.9 (pma 2.17) For $a, b \in \mathbb{R}$,

(1) (a, b): ***segment***

(2) $[a, b]$: ***interval***

(3) If $a_i < b_i$ for $i = 1, \dots, k$, the set of all points $x = (x_1, \dots, x_k)$ in $\mathbb{R}^k$ where $a_i \leq x_i \leq b_i$ for $(1 < i < k)$ is called ***k-cell***.

(4) An ***open(or closed) ball*** with center $x \in \mathbb{R}^n$ and radius $r > 0$ is the set of all $y \in \mathbb{R}^n$ such that $|y - x| < r$(or $|y - x| \leq r$).

(5) A set $E \subset \mathbb{R}^n$ is ***convex*** if $\lambda x + (1 - \lambda)y \in E$ whenever $x, y \in E$ and $0 < \lambda < 1$.

Remark (pma 2.17) For y, z in a ball, $|\lambda y + (1-\lambda)z - x| = |\lambda(y-x) + (1-\lambda)(z-x)| \leq \lambda|y-x| + (1-\lambda)|z-x| < \lambda r + (1-\lambda)r = r$. In other words, a ball is convex. Likewise, k-cells are convex.

Definition 1.10 (pma 2.18) Let X be a metric space.

(1) A ***neighborhood*** of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$ for some $r > 0$.

(2) A point p is a ***limit point*** of E if $N_r(p) \cap E \backslash \{p\} \neq \emptyset$ for every $r > 0$. The set of all limit points of E is denoted by E'.

(3) A point p is a ***isolate point*** if $p \in E$, $p \notin E'$.

(4) A point p is a ***interior point*** of E if $N_r(p) \subset E$ for some r.

(5) A ***closer*** of E is $E \cup E'$ and is denoted by $\overline{E}$.

(6) E is ***closed*** if $E' \subset E$.

(7) E is ***open*** if every point of E is an interior point.

(8) E is ***perfect*** if E is closed and has no isolate points.

(9) E is ***bounded*** if there is a real number M and $q \in X$ such that $d(p,q) < M$ for all $p \in E$.

(10) E is ***dense*** if $X = E \cup E'$.

Theorem 1.11 (pma 2.19) Every neighborhood is an open set.

Theorem 1.12 (The neighborhood of limit points (pma 2.20)) The neighborhood of a limit point of a set E contains infinite many point of E.

Corollary (pma 2.20) Every finite set has no limit points.

Theorem 1.13 (pma 2.23) A set E is open ***if and only if*** its complement is closed.

Proof) Consider that every point of E is not a limit point of E^c and every point of E^c is not an interior point of E. □

Theorem 1.14 (pma 2.24) Every finite intersection and arbitrary union of open set is open. And every finite union and arbitrary intersection of closed set is closed.

Theorem 1.15 (pma 2.27) For $E \subset X$,

(1) $\overline{E}$ is closed.

(2) $E = \overline{E}$ ***if and only if*** E is closed.

(3) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

Proof)

a) If $x \in (\overline{E})^c$, then there exists some $r > 0$ such that $D(x,r) \cap E = \emptyset$. Consequently, $D(x,r) \cap E' = \emptyset$; otherwise, the neighborhood of any $z \in D(x,r) \cap E'$ contains some points of E, leading to a contradiction. Therefore, $D(x,r) \subset (\overline{E})^c$.

c) $E \subset F$ & $E' \subset F' \subset F$. □

Theorem 1.16 (pma 2.28) Let E be a nonempty set of real numbers which is bounded above. Then $\sup E$ is in $\overline{E}$, i.e., $\sup E \in E$ if E is closed.

Definition 1.17 (pma 2.29) Let $E \subset Y \subset X$ where X is a metric space. Then E is ***open relative to*** Y if for every $p \in E$, there exists some $r > 0$ such that $q \in E$ whenever $d(p,q) < r$ and $q \in Y$.

Theorem 1.18 (pma 2.30) E is open relative to Y ***if and only if*** $E = Y \cap G$ for some open subset G of X.

Proof)

⇒) $G = \bigcup_{p \in E} D(p, r_p)$.

⇐) trivial. □

Compactness

Definition 1.19 (pma 2.31, 2.32) Let X be a metric space and $K \subset X$. By an **open cover** of K we mean a collection $\{G_\alpha\}$ of open subsets of X such that $K \subset \bigcup_\alpha G_\alpha$. K is **compact** if every open cover of K contains a finite subcover.

Theorem 1.20 (pma 2.33) Suppose $K \subset Y \subset X$. K is compact relative to X ***if and only if*** K is compact relative to Y.

Proof)

(⇒) Let $\{U_\alpha\}$ be a collection of sets that are open relative to and covers E. Since each U_α is open relative to Y, there exists an open set $G_\alpha \subset X$ such that $U_\alpha = G_\alpha \cap Y$. Consequently, there exists a collection $\{G_{\alpha_n}\}$ that covers E. Given that $E \subset \bigcup_n G_{\alpha_n}$ and $E \subset Y$, it follows that $E \subset \bigcup_n (G_{\alpha_n} \cap Y) = \bigcup_n U_{\alpha_n}$.

(⇐) Let $\{U_\alpha\}$ be a collection of open sets in X that cover E. Since each intersection $U_\alpha \cap Y$ is open in Y, the collection $\{U_\alpha \cap Y\}$ forms an open cover of E in Y.Consequently, there exists a subcollection $\{U_{\alpha_n}\}$ also covers E in X. □

Theorem 1.21 (pma 2.34) Compact subsets of metric spaces are closed.

Proof) Let K be a compact set in a metric space X. Given $q \in K^c$, difine $r_p = \frac{1}{2} d(p,q)$ for each $p \in K$. Then, $\{D(p, r_p)\}$ forms an open cover of K. Consequently, there exists a finite subcover $\{D(p_n, r_{p_n})\}$. Clearly. $\bigcap D(q, r_{p_n}) \subset K^c$. Thus, q is an interior point of K^c, i.e., K^c is open. □

Theorem 1.22 (pma 2.35) Closed subsets of compact sets are compact.

Proof) Let X be a metric space, let $K \subset X$ be a compact set and let $F \subset K$ be a closed set. Suppose $\{U_k\}$ is an open cover of F. Then, $\{U_k\} \cup F^c$ forms an open cover of K. This cover admits a finite subcover of K, clearly containing F. □

Theorem 1.23 (pma 2.36) If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, i.e., if they satisfy the Finite Intersection Property (FIP), then $\bigcap K_\alpha$ is nonempty.

Proof) Suppose, for contradiction, that $\bigcup K_\alpha$ is empty. Then $\exists \beta$ s.t. $K_\beta \not\subset \bigcup_{\alpha \neq \beta} K_\alpha$. Consequently, $K_\beta \subset \bigcup_{\alpha \neq \beta} K_\alpha^c$, which forms a finite subcover of K_β. Therefore, the intersection of its complement and K_β is empty, leading to a contradiction. □

Corollary (pma 2.36) If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$, then $\cap_1^\infty K_n$ is not empty.

Bolzano-Weierstrass theorem

Theorem 1.24 (Bolzano-Weierstrass theorem in the context of compact sets (pma 2.37)) If E is an infinite subset of a compact set K, then E has a limit point in K.

Proof) Suppose, for contradiction, that there are no limit points of E. This implies that for each $x \in E$, there exists a real number $r_x > 0$ s.t. $D(x, r_x) \cap (E \backslash \{x\}) = \emptyset$, i.e., $D(x, r_x)$ contains only the point x. Consequently, the open cover $\{D(x, r_x)\}$ does not form a finite subcover of E. □

Theorem 1.25 (pma 2.38) If $\{I_n\}$ is a sequence of intervals in $\mathbb{R}^1$, such that $I_n \supset I_{n+1}$ $(n = 1, 2, \ldots)$, then $\cap_1^\infty I_n$ is not empty.

(pf) Let $X = \sup\{x_k\}$. We will show that $x \in I_m$ for all $m \geq 1$. For positive integers n, m, we have $a_n \leq a_{n+m} \leq b_{n+m} \leq b_m$. Thus, $x \leq b_m$ for each m, and clearly $a_m \leq x$. $\square$

Theorem 1.26 (pma 2.39) Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cell such that $I_n \supset I_{n+1}$ $(n = 1, 2, \dots)$, then $\bigcap_1^\infty I_n$ is not empty.

Theorem 1.27 (pma 2.40) Every k-cell is compact.

(pf) Suppose, for contradiction, that there are no open covers which form a finite subcover containing I. Let $\{G_\alpha\}$ be an arbitrary open cover of I. Without loss of generality, we may assume $k = 1$ and $I = [a, b]$. Let $c = \frac{a+b}{2}$. Then at least one of the intervals $[a, c]$ or $[c, b]$ is not compact. Denote this interval as I_1. For $n > 1$, define I_n in the same manner. According to the previous theorem, there exists an $x^* \in I_n \subset I \subset \bigcup G_\alpha$ for all $n = 1, 2, \dots$. Clearly, $x^* \in G_\alpha$ for some α. Since G_α is open, there exists $r > 0$ such that $D(x^*, r) \subset G_\alpha$. If n is large enough, by the Archimedian property, $I_n \subset D(x^*, r) \subset G_\alpha$, leading to a contradiction. $\square$

Theorem 1.28 (**Heine–Borel theorem** (pma 2.41)) If a set E in $\mathbb{R}^k$ has one of the following three properties, then it has the other two:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

(pf)

(a) $\Rightarrow$ (b) E is in a k-cell.

(b) $\Rightarrow$ (c) By previous theorem.

(c) $\Rightarrow$ (a) Suppose, for contradiction, that E is neither bounded nor closed. In the first case, if E is not bounded, there exist points $x_n \in E$ such that $|x_n| > n$ for each $n = 1, 2, \dots$. Clearly there are no limit points in the collection $\{x_n\}$. In the second case, assume there exists a limit point $x \in E^c$. Choose $x_n \in E$ so that $d(x, x_n) < \frac{1}{n}$ for $n = 1, 2, \dots$. Now suppose there is a limit point y of $\{x_n\}$ such that $y \neq x$. Then for large enough n, $d(y, x_n) \geq d(y, x) - d(x, x_n) \geq \frac{1}{2}d(y, x)$, leading to a contradiction. $\square$

Theorem 1.29 (**Bolzano-Weierstrass theorem** (pma 2.42)) Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Perfect sets

Theorem 1.30 (pma 2.43) Let P be a nonempty perfect set in $\mathbb{R}^k$. Then P is uncountable.

(pf) Construct $\{K_n\}$ as follows: Let U_1 be any neighborhood of x_1. Suppose that U_n has been constructed, so that $U_n \cap P$ is not empty. Then, choose a neighborhood U_{n+1} of x_{n+1} such that $\overline{U_n} \subset U_{n+1}$, $x_n \notin \overline{U_{n+1}}$, and $U_{n+1} \cap P$ is not empty. If $K_n = \overline{U_n} \cap P$, then no points of P lie in $\bigcap_1^\infty K_n$. This contradicts the previous theorem. $\square$

Definition 1.31 (pma 2.44) Let E_0 be the interval $[0, 1]$. Suppose that E_n has been constructed. Define E_{n+1} by removing the segments $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$ for each nonnegative integer k from E_n. Then $P = \bigcap_{n=1}^{\infty} E_n$ is called the ***Cantor set***.

Theorem 1.32 (pma 2.44) The Cantor set has no segment

(pf) Suppose there exists a segment (a, b) within the Cantor set P. Given the method of the construction of P, it is established that the segment $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$ does not intersect with P. Therefore, if m large enough such that $3^{-m} < \frac{b-a}{4}$, there exists integer k such that the interval (a, b) includes the interval $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$. $\square$

Theorem 1.33 (pma 2.44) The Cantor set is perfect.

Proof) To show that for each $x \in P$ and each $r > 0$, there exists $y \in P$ such that $y \in D(x, r)$: Given $x \in P$ and $r > 0$, let I_{n_k} is the interval in E_n that contains x. If n is large enough, $I_n \subset D(x, r)$, and it is evident that $y = \sup I_k \in P$. Therefore, x is a limit point of P. $\square$

Connected sets

Definition 1.34 (pma 2.45) Two sets $A, B \subset X$ are said to be ***separated*** if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty. A set $E \subset X$ is called ***connected*** if E is not a union of two nonempty separated sets.

Remark (pma 2.46) Disjoint sets $\subset$ separated sets.

Theorem 1.35 (pma 2.47) A set $E \in \mathbb{R}$ is connected ***if and only if*** it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

(pf)

($\Rightarrow$) Sup, for contradiction, that $\exists z \notin E$ s.t. $x < z < y$. Define $A = (-\infty, z)$ and $B = (z, \infty)$. Then $A \cap E$ and $B \cap E$ separate E, leading to a contradiction.

($\Leftarrow$) Suppose $E = A \cup B$ for some separated set A, B in $\mathbb{R}$. Let $x \in A$, $y \in B$ and without loss of generality, assume $x < y$. Let $z = \sup(A \cap [x, y])$. If $z \notin A$, then $z \notin E$, otherwise $z \in B$, i.e., $\overline{A} \cap B \neq \emptyset$. On the other hand, if $z \in A$, by the definition of a limit point, for every $r > 0$, $[z, z + r] \cap B = \emptyset$(otherwise z is also a limit point of B, leading to a contradiction), indicating that there exists $z' \in [z, z + r]$ such that $z' \notin A \cup B$. Thus, in both cases, there exists a point within $[x, y]$ that does not belong to E. $\square$

Section 2. SEQUENCE AND SERIES

Convergent sequence

Definition 2.1 (The definition of convergence of sequences (pma 3.1)) Let $\{p_n\}$ be a sequence in a metric space X.

(1) $\{p_n\}$ is said to **converge** if there is a point $p \in X$ with the following property:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } n \geq N \Rightarrow d(p_n, p) < \epsilon$$

In this case, we write $p_n \to p$, or $\lim_{n\to\infty} p_n = p$.

(2) The set of all point p_n is called the **range** of the sequence.

(3) A sequence is **bounded** if its range is bounded.

Theorem 2.2 (The properties of convergent sequences in metric space (pma 3.2)) Let $\{p_n\}$ be a sequence in a metric space X.

(a) The $\{p_n\}$ converges to $p \in X$ ***if and only if*** every neighborhood of p contains p_n for all but finitely many n.

(b) If $\{p_n\}$ converges, then its limit is unique.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

(d) If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.

Proof)

(b) Given $\epsilon > 0$, choose $N > 0$ such that $n \geq N$ implies $d(p, p_n) < \frac{\epsilon}{2}$ and $d(p', p_n) < \frac{\epsilon}{2}$. Then $d(p, p') \leq d(p, p_n) + d(p', p_n) < \epsilon$.

(c) Given $\epsilon > 0$, there exists $N > 0$ such that $n \geq N$ implies $d(p, p_n) < \epsilon$. Thus, the range of $\{p_n\}$ is bounded by $\max\{d(p, p_1), d(p, p_2), \ldots, d(p, p_{N-1}), \epsilon\}$.

(d) Choose p_n such that $d(p, p_n) < \frac{1}{n}$. Given $\epsilon > 0$, by Archimedian property, there exists a integer N such that $\frac{1}{N} < \epsilon$. If $n \geq N$, then $d(p, p_n) < \epsilon$. □

Theorem 2.3 (Limit operations for complex sequences (pma 3.3)) Suppose $\{s_n\}$, $\{t_n\}$ are complex sequence, and $s_n \to s$, $t_n \to t$ as $n \to \infty$. Then

(1) $\lim_{n\to\infty} (s_n + t_n) = s + t$.

(2) $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty}(c + s_n) = c + s$ for any number c.

(3) $\lim_{n\to\infty} (s_n t_n) = st$.

(4) $\lim_{n\to\infty} \dfrac{1}{s_n} = \dfrac{1}{s}$, provided $s_n \neq 0$ $(n = 1, 2, \ldots)$, and $s \neq 0$.

Theorem 2.4 (The properties of convergent sequences in Euclidean space (pma 3.4))

(a) Suppose $x_n \in \mathbb{R}^k$ $(n = 1, 2, \ldots)$ and $x_n = (a_{1,n}, \ldots, a_{k,n})$. Then $\{x_n\} \to x = (a_1, \ldots, a_k)$ ***if and only if*** $\lim_{n\to\infty} a_{j,n} = a_j$ $(1 \leq j \leq k)$.

(b) Suppose $\{x_n\}$, $\{y_n\}$ are sequences in $\mathbb{R}^k$, $\{\beta_n\}$ is a sequence of real numbers, and $x_n \to x$, $y_n \to y$, $\beta_n \to \beta$. Then $\lim_{n\to\infty}(x_n + y_n) = x + y$, $\lim_{n\to\infty} x_n \dot{y}_n = x\dot{y}$, $\lim_{n\to\infty} \beta_n x_n = \beta x$.

Subsequences

Definition 2.5 (The definition of subsequences (pma 3.5)) Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integer such that $n_1 < n_2 < \ldots$. Then the sequence $\{p_{n_1}\}$ is called a **subsequence** of $\{p_n\}$. If $\{p_{n_i}\}$ converges, its limit is called a **subsequential limit** of $\{p_n\}$.

Theorem 2.6 (**Bolzano–Weierstrass theorem** (pma 3.6))

(a) A sequence in a compact metric space has a subsequences such that converges a point of X.

