M/STAT 501 Fall 2022: Homework 6

Due *Friday*, Nov 11 by 5:00pm in Gradescope

Directions: For this homework, there are three different types of problems: 1) *Graded for Accuracy Problems*, 2) *Graded for Completion Problems*, and 3) *Extra Practice Problems.* Please turn in your work for the *Graded for Accuracy Problems* and the *Graded for Completion Problems*, as well as any *Sources Used* by the due date via Gradescope. Show work and/or provide justification for credit—neatness counts! *The Extra Practice Problems* do not need to be turned in, and the solutions will be posted for you to check and compare your work.

You are encouraged to use R Markdown for your homework, but it is not required. However, **you must use R Markdown for any problems that require the use of R; include both your R code and the R output on the rendered pdf**.

Graded for Accuracy Problems (turn in)

1. An environmental engineer measures the amount (by weight) of particulate pollution in air samples (of a certain volume) collected over the smokestack of a coal-fueled power plant. Let X denote the amount of pollutant per sample when a certain cleaning device on the stack is not operating, and let Y denote the amount of pollutants per sample when the cleaning device is operating under similar environmental conditions. It is observed that X is always greater than $2Y$, and the joint distribution of (X, Y) can be modeled as:

$$f(x,y) = \begin{cases} k & 0 \le x \le 2, 0 \le y \le 1, 2y \le x \\ 0 & \text{else} \end{cases}$$

   a. Find the value of k that makes this a probability density function.
   b. Find the probability that the cleaning device will reduce the amount of pollutant by 1/3 or more.
   c. Find the marginal probability density function of X.
   d. Are X and Y independent? Justify your answer.
   e. Find the conditional pdf of Y given $X = x$, $f_{Y|X}(y|x)$.

2. Let X be the number of customers arriving at a given minute at the drive-up window of a local bank. Let Y be the number who make withdrawals. Assume X is Poisson with $E(X) = 3$, and that the conditional expectation and variance of $Y|X = x$ are $E(Y|X = x) = x/2$ and $Var(Y|X = x) = (x + 1)/3$.
   a. Find $E(Y)$.
   b. Find $Var(Y)$.
   c. Find $E(XY)$.
   d. Name a probability distribution that would be a reasonable model for the conditional distribution of $Y|X = x$. Justify your answer.

Graded for Completion Problems (turn in)

1. Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ be random variables, all of whose expectations and variances exist. Let $a_1, \dots, a_n$, $b_1 \dots, b_n$, $c_1 \dots, c_m$, and $d_1, \dots, d_m$ be constants not all equal to zero.
   a. Show

$$Cov\left(\sum_{i=1}^{n}(a_iX_i + b_i), \sum_{j=1}^{m}(c_jY_j + d_j)\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_ic_jCov(X_i, Y_j).$$

b. Use the result in part a. to show that

$$\begin{aligned} Var(\sum_{i=1}^{n}(a_iX_i + b_i)) &= \sum_{i=1}^{n} a_i^2 Var(X_i) + \sum\sum_{i \neq j} a_i a_j Cov(X_i, X_j) \\ &= \sum_{i=1}^{n} a_i^2 Var(X_i) + 2\sum\sum_{i<j} a_i a_j Cov(X_i, X_j). \end{aligned}$$

Sources Used (turn in)

At the end of your assignment, list and cite **all** sources used when working on this assignment, including individuals with whom you discussed the problems, old homework assignments you may have found, discussion boards, websites, etc. If you did not use any sources besides our textbook or the course notes, please note this.

Extra Practice Problems (do not turn in, solutions posted in D2L)

1. A mountain rescue service studied the behavior of lost hikers so that more effective search strategies could be devised. They decided to determine both the direction traveled and the experience level of hikers. From this study, it is known that the joint probabilities of a hiker being experienced or not and of going uphill or downhill or remaining in the same place are shown in the table below.

	Direction		
	Uphill	Downhill	Remain in same place
Novice	0.10	0.25	0.25
Experienced	0.05	0.10	0.25

Let X be the random variable associated with experience and Y be the random variable associated with direction.

a. Define the random variables X and Y in such a way that they meet the definition of a random variable.
b. Find the marginal distribution of the direction that a lost hiker travels.
c. Are X and Y independent? Justify your answer.

2. Let X and Y denote the values of two stocks at the end of a five-year period. X is uniformly distributed on the interval $(0, 12)$. Given $X = x$, Y is uniformly distributed on the interval $(0, x)$. Find the covariance and correlation between X and Y.