

# STAT 501 Fall 2022 Course Notes

## Chapter 4: Multiple Random Variables

**Motivation:**  $Y$  is a RV that denotes a random numerical outcome of an “experiment”. A random variable was defined to be a function from a sample space  $S$  into the real numbers. Often we are interested *simultaneously* in two or more outcomes of a random experiment like...

- $Y_1$  = the number of eggs in a hen Mallard’s nest, AND  $Y_2$  = the number of eggs that survive and hatch
- $Y_1$  = GRE, AND  $Y_2$  = GPA of a prospective graduate student
- A 501 student’s  $Y_1$  = number of hours spent on extra practice problems, AND  $Y_2$  = midterm score

In each of these situations, we are interested in the 2-dimensional *random vector*  $(Y_1, Y_2)$ . A natural extension of this is to more than two dimensions. In fact, consider taking a random sample of  $n$  MSU students who live in the dorms and measure whether or not they are registered to vote (1 = yes, 0 = no; assume 18+ years old), define

$$Y_i =$$

Then, taken together, we have an *n-dimensional* random vector of voter registrations

$$\mathbf{Y} =$$

and the observed values as,

$$\mathbf{y} =$$

How could we represent the collection of these outcomes as an event?

Another way to write the intersection of  $n$  events is:

or even more simply as,

We can use *multivariate probability distributions* as models for random samples to make inference about the population from which the sample was drawn.

## 4.1 Joint and Marginal Distributions

**DEF 4.1.1 ( $n$ -dimensional random vector)** An  $n$ -dimensional random vector is a function from a sample space  $S$  into  $\mathbb{R}^n$ ,  $n$ -dimensional Euclidean Space.

For simplicity, we'll first focus on *bivariate random variables*, denoted  $(X, Y)$ .

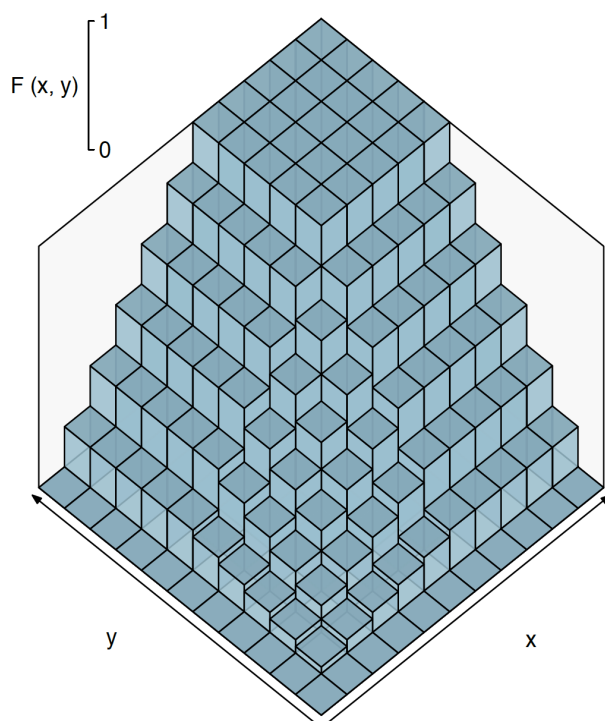
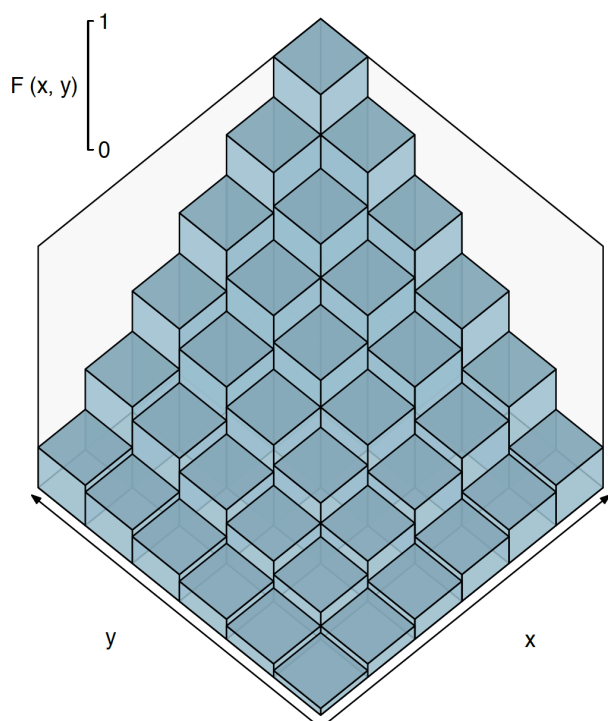
**DEF p. 147 (Bivariate cdf)** Let  $X$  and  $Y$  be two random variables. The joint cumulative distribution function (**bivariate cdf**) of  $X$  and  $Y$  is

$$F_{XY}(x, y) =$$

This extends to  $n$ -dimensional vectors  $\mathbf{X} = (X_1, \dots, X_n)$ :

$$F_{\mathbf{X}}(x_1, \dots, x_n) =$$

**Bivariate Die-roll Example** Suppose we roll two 6-sided fair dice and let  $(X, Y)$  be the random vector of the two outcomes. The visualization of the cdf,  $F_{X,Y}(x, y)$ , below helps provide some intuition about properties of bivariate cdfs.