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

Proof)

(a) Consider two cases: the range of $\{p_n\}$ is either finite or infinite. The first case is trivial. In the second case, there exists a point $p_{n_k} \in D(p, \frac{1}{k})$ for each $k = 1, 2, \ldots$ (thm 1.24), and it is guaranteed that $n_k \leq n_{k+1}$ (thm 1.12).

(b) By (a) and thm 1.29 □

Theorem 2.7 (The set of subsequential limits is closed (pma 3.7)) The subsequential limits of sequence $\{p_n\}$ in a metric space X forms a closed subset of X.

(pf) Let E' be the set of all subsequential limits of $\{p_n\}$. Suppose $r = d(q, p_{n_1})$, where q is a limit point of E' and $q \neq p_{n_1}$. Then there exist a point $x \in E'$ and a point p_{n_2} such that $d(q, x) < \frac{r}{2^2}$ and $d(x, p_{n_2}) < \frac{r}{2^2}$. Consequently, $d(q, p_{n_2}) < \frac{r}{2}$. By induction, we can show that $d(q, p_{n_k}) < \frac{r}{2^{k-1}}$ for each k. Therefore $q \in E'$. □

Cauchy sequences

Definition 2.8 (The definition of Cauchy sequences and diameter (pma 3.8, 3.9)) Let X be a metric space.

(1) A sequence $\{p_n\}$ in X is said to be a **Cauchy sequence** if

$$\forall \epsilon > 0, \exists N \in \mathbb{Z} \text{ s.t. } n, m \geq N \Rightarrow d(p_n, p_m) < \epsilon$$

(2) Let E be a nonempty subset of X, and let S be the set of all real numbers of the form $d(p, q)$, with $p, q \in E$. The $\sup S$ is called the **diameter** of E.

If E_N consists of the points $P_N, P_{N+1}, \ldots$, then $\{p_n\}$ is a Cauchy sequence ***if and only if*** $\lim_{N\to\infty}$ diam $E_N = 0$.

Theorem 2.9 (Properties of diameter (pma 3.10)) Let X be a metric space, let $E \subset X$ be a set, and let K_n be a sequence of compact sets in X such that $K_n \supset K_{n+1}$ for $n = 1, 2, \ldots$.

(1) diam $\overline{E}$ = diam E

(2) If $\lim_{n\to\infty}$ diam $K_n = 0$, then $\bigcap_1^\infty K_n$ consists of exactly one point

Proof)

a) Obviously diam $E <$ diam $\overline{E}$. On the other hand, for $p, q \in \overline{E}$, there exist $p', q' \in E$ such that $d(p,p') < \epsilon$ and $d(q,q') < \epsilon$. Thus $d(p,q) < 2\epsilon +$ diam E. $\square$

Theorem 2.10 (**Cauchy criterion** (pma 3.11))

(a) In any metric space X, every convergent sequence is a Cauchy sequence.

(b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.

(c) (Cauchy criterion) In $\mathbb{R}^k$, every Cauchy sequence converges.

(pf)

(b) There exists a subsequence $p_{n_k} \to p$. Given $\epsilon > 0$, choose N_1 such that $n_k \geq N_1$ implies $d(p_{n_k}, p) < \epsilon$, and choose N_2 such that $n, n_k \geq N_2$ implies $d(p_n, p_{n_k}) < \epsilon$. If $n, n_k \geq \max(N_1, N_2)$, then $d(p_n, p) < 2\epsilon$.

(c) Let E_N be the set consisting of $p_{N+1}, p_{N+2}, \ldots$ such that diam $E_N < 1$. Then the range of $\{p_n\}$ is bounded by the union of $p_1, p_2, \ldots, p_N$ and E_N. Since a bounded set in $\mathbb{R}^k$ is contained in some k-cell, assertion (b) implies (c). $\square$

Completeness

Definition 2.11 (The definition of completeness (pma 3.12)) A metric space in which every Cauchy sequence converges is said to be **complete**.

Remark (1) Every compact metric space is complete.

(2) Every Euclidean space is complete.

(3) Every closed subset of a complete metric space is complete.

Definition 2.12 (The definition of monotone sequences (pma 3.13)) A sequence $\{s_n\}$ of real numbers is said to be

(a) **monotonically increasing** if $s_n \leq s_{n+1}$ for $n = 1, 2, \ldots$.

(b) **monotonically decreasing** if $s_n \geq s_{n+1}$ for $n = 1, 2, \ldots$.

Theorem 2.13 (**Monotone Convergence theorem** (pma 3.14)) A monotonic sequence converges ***if and only if*** it is bounded.

Proof)

⇒) Suppose $s_n \to s$, then $|s - s_n| < 1$ all but finitely many n; hence, $\{s_n\}$ is bounded by $\max\{1+s, s_1+s, \ldots, s_n+s\}$.

⇐) Let s be a least upper bound of $\{s_n\}$ and let $\epsilon > 0$ be given. Since s is a least upper bound, there exists some N such that $s - \epsilon < s_N \leq s$. Since $\{s_n\}$ increases, $n \geq N$ implies $s - \epsilon < s_n \leq s$. $\square$

Upper and lower limits

Definition 2.14 (Divergence to infinity (pma 3.15)) If $\forall M \in \mathbb{R}, \exists N \in \mathbb{Z}$ such that $n \geq N$ implies $s_n \geq M$, then we write $s_n \to +\infty$. On the other hand, if $s_n \leq M$, then we wirte $s_n \to -\infty$.

Definition 2.15 (Upper and lower limits (pma 3.16)) Let $\{s_n\}$ be a sequence of real numbers, and let E be the set of all subsequential limits, including possibly $+\infty, -\infty$. Then $\sup E$ and $\inf E$ are called the **upper** and **lower limits** of $\{s_n\}$; we write $\limsup_{n\to\infty} s_n = \sup E := s^*$, $\liminf_{n\to\infty} s_n = \inf E := s_*$.

Theorem 2.16 (Properties of upper limits (pma 3.17)) In above definition, s^* has the follwing properties:

(a) $s^* \in E$

(b) If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.

Moreover, s^* is the only number with these properties.

(pf)

(a) If $s^* = +\infty$, then E is not bounded above; hense s_n is not bounded above(thm 2.6), and there exists $\{s_{n_k}\}$ such that $s_{n_k} \to +\infty$. If s^* is real, then (a) follows from thm 2.7 and thm 1.16. If $s^* = -\infty$, then there is no subsequential limit. Thus $s_n \to -\infty$.

(b) If there exists infinitely many n such that $s_n \geq x$, then a number $y \in E$ exists such that $y \geq x > s^*$, leading to a contradiction.

(Uniqueness) If p, q satisfy (a) and (b), then we may assume $p < q$. There exists a number x such that $p < x < q$. Since p satisfies (b), q cannnot satisfies (a). $\square$

Theorem 2.17 (Condition of convergence (pma 3.16)) A real-valued sequence converges ***if and only if*** its upper limit and lower limit are the same.

Example 2.18 (pma 3.20) Some speical sequences:

(a) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(b) If $p > 0$, then $\lim_{n\to\infty} p^{\frac{1}{n}} = 1$.

(c) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(d) If $p > 0$ and $a \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{n^a}{(1+p)^n} = 0$.

(e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Series

Definition 2.19 (pma 3.21) Let $\{a_k\}$ be a sequence of complex numbers.

(1) $\sum_{k=1}^{\infty} a_k (= \sum a_k)$ is called a(n) **(infinite) series**.

(2) $s_n = \sum_{k=1}^{n} a_k$ is called a **partial sum** of the series.

(3) If the series converges, then $s = \sum a_k$ is called the sum of the series.

Theorem 2.20 (pma 3.22) By Cauchy criterion(thm 2.10), the series $\sum a_k$ converges ***if and only if*** for every $\epsilon > 0$, there exists an integer N such that $|s_n - s_m| = |\sum_{k=m+1}^{n} a_k| < \epsilon$ whenever $n \geq m \geq N$.

Theorem 2.21 (pma 3.23) If $\sum a_k$ converges, then $\lim_{n\to\infty} a_n = 0$ (taking $m = n$).

Theorem 2.22 (pma 3.24) A series of nonnegative(real) terms converges ***if and only if*** its partial sums form a bounded sequence.

Theorem 2.23 (pma 3.25) (Comparison test) If $|a_n| \leq c_n$ for $n > N$ where N is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges. On the other hand, if $a_n \geq d_n \geq 0$ for $n > N$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

Proof) Given $\epsilon > 0$, there exists an integer N' such that $n \geq m \geq N'$ implies $\sum_{k=m+1}^{n} c_k < \epsilon$. Then if $n \geq m \geq max(N, N')$, then $|\sum_{k=m+1}^{n} a_k| \leq \sum_{k=m+1}^{n} |a_k| \leq \sum_{k=m+1}^{n} c_k < \epsilon$. The other assertion is derived using similar logic. □

Series of nonnegative terms

Theorem 2.24 (pma 3.26) **(Geometric Series)** If $0 \leq x < 1$, then $\sum x^n = \frac{1}{1-x}$. If $x \geq 1$, the series diverges.

Proof)

□

Theorem 2.25 (pma 3.27) Suppose $a_1 \geq a_2 \geq \cdots \geq 0$. Then $\sum a_n$ converges ***if and only if*** $\sum 2^k a_{2^k}$ converges.

Proof)

□

Theorem 2.26 (pma 3.28) $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Theorem 2.27 (pma 3.29) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

The number e

Definition 2.28 (pma 3.30) $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Theorem 2.29 (pma 3.31) $\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$.

(pf) Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$, $t_n = (1+\frac{1}{n})^n$. Then $t_n = 1+1+\frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{n-1}{n})$. Hense $t_n \leq s_n$, so that $\limsup_{n\to\infty} t_n \leq e$. Next, if $n \geq m$, $t_n \geq 1+1+\frac{1}{2!}(1-\frac{1}{n}) + \cdots + \frac{1}{m!}(1-\frac{1}{n})\ldots(1-\frac{m-1}{n})$. Let $n \to \infty$, keeping m fixed. We get $\liminf_{n\to\infty} t_n \geq 1+1+\frac{1}{2!}+\cdots+\frac{1}{m!}$, so that $s_m \leq \liminf_{n\to\infty} t_n$. Letting $m \to \infty$, we finally get $e \leq \liminf_{n\to\infty} t_n$. □

Remark (pma 3.32) $e - s_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{(n+1)!}\{1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \ldots\} = \frac{1}{n!n}$.

Theorem 2.30 (pma 3.32) e is irrational.

(pf) Suppose $e = p/q$. Then $0 < q!(e - s_q) < 1/q$. By our assumption, $q!(e-s_q)$ is an integer, leading to a contradiction. □

The root and ratio tests

2.31 Theorem: Root test

Given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. Then

(a) if $\alpha < 1$, $\sum a_n$ converges;

(b) if $\alpha > 1$, $\sum a_n$ diverges;

(c) if $\alpha = 1$, the test gives no information.

PMA 3.33 (a) If $\alpha < \beta < 1$, the comparison test show the convergence of $\sum a_n$. (b) If $\alpha > 1$, by the definition of limsup, there exist infinitely many n such that $|a_n| > 1$, so that the condition $a_n \to 0$(necessaty for convergence of $\sum a_n$) does not hold. □

2.32 Theorem: Raito test

The series $\sum a_n$

(a) converges if $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$,

(b) divergess if $|\frac{a_{n+1}}{a_n}| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.

PMA 3.34 (a) Suppose for some $N \in \mathbb{Z}$, $\left|\frac{a_{n+1}}{a_n}\right| < \beta < 1$ for $n \geq N$. Then $|a_{N+p}| < \beta|a_{N+p-1}| < \cdots < \beta^p|a_N|$. If we set $n = N+p$, then $|a_n| < |a_N|\beta^{n-N}$ for $n \leq N$, and $\sum a_n$ converges by the comparison test. □

2.33 Proposition

For any sequence $\{c_n\}$ of positive numbers,

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n},$$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

pma 3.37 □

Power series

Definition 2.34 (pma 3.38) Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a **power series**. The numbers c_n are called the **coefficient** of the series; z is a complex number.

Theorem 2.35 (pma 3.39) Put

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \quad R = \frac{1}{\alpha}$$

Then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$.

Summation by parts

Theorem 2.36 (pma 3.41) Given two sequence $\{a_n\}$, $\{b_n\}$, put $A_n = \sum_{k=0}^{n} a_k$ if $n \ge 0$; put $A_{-1} = 0$. Then, if $0 \le p \le q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Proof) Computation. □

Theorem 2.37 (pma 3.42) Suppose

(a) A_n form a bounded sequence;

(b) $b_0 \ge b_1 \ge \dots$;

(c) $\lim_{n\to\infty} b_n = 0$.

Then $\sum a_n b_n$ converges.

Proof)

□

Corollary (pma 3.43) Suppose

(a) $|c_1| \ge |c_2| \ge \dots$;

(b) $c_{2m-1} \ge 0$, $c_{2m} \le 0$ for $m = 1, 2, \dots$;

(c) $\lim_{n\to\infty} c_n = 0$.

Then $\sum c_n$ converges.

Theorem 2.38 (pma 3.44) Suppose

(a) the radius of convergence of $\sum c_n z^n$ is 1;

(b) $c_0 \ge c_1 \ge \dots$;

(c) $\lim_{n\to\infty} c_n = 0$.

Then $\sum c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at $z = 1$.

Proof)

□

Absolute convergence

Definition 2.39 (pma 3.45) The series $\sum a_n$ is said to **converge absolutely** if the series $\sum |a_n|$ converges.

Theorem 2.40 (pma 3.45) If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Addition and multiplication of series

Theorem 2.41 (pma 3.47) If $\sum a_n = A$, and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$, and $\sum c a_n = cA$, for any fixed c.

Definition 2.42 (pma 3.48) Given $\sum a_n$ and $\sum b_n$, we put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0, 1, \dots)$$

and call $\sum c_n$ the **product** of two given series, in other word, the **Cauchy product** (consider the equation $(a_0 + a_1 z + a_2 z^2 + \dots)(b_0 + b_1 z + b_2 z^2 + \dots) = a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots$).

Example 2.43 (pma 3.49) A counterexample to the assertion that the Cauchy product of two convergent series converges:

$$A_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

Consider the Cauchy product of A_n itself:

$$c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}}$$

Since

$$(n-k+1)(k+1) = (\frac{n}{2}+1)^2 - (\frac{n}{2}-k)^2 \le (\frac{n}{2}+1)^2$$

we have

$$|c_n| \ge \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2}$$

Theorem 2.44 (pma 3.50) If $\sum a_n$ converges absolutely, $\sum a_n = A$, $\sum b_n = B$, then the Cauchy product $\sum c_n = AB$.

(pf) □

Theorem 2.45 (pma 3.51) If the series $\sum a_n$, $\sum b_n$, $\sum c_n$ converge to A, B, C, then $C = AB$.

Proof) Latter □

Rearrangements

Definition 2.46 (pma 3.52) Let $\{k_n\}$ be a 1-1 function from $\mathbb{N}$ to $\mathbb{N}$. Putting $a'_n = a_{k_n}$, we say that $\sum a'_n$ is a **rearrangement** if $\sum a_n$.

Theorem 2.47 (pma 3.54) Let $\sum a_n$ be a series of real numbers which converges, but not absolutely. Suppose $-\infty \le \alpha \le \beta \le \infty$. Then there exists a rearrangement $\sum a'_n$ with partial sums s'_n such that $\liminf_{n\to\infty} s'_n = \alpha$, $\limsup_{n\to\infty} = \beta$.

(pf) □

Theorem 2.48 (pma 3.55) If $\sum a_n$ is a series of complex numbers which converges absolutely, then everyy rearrangement of $\sum a_n$ converges,and they all converge to the same sum.

(pf) □

Section 3. CONTINUITY

LIMITS OF FUNCTIONS

Definition 3.1 Suppose X and Y are metric spaces, $E \subset X$, $p \in E'$, $f : E \to Y$. Then we write $\lim_{x \to p} f(x) = q$ if there is a point $q \in Y$ with following property:

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } x \in E \wedge d_X(x,p) < \delta \Rightarrow d_Y(f(x), q) < \epsilon$$

Theorem 3.2 $\lim_{x \to p} f(x) = q$ ***if and only if*** $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$ and $\lim_{n \to \infty} p_n = p$.

Proof)

(⇒) Given $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ if $x \in E$ and $0 < d_X(x,p) < \delta$. Also, there exists N such that $n > N$ implies $0 < d_X(p_n, p) < \delta$.

(⇐) Suppose, for contradiction, that there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists a point $x_\delta \in E$, for which $d_Y(f(x_\delta), q) \geq \epsilon$ but $0 < d_X(x_\delta, p) < \delta$. Take $\delta_n = 1/n$. □

Corollary If f has a limit at p, this limit is unique.

Theorem 3.3 Suppose f, g are complex functions on E, and $\lim_{x \to p} f(x) = A$, $\lim_{x \to p} g(x) = B$. Then

(a) $\lim_{x \to p} (f+g)(x) = A + B$;

(b) $\lim_{x \to p} (fg)(x) = AB$;

(c) $\lim_{x \to p} (\frac{f}{g})(x) = \frac{A}{B}$, if $B \neq 0$.