**Bivariate cdf properties:** The function  $F_{XY}(x, y)$  is a bivariate cdf *iff* the following four conditions are met.

1. If  $a \leq b$  and  $c \leq d$ , then  $F_{XY}(b, d) - F_{XY}(a, d) - F_{XY}(b, c) + F_{XY}(a, c) \geq 0$ .

Write this expression as a probability:

2.  $F_{XY}(x, y)$  is right continuous in each variable, i.e.,  $\lim_{h \rightarrow 0^+} F_{XY}(x + h, y) = \lim_{h \rightarrow 0^+} F_{XY}(x, y + h) = F_{XY}(x, y)$  for all  $x, y$ .

3.  $\lim_{x \rightarrow -\infty, y \rightarrow -\infty} F_{XY}(x, y) = \lim_{x \rightarrow -\infty} F_{XY}(x, y) = \lim_{y \rightarrow -\infty} F_{XY}(x, y) =$

4.  $\lim_{x \rightarrow \infty, y \rightarrow \infty} F_{XY}(x, y) =$  ;  $\lim_{x \rightarrow \infty} F_{XY}(x, y) =$  ;  $\lim_{y \rightarrow \infty} F_{XY}(x, y) =$

## Bivariate Discrete Random Variables

**DEF 4.1.3 (Joint/Bivariate pmf)** Let  $(X, Y)$  be a discrete bivariate random vector. Then the function  $f_{XY}(x, y)$  from  $\mathbb{R}^2$  to  $\mathbb{R}$  defined by  $f_{XY}(x, y) = P(X = x, Y = y)$  is called the *joint probability mass function* — **joint pmf** — of  $(X, Y)$ .

**Note:** A joint pmf can be used to compute the probability of any event in terms of  $(X, Y)$ . Let  $A$  be any subset of  $\mathbb{R}^2$ . Then,

$$P((X, Y) \in A) = \sum_{\{(x, y): (x, y) \in A, f_{XY}(x, y) > 0\}} f_{XY}(x, y)$$

### Properties of a Joint pmf:

1.

2.

**Recall:** How do we use a univariate *pmf* to find a univariate *cdf*?

Joint *pmf* and *cdf* relationship: If  $X$  and  $Y$  are jointly distributed discrete random variables, then the support,  $\{(x_1, y_1), (x_2, y_2), \dots\}$ , is countable, and

$$F_{XY}(x, y) =$$

where  $f_{XY}(x, y)$  is a joint (bivariate) pmf.

## Bivariate Continuous Random Variables

**DEF 4.1.10 (Joint/Bivariate pdf)** A function  $f_{XY}(x, y)$  from  $\mathbb{R}^2$  to  $\mathbb{R}$  is called a *joint probability density function* — **joint pdf** — of the continuous bivariate random vector  $(X, Y)$  if, for every  $A \subset \mathbb{R}^2$ ,

$$P((X, Y) \in A) = \int \int_A f_{XY}(x, y)$$

.

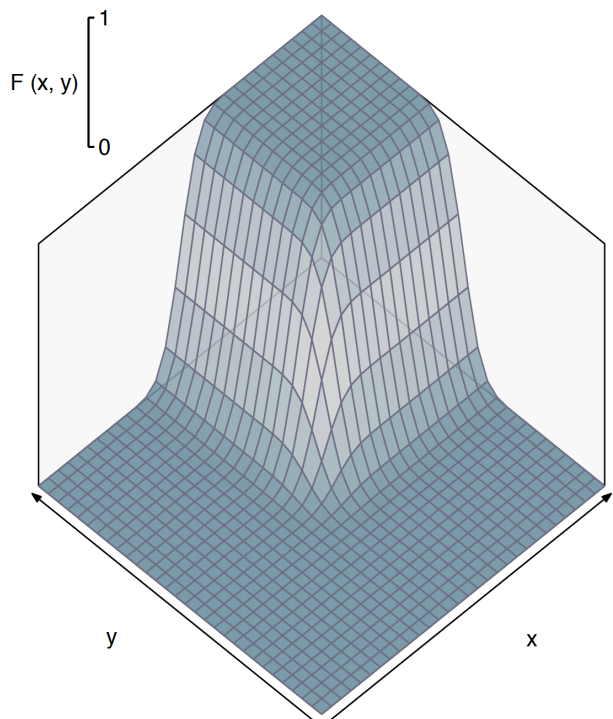