CONTINUOUS FUNCTIONS

Definition 3.4 Suppose X and Y are metric space, $E \subset X$, $p \in E$, and $f : E \to Y$. Then f is said to be **continuous at** p if

$$\forall \epsilon > 0, \exists \delta > 0 \forall x \in E : d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \epsilon$$

Theorem 3.5 Assume also that $p \in E \cap E'$. Then f is continuous at p ***if and only if*** $\lim_{x \to p} f(x) = f(p)$.

Theorem 3.6 Suppose X, Y, Z are metric space, $E \subset X$, $f : E \to Y$, $g : f(X) \to Z$, and $h : E \to Z$ given by $h(x) = g(f(x))$ for $x \in E$. If f is continuous at $p \in E$ and g is continuous at $f(p)$, then h is continuous at p. h is called the **composition** or **composite** of f and g. We write $h = g \circ f$.

Proof) Given $\epsilon > 0$, since g is continuous at $f(p)$, $\exists \eta > 0$ s.t. $y \in f(E) \wedge d_Y(y, f(p)) < \eta \Rightarrow d_Z(g(y), g(f(p))) < \epsilon$. Since f is continuous at p, $\exists \delta > 0$ s.t. $x \in E \wedge d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \eta$. □

Theorem 3.7 $f : X \to Y$ is a **continuous on** X ***if and only if*** $f^{-1}(V)$ is open in X for every open set V in Y.

Proof)

(⇒) Suppose $p \in E$ and $f(p) \in V \subset Y$. Since V is open, there exists $\epsilon > 0$ s.t. $D_Y(f(p), \epsilon) \subset V$. Also, by definition of continuity, there exists $\delta > 0$ s.t. $x \in X \wedge d_X(x,p) < \delta$ implies $d(f(x), f(p)) < \epsilon$. Thus $D_X(p, \delta) \subset f^{-1}(V)$, i.e. p is an interior point.

(⇐) Given $p \in X$ and $\epsilon > 0$, let $V = D_Y(f(p), \epsilon)$. Then V is open in Y; hence $f^{-1}(V)$ is open; hence there exists $\delta > 0$ such that $(D_X(p,\delta) \cap X) \subset f^{-1}(V)$. □

Corollary f is continuous ***if and only if*** $f^{-1}(C)$ is closed in X for every closed set C in Y. (Consider that $f^{-1}(E^c) = [f^{-1}(E)]^c$)

Theorem 3.8 Let f and g be complex continuous functions on a metric space X. Then $f+g$, fg, f/g are continuous.

Theorem 3.9 (a) Let $f_1, \ldots, f_k$ be real functions on a metric space X, and let f be the mapping of X into $\mathbb{R}^k$ defined by $f(x) = (f_1(x), \ldots, f_k(x))$. Then f is continuous ***if and only if*** each of the functions $f_1, \ldots, f_k$ is continuous.

(b) If f and g are continuous mapping on X into $\mathbb{R}^k$, then $f+g$ and $f\dot{g}$ are continuous on X.

Proof)

(a)

(⇒) $|f_j(x) - f_j(y)| \leq |f(x) - f(y)|$

(⇐) Given $x_0 \in X$ and $\epsilon > 0$, choose $\delta > 0$ s.t. $d(x_0, x) < \delta$ implies $d(f_j(x_0), f_j(x)) < \epsilon/\sqrt{k}$ for $j = 1, \ldots, k$. □

CONTINUITY AND COMPACTNESS

Definition 3.10 $f : E \to \mathbb{R}^k$ is said to **bounded** if there is a real number M such that $|f(x)| \leq M$ for all $x \in E$.

Theorem 3.11 Let X be a compact metric space, Y a metric space, $f : X \to Y$ continuous. Then $f(X)$ is compact.

Proof) Let $\{V_n\}$ be an open cover of $f[X]$. Then $f^{-1}[V_n]$ is open in X; hence there exists a subcover $\{f^{-1}[V_{n_\alpha}]\}$ that covers X. Since $f[f^{-1}[E]] \subset E$ for every $E \subset Y$, the assertion holds. □

Theorem 3.12 Let X be a compact metric space and suppose $f : X \to \mathbb{R}^k$ be a continuous function. Then $f[X]$ is closed and bounded, meaning that f is bounded.

Theorem 3.13 Let X be a compact metric space and suppose $f : X \to \mathbb{R}$ be a continuous function. Then there exist the **maximum** point p and **minimum** point q in X such that $f(q) \leq f(x) \leq f(p)$ for all $x \in X$.

Theorem 3.14 Let X be a compact metric space, Y a metric space, and $f : X \to Y$ a continuous bijection. Then the **inverse map** $f^{-1} : Y \to X$, defined by $f^{-1}(f(x)) = x$ for $x \in X$, is continuous.

Proof) Let V be an open set in X, then V^c is compact, and so is $f[V^c]$. Since $f[V] = (f[V^c])^c$, $f[V]$ is open. By theorem 3.7, the assertion holds. □

Definition 3.15 Let X and Y be metric spaces. We say $f : X \to Y$ **uniformly continuous** on X if

$$\forall \epsilon > 0, \exists \delta > 0, \forall p, q \in X : d_X(p, q) < \delta \Rightarrow d_Y(f(p), f(q)) < \epsilon$$

Theorem 3.16 Let X be a compact metric space, Y a metric space, $f : X \to Y$ a continuous function. Then f is uniformly continuous on X.

Proof) Given $\epsilon > 0$, for each $p \in X$ there exists $\delta_p > 0$ such that $q \in X$, $d_X(p, q) < \delta_p$ implies $d_Y(f(p), f(q)) < \frac{1}{2}\epsilon$. Since the collection $\{D(p, \frac{1}{2}\delta_p)\}$ is an open cover of X, there exists a finite subcover $\{D(p_n, \frac{1}{2}\delta_{p_n})\}$. Put $\delta = \frac{1}{2}\min\{D(p_n, \delta_{p_n}/2)\}$. Let $p, q \in X$ such that $d_X(p, q) < \delta$. Then there exists $1 \leq m \leq n$ such that $p \in D(p_m, \frac{1}{2}\delta_{p_m})$, which implies $d_X(q, p_m) \leq d_X(p, q) + d_x(p, p_m) < \delta + \frac{1}{2}\delta_{p_m} \leq \delta_{p_m}$. Finally, $d_Y(f(p), f(q)) \leq d_Y(f(p), f(p_m)) + d_Y(f(p_m), f(q)) < \epsilon$. □

Theorem 3.17 Let E be a noncompact set in $\mathbb{R}$. Then

(a) there exists a continuous function on E which is not bounded;

(b) there exists a continuous and bounded function on E which has no maximum;

(c) if E is bounded, then there exists a continuous function on E which is not uniformly continuous

Proof) If E is bounded, there exists a point $x_0 \in E' \backslash E$.

(a) (E bounded) $f(x) = \frac{1}{x - x_0}$; (unbounded) $f(x) = x$.

(b) (E bounded) $f(x) = \frac{1}{1+(x-x_0)^2}$; (unbounded) $f(x) = \frac{x^2}{1+x^2}$

(c) The first function in (a). □

CONTINUITY AND CONNECTEDNESS

Theorem 3.18 Let X, Y be metric spaces, $f : X \to Y$ a continuous map. If $E \subset X$ is connected, the $f[E]$ is connected.

Proof) Let Y_1, Y_2 be sets which seperate $f[E]$. Then clearly $X_1 \subset f^{-1}[Y_1]$, and since f is continuous and $f^{-1}[\overline{Y_1}]$ is closed, $\overline{X_1} \subset f^{-1}[\overline{Y_1}]$. Therefore, $\overline{Y_1} \cap Y_2 = \emptyset$ implies $\overline{X_1} \cap X_2 = \emptyset$. Similary, $X_1 \cap \overline{X_2} = \emptyset$, leading to a contradiction. □

Corollary (***Intermediate value theorem***) Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f has the **intermediate value property**: If $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

DISCONTINUITIES

Definition 3.19 Let $f : (a, b) \to (a, b)$, $a \leq x < b$, and $\{t_n\}$ a sequence in (x, b) such that $t_n \to x$. If $f(t_n) \to q$ as $t_n \to x$, then we write $f(x+) = q$.

Definition 3.20 Suppose f is discontinuous at a point x. If $f(x+)$ and $f(x-)$ exist, then f is said to have a discontinuity of the **first kind**, or a **simple discontinuity** at x. Otherwise the discontinuity is said to be of the **second kind**.

MONOTONIC FUNCTIONS

Definition 3.21 Let f be real on (a, b). Then f is said to be **monotonically increasing** on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$.

Theorem 3.22 Let f be monotonically inceasing on (a, b). Then for every point $x \in (a, b)$, $\sup\limits_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf\limits_{x<t<b} f(t)$. Furthermore, if $a < x < y < b$, then $f(x+) \leq f(y-)$.

Proof) Let $A = \sup_{a<t<x} f(t)$. Given $\epsilon > 0$, there exists $\delta > 0$ such that $A - \epsilon < f(x - \delta) \leq A$. Since f is monotonic, it follows that $A - \epsilon < f(x - \delta) \leq f(t) \leq A$ for $x - \delta < t < x$, i.e., $0 < f(t) - A < \epsilon$. The second half of the statement holds since $f(x+) = \inf_{x<t<y} f(t) \leq \sup_{x<t<y} f(t) = f(y-)$. □

Corollary Monotonic functions have no discontinuities of the seconed kind.

Theorem 3.23 Let f be monotonic on (a, b). Then the set of points of (a, b) at which f is discontinuous is at most countable.

Proof) Let E be the set of points at which f is discontinuous. Then for each $x \in E$, there exists a rational number r_x such that $f(x-) < r_x < f(x+)$. Since $x_1 < x_2 \Rightarrow f(x_1+) \leq f(x_2-)$, we see that $r_{x_1} \neq r_{x_2}$ whenever $x_1 \neq x_2$. Therefore, we have a bijective mapping between E and the set $\{r_x\}$. □

Example 3.24 Let $\{c_n\}$ be a set of positive numbers which $\sum c_n$ converges, E the set of rational number in (a, b) arranged in $\{x_n\}$, and $f(x) = \sum\limits_{x_n<x} c_n$ for $a < x < b$. Then

(1) f is monotonically increasing on (a, b);

(2) f is discontinuous at every point of E;

(3) f is continuous at every point of E^c.

Proof)

(2) $f(x_n+) - f(x_n-) = c_n$.

(3) Given $\epsilon > 0$ and $x \in E^c$, choose $N \in \mathbb{Z}^+$ such that $\sum\limits_{n=N+1}^{\infty} < \epsilon$. Let $\delta = \min\{|x - x_n| : 1 \leq n \leq N\}$ ($\delta > 0$ because $x \in E^c$). Then $|x - y| < \delta$ implies $|f(x) - f(y)| \leq \sum_{n=N+1}^{\infty} c_n < \epsilon$ (if $|x - y| < \delta$, then x_n does not lies in the interval x and y, i.e., c_n does not appear in the difference). □

INFINITE LIMITS AND LIMITS AT INFINITY

The concept of 'neighborhood' is extended to infinity, but I am not sure where it is applied.

Definition 3.25 For any $c \in \mathbb{R}$, $(c, +\infty)$ is a neighborhood of $+\infty$. The same applies to $-\infty$.

Definition 3.26 $f(t) \to A$ as $t \to x$ where A and x are in the extended real number system, if for every neighborhood U of A, there is a neighborhood V of x such that $V \cap E$ is not empty and $f(t) \in U$ for all $t \in V \cap E$, $t \neq x$.

Section 4. DIFFERENTIATION

2024-08-16: only def and thm : have to pf

REARRANGEMENTS

Definition 4.1 Let $f : [a, b] \to \mathbb{R}$. $f'(x) = \lim_{t \to x} \dfrac{f(t) - f(x)}{t - x}$ for any $x \in [a, b]$. f' is called the **derivative** of f, and f is **differentiable** at a point x if f' is defined at x.

Theorem 4.2 Differentiability implies continuity.

Proof)

$\square$

Theorem 4.3 If f, g differentiable at x,

(a) $(f + g)'(x) = f'(x) + g'(x)$;

(b) $(fg)'(x) = (f'g)(x) + (fg')(x)$;

(c) $\left(\dfrac{f}{g}\right)'(x) = \dfrac{(f'g)(x) - (fg')(x)}{g^2(x)}$ if $g'(x) \neq x$

Proof)

$\square$

Theorem 4.4 (Chain rule) Let f be a continuous function on $[a, b]$, I a interval containing $[a, b]$, g a function defined on I. Suppose $f'(x)$ exists at some point $x \in [a, b]$, and $g'(f(x))$ exists. If $h(t) = g(f(t))$ for $a \leq t \leq b$, then h is differentiable at x, and $h'(x) = g'(f(x))f'(x)$.

Proof)

$\square$

MEAN VALUE THEOREM

Definition 4.5 Let f be a real function defined on a metric space X. We say f has a **local maximum** at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.

Theorem 4.6 Let f be defined on $[a, b]$. If f has a local maximum at a point $x \in (a, b)$ and if $f'(x)$ exists, then $f'(x) = 0$.

Proof)

$\square$

Theorem 4.7 (Cauchy mean value theorem) If f, g be continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $c \in (a, b)$ at which $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$.

Proof) Put $h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t)$ for $a \leq t \leq b$. $\square$

Corollary (Mean value theorem) If $g(x) = x$, there exists a point $c \in (a, b)$ at which $f(b) - f(a) = (b - a)f'(c)$.

Theorem 4.8 Suppose f is differentiable in (a, b).

(a) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.

(b) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

(c) If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.

Proof) For $a < x_1 < x_2 < b$, there is a point $x \in (x_1, x_2)$ such that $f(x_2) = f(x_1) = (x_2 - x_1)f'(x)$. $\square$

THE CONTINUITY OF DERIVATIVES

Theorem 4.9 (***Darboux's theorem***) Let I be a closed interval, $f : I \to \mathbb{R}$ a differentiable function. Then f' has the intermediate value property.

Proof)

$\square$

Corollary f' cannot have any simple discontinuities on $[a, b]$.

L'HOSPITAL'S RULE

Theorem 4.10 Suppose f, g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose $\dfrac{f'(x)}{g'(x)} \to A$ as $x \to a$. If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, or if $g(x) \to +\infty$ as $x \to a$, then $\dfrac{f(x)}{g(x)} \to A$ as $x \to a$.

(pf) $\square$

DERIVATIVES OF HIGHER ORDER

Definition 4.11

TAYLOR'S THEOREM

Theorem 4.12 (***Taylor's theorem***) Suppose

(a) $f : [a, b] \to \mathbb{R}$;

(b) $f^{(n-1)}$ continuous on $[a, b]$;

(c) $f^{(n)}$ is differentiable on (a, b);

(d) α, β are distinct points of $[a, b]$;

(e) $P(t) := \displaystyle\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)$.

Then there exists a point $x \in (\alpha, \beta)$ such that $f(\beta) = P(\beta) + \dfrac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$. The last term in the second part of the equation is called the **Lagrange form** of the remainder.

(pf) Let M be the number defined by

$$f(\beta) = P(\beta) + M(\beta - \alpha)^n$$

and put

$$g(t) = f(t) - P(t) - M(t-\alpha)^n \quad (a \le t \le b)$$

We now see

$$g^{(n)}(t) = f^{(n)}(t) - n!M$$

Hence it is enough to show that $g^{(n)}(x) = 0$ for some $x \in (\alpha, \beta)$. Note that

$$g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$$

We can repeatedly apply the mean-value theorem to the above equation. That is, since $g(\alpha) = g(\beta) = 0$, there exists some x_1 in (α, β) such that $g'(x_1) = 0$. Similarly, $g''(x_2) = 0$ for some $x_2 \in (\alpha, x_1)$, and so on. □

DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS

Remark Mean-value theorem does not holds for vector-valued function. Consider the function $f(x) = e^{ix}$.

Remark L'Hospital's rule does not holds for vector-valued function. Consider the function $f(x) = x$ and $g(x) = x + x^2 e^{i/x^2}$.

Theorem 4.13 Suppose $f : [a, b] \to \mathbb{R}^k$ is continuous and f is differentiable in (a, b). Then there exists $x \in (a, b)$ such that $|f(b) - f(a)| \le (b-a)|f'(x)|$.

(pf) □

Section 5. THE RIEMANN-STIELTJES INTEGRAL

DEFINITION AND EXISTSANCE OF THE INTEGRAL

Definition 5.1 Let $[a,b]$ be a given interval. By a **partition** P of $[a,b]$ we mean a finite set of points $x_0, x_1, \ldots, x_n$ where $a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b$. We write $\Delta x_i = x_i - x_{i-1}$ for $i = 1, \ldots, n$. Suppose f is a bounded real function on $[a,b]$. Corresponding to each partition P of $[a,b]$ we put $M_i = \sup_{x_{i-1} \le x \le x_i} f(x)$, $m_i = \inf_{x_{i-1} \le x \le x_i} f(x)$, $U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$, and $L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$. $\overline{\int_a^b} f dx = \inf U(P,f)$ and $\underline{\int_a^b} f dx = \sup L(P,f)$ are called the **upper** and **lower Riemann integrals** of f over $[a,b]$, repectively. If the upper and lower integrals are equal, we say that f is **Riemann integrable** on $[a,b]$, we write $f \in \mathscr{R}$($\mathscr{R}$ denotes the set of Riemann integrable functions), and we denoted the common value of them by $\int_a^b f dx$, which is called the **Riemann integral** of f over $[a,b]$. Clearly, if f is bounded, then $L(P,f)$ and $U(P,f)$ exist.

Definition 5.2 Let α be a monotonically increasing function on $[a,b]$. Corresponding to each partition P of $[a,b]$, we write $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. For any real function f which is bounded on $[a,b]$ we put $U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i$ and $L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i$. Then $\int_a^b f d\alpha$ is called the **Riemann-Stieltjes integral** of f with respect to α over $[a,b]$. If it exists, we say f is integrable with respect to α, in the Riemann sense, and write $f \in \mathscr{R}(\alpha)$.

Definition 5.3 We say that the partition P^* is a **refinement** of P is $P^* \supset P$. If $P^* = P_1 \cup P_2$, it is called the **common refinement** of P_1 and P_2.

Theorem 5.4 If P^* is a refinement of P, then $L(P,f,\alpha) \le L(P^*,f,\alpha) \le U(P^*,f,\alpha) \le U(P,f,\alpha)$.

Proof)

□

Theorem 5.5 $\underline{\int_a^b} f d\alpha \le \overline{\int_a^b} f d\alpha.$

Proof)

□

Theorem 5.6 $f \in \mathscr{R}(\alpha)$ on $[a,b]$ ***if and only if*** for every $\epsilon > 0$ there exists a partition P such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$.

Proof)

□

Theorem 5.7 If the right-hand side of thoerem 5.6 holds

(a) for some P and some ϵ, then it also holds with the same ϵ for every refinement of P;

(b) for $P = \{x_0, \ldots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then $\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha < \epsilon$.

(c) If $f \in \mathscr{R}(\alpha)$ and (b) holds, then $\left| \sum_{i=1}^{n} f(t_i) \Delta \alpha_i - \int_a^b f dx \right| < \epsilon$.

Proof)

□

Theorem 5.8 If f is continuous on $[a,b]$ then $f \in \mathscr{R}(\alpha)$ on $[a,b]$.

Proof)

□

Theorem 5.9 If f is monotonic on $[a,b]$ and if α is continuous on $[a,b]$, then $f \in \mathscr{R}(\alpha)$.

Proof)

□

Theorem 5.10 Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathscr{R}(\alpha)$.

Proof)

□

Theorem 5.11 Suppose $f \in \mathscr{R}(\alpha)$ on $[a,b]$, $m \le f \le M$, ϕ is continuous on $[m,M]$, and $h(x) = \phi(f(x))$ on $[a,b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a,b]$.

Proof)

□

Section 6. SEQUENCE AND SERIES OF FUNCTIONS

The functions mentioned in this chapter are all complex-valued functions.

Convergence of a sequence of functions

Definition 6.1 A sequence functions $\{f_n\}$ on a set E converges **pointwise** on E to a function f(we write $f_n \to f$):

$$\lim_{n\to\infty} f_n(x) = f(x),\ \forall x \in E$$

Definition 6.2 $\{f_n\}$ converges **uniformly** to f(we write $f_n \rightrightarrows f$):

$$\forall \epsilon > 0, \exists N \in \mathbb{Z} : \forall x \in E, n \geq N \implies |f_n(x) - f(x)| < \epsilon$$

In other word,

$$\forall \epsilon > 0, \exists N \in \mathbb{Z} : n \geq N \implies \sup_{x\in E} |f_n(x) - f(x)| < \epsilon$$

Alternative definition(Cauchy criterion in $\mathbb{R}$):

$$\forall \epsilon > 0, \exists N \in \mathbb{Z} : \forall x \in E, n, m \geq N \implies |f_n(x) - f_m(x)| < \epsilon$$

On the other hand, the series $\sum f_n(x)$ converges uniformly if the sequence $\{s_n\}$ defined by $\sum_{i=1}^{n} f_i(x) = s_n$ converges uniformly.

Theorem 6.3 (***Weierstrass M-test***) Suppose $|f_n(x)| \leq M_n$. Then $\sum f_n$ converges uniformly if $\sum M_n$ converges.

Uniform converges and continuity

Theorem 6.4 Suppose $f_n \rightrightarrows f$ on a set E in a metric space, and let $x \in E'$. Then

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$

Proof) Let $\epsilon > 0$ be given, and denote $\lim_{t\to x}$ by A_n. There exists N such that $n, m \geq N$, $t \in E$ implies $|f_n(t) - f_m(t) < \epsilon$. Letting $t \to x$, we obtain $|A_n - A_m| < \epsilon$. By the Cauchy criterion, $\lim_{n\to\infty}\lim_{t\to x} f_n(t)$ exists, say A. Then $|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| < \epsilon$ for some N and δ, whenever $n \geq N$ and $t \in D(x, \delta) \cap E$. □

$C(E)$ denotes the set of functions with domain E and codomain $\mathbb{C}$.

Theorem 6.5 If $\{f_n\}$ is a sequence in $C(E)$, and if $f_n \rightrightarrows f$, then $f \in C(E)$.

Proof)

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{t\to x} f(t)$$

On the other hand,

$$\lim_{n\to\infty}\lim_{t\to x} f_n(t) = \lim_{n\to\infty} f_n(x) = f(x)$$

By the previous theorem, $\lim_{t\to x} f(t) = f(x)$, which proves the proposition. □

Theorem 6.6 Suppose K is compact, $\{f_n\}$ is a sequence in $C(K)$, $f_n \to f$ where $f \in C(K)$, and $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, \ldots$. Then $f_n \rightrightarrows f$.

Proof) Put $g_n = f_n - f$. Let $\epsilon > 0$ be given and let K_n be the set off all $x \in K$ with $g_n(x) \geq \epsilon$. Since g_n is continuous, $K_n (= g_n^{-1}[\epsilon, +\infty])$ compact(thm 3.7), hense compact(thm 1.22). Since $g_n(x) \to 0$, for fixed $x \in K$, $x \notin K_n$, i.e., $\bigcap K_n$ is empty. By theorem 1.23, K_N is empty for some N. It follows that $0 \leq g_n(x) < \epsilon$ for all $x \in K$ and all $n \geq N$. □

Section 7. SOME SPECIAL FUNCTIONS

Definition 7.1 (123) test
test2

Definition 7.2 test3

Definition 7.3 (Test) test4

Section 8. Differentiable mapping

Definition of the Derivative

Definition 8.1 (Definition of the derivative (marsden 6.1.1)) A map $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be differentiable at $x_0 \in A$ if there is a linear function, denoted $Df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ and called the **derivative** of f at x_0, such that

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - Df(x_0)(x - x_0)\|}{\|x - x_0\|} = 0$$

In other words, for every $\epsilon > 0$ there is a $\delta > 0$ such that $x \in A$ and $\|x - x_0\| < \delta$ implies

$$\|f(x) - f(x_0) - Df(x_0)(x - x_0)\| \leq \epsilon \|x - x_0\|$$

$x \mapsto f(x_0) + Df(x_0)(x - x_0)$ is supposed to be the **best affine approximation** to f near the point x_0.

Theorem 8.2 (Uniqueness of the derivative (marsden 6.1.2)) Let A be an open set in $\mathbb{R}^n$ and suppose $f : A \to \mathbb{R}^m$ is differentiable at x_0. Then $Df(x_0)$ is uniquely determined by f.

Proof)

□

Example 8.3 If $A = \{x_0\}$, the theorem does not hold.

Matrix representation

Definition 8.4 (Definition of the partial derivative (marsden 6.2.1)) The **partial derivative** $\partial f_j / \partial x_i$ is given by the following limit, when the limit exists:

$$\frac{\partial f_j}{\partial x_i}(x_1, \ldots, x_n) = \lim_{h \to 0} \left\{ \frac{f_j(x_1, \ldots, x_i + h, \ldots, x_n) - f_j(x_1, \ldots, x_n)}{h} \right\}$$

Theorem 8.5 (Definition of the Jacobian metrix (marsden 6.2.2)) Suppose $A \subset \mathbb{R}$ is an open set and $f : A \to \mathbb{R}^m$ is differentiable on A. Then the partial derivatives $\partial f_j / \partial x_i$ exist, and the matrix of the linear map $Df(x)$ with respect to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$ is given by

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

where each partial derivative is evaluated at $x = (x_1, \ldots, x_n)$. This matrix called the **Jacobian matrix** of f or the **derivative metrix**.

Definition 8.6 (Definition of the gradient (marsden 6.2.2)) In the case where $m = 1$, $Df(x)$ is a $1 \times n$ matrix, called the **gradient** of f and denoted by grad$f$ or f. Then,

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix}$$

Example 8.7 Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then $DL(x) = L$.

Continuity of differentiable mappings; differentiable paths

Theorem 8.8 (Local Lipschitz property (marsden 6.3.1)) Suppose $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable on A. Then f is continuous. In fact, for each $x_0 \in A$, there are a constant $M > 0$ and a $\delta_0 > 0$ such that $\|x - x_0\| < \delta_0$ implies $\|f(x) - f(x_0)\| \leq M \|x - x_0\|$, which is called the **local Lipschitz property**.

Definition 8.9 (Curve or Path (marsden 6.3.1)) By a **curve** or **path**, we mean a continuous function $c : \mathbb{R} \to \mathbb{R}^m$. If c is differentiable, then $Dc(t) : \mathbb{R} \to \mathbb{R}^m$ is represented by the vector associated with the single-column matrix

$$\begin{pmatrix} \frac{dc_1}{dt} \\ \vdots \\ \frac{dc_m}{dt} \end{pmatrix}$$

where $c(t) = (c_1(t), \ldots, c_m(t))$. This vector is denoted by $c'(t)$ and is called the **tangent vector** or **velocity** vector to the curve.

Conditions for differentiablity

Theorem 8.10 (Condition for differentiablity (marsden 6.4.1)) Let $A \subset \mathbb{R}$ be open and $f : A \to \mathbb{R}^m$. Suppose $f = (f_1, \ldots, f_m)$ and each of the partials $\frac{\partial f_j}{\partial x_i}$ exists and continuous on A. Then f is differentiable on A.

Definition 8.11 (Definition of directional derivatives (marsden 6.4.2)) Let f be real-valued and defined in a neighborhood of $x_0 \in \mathbb{R}^n$, and let $e \in \mathbb{R}^n$ be a unit vector. Then

$$\frac{d}{dt} f(x_0 + te)|_{t=0} = \lim_{t \to 0} \frac{f(x_0 + te) - f(x_0)}{t}$$

is called the **directional derivative** of f at x_0 in the direction e. If f is differentiable, then

$$\frac{d}{dt} f(x_0 + te)|_{t=0} = Df(x_0) \cdot e$$

Definition 8.12 (Definition of tangent plane (marsden 6.4.2)) The **tangent plane** to the graph of f at $(x_0, f(x_0))$ described by the equation

$$z = f(x_0) + Df(x_0) \cdot (x - x_0)$$

Example 8.13 ((marsden 6.4.3)) Existence of all directional derivatives at a point does not imply differentiablity. Consider the function

$$f = \begin{cases} 1 & \text{if } 0 < y < x^2, \\ 0 & \text{otherwise} \end{cases}$$

Let a unit vector $e = (e_1, e_2)$ be given. $f(e_1 t, e_2 t)$ is 0 for sufficiently small t. Hence, the directional derivative of f at $(0, 0)$ is 0 regardless of the direction of e.

The chain rule

8.14 Theorem: Chain rule

(Marsden 6.5.1) Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}^m$ be differentiable at $x_0 \in A$. Let $B \subset \mathbb{R}^m$ be open, $f[A] \subset B$, and $g : B \to \mathbb{R}^p$ be differentiable at $f(x_0)$. Then the composite $g \circ f$ is differentiable at x_0 and $D(g \circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0)$.

Example 8.15 Suppose

$$f : \mathbb{R}^2 \to \mathbb{R} \text{ given by } (x, y) \mapsto f(x, y)$$

$$g : \mathbb{R}^2 \to \mathbb{R}^2 \text{ given by } (r, \theta) \mapsto (r \cos\theta, r \sin\theta)$$

$$h = f \circ g : \mathbb{R}^2 \to \mathbb{R} \text{ given by } (r, \theta) \mapsto f(r\cos\theta, r\sin\theta)$$

Then

$$Dh(r, \theta) = \begin{pmatrix} \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta & -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta \end{pmatrix}$$

8.16 Proposition

(Marsden 6.6.1) Let $A \subset \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}^m$ and $g : A \to \mathbb{R}$ be differentiable. Then

(a) gf is differentiable.

(b) For $x \in A$, $D(gf)(x) : \mathbb{R}^n \to \mathbb{R}^m$ is given by $D(gf)(x) \cdot e = g(x)(Df(x) \cdot e) + (Dg(x) \cdot e)f(x)$ for all $e \in \mathbb{R}^n$.

8.17 Proposition

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable.

(a) $\nabla f(x)$ is orthogonal to the surface defined by $f(x) =$ constant.

(b) $\nabla f(x)$ is the direction of greatest rate of increase of $f(x)$.

The mean value theorem

8.18 Theorem: Mean value theorem on Euclidean space

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable on open set A. For any $x, y \in A$ such that line segment joining x and y lies in A, there is a point c on that segment such that

$$f(y) - f(x) = Df(c) \cdot (y - x)$$

Proof) Let $h : [0, 1] \to \mathbb{R}$ be given by $h(t) = f((1 - t)x + y)$. □

Taylor's thoerem and higher derivatives

$L(\mathbb{R}^n, \mathbb{R}^m)$ denotes the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.

8.19 Definition: Derivatives of higher order

(Marsden 6.8.1) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable. Then $Df : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^m)$; hence $D(Df)(x_0)$ is a linear map from $\mathbb{R}^n$ to $L(\mathbb{R}^n, \mathbb{R}^m)$. We define $B_{x_0} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ by setting $B_{x_0}(x_1, x_2) = [D^2 f(x_0)(x_1)](x_2)$.

By a ***bilinear map*** $B : E \times F \to G$, where E, F, G are vector space, we mean a map that is linear in each variable seperately.

Given a bilinear map $B : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, if

$$x = \sum_{i=1}^{n} x_i e_i \quad y = \sum_{j=1}^{m} y_j e_j,$$

then

$$B(x, y) = (x_1, \ldots, x_n) \begin{pmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nm} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

8.20 Theorem

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be twice differentiable on the open set A. Then the matrix of $D^2 f(x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with respect to standard basis is given by

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_2} & \ldots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \ldots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$

where each partial derivative is evaluated at the point $x = (x_1, \ldots, x_n)$.

8.21 Theorem: Symmetry of mixed partials

Let $f : A \to \mathbb{R}^m$ be twice differentiable on the open set A with $D^2 f$ continuous. Then $D^2 f$ is symmetric; that is,

$$D^2 f(x)(x_1, x_2) = D^2 f(x)(x_2, x_1),$$

or in terms of components,

$$\frac{\partial^2 f_k}{\partial x_i \partial x_j} = \frac{\partial^2 f_k}{\partial x_j \partial x_i}.$$

8.22 Definition

A function is said to be ***of class*** C^r if the first r derivatives exist and are continuous. Also, it said to be ***smooth*** or ***of class*** C^∞ if it is of class C^r for all positive integers r.

8.23 Theorem: Taylor's theorem

Let $f : A \to \mathbb{R}$ be of class C^r for $A \subset \mathbb{R}^n$ an open set. Let $x, y \in A$ and suppose that the segment joining x and y lies in A. Then there is a point c on that segment such that

$$f(y) - f(x) = \sum_{k=1}^{r-1} \frac{1}{k!} D^k f(x)(y-x, \ldots, y-x) + \frac{1}{r!} D^r f(c)(y-x, \ldots, y-x),$$

where

$$D^k f(x)(y-x, \ldots, y-x) = \sum_{i_1, \ldots, i_k = 1}^{n} \left(\frac{\partial^k f}{\partial x_{i_1} \cdots x_{i_k}} \right) (y_{i_1} - x_{i_1}) \cdots (y_{i_k} - x_{i_k}).$$

Setting $y = x + h$, we can write the Taylor formula as

$$f(x+h) = f(x) + Df(x) \cdot h + \cdots + \frac{1}{(r-1)!} D^{r-1} f(x) \cdot (h, \ldots, h) + R_{r-1}(x, h)$$

where $R_{r-1}(x, h)$ is the remainder such that

$$\frac{R_{r-1}(x,h)}{\|h\|^{r-1}} \to 0 \quad \text{as} \quad h \to 0.$$

By $r \to \infty$, we are led to form the ***Taylor series*** about x_0. If the Taylor converges in a neighborhood of x_0, that is, the reaminder converges to 0 as $r \to \infty$, we say that f is ***real analytic*** at x_0.

Section 9. FUNCTIONS OF SEVERAL VARIABLES

Section 10. INTEGRATION OF DIFFERENTIAL FORMS

Section 12. 0905 Lecture

12.1 Definition

Let $\Omega \subset \mathbb{R}^n$ be open, $f : \Omega \to \mathbb{R}^m$. f is ***differentiable*** if there exists a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that given $\epsilon > 0$,

$$0 < |x - x_0| < \delta \implies |f(x) - f(x_0) - T(x - x_0)| < \epsilon$$

In this case, T is called a ***derivative*** of f at x_0, denoted by $T = Df(x_0)$.

12.2 Definition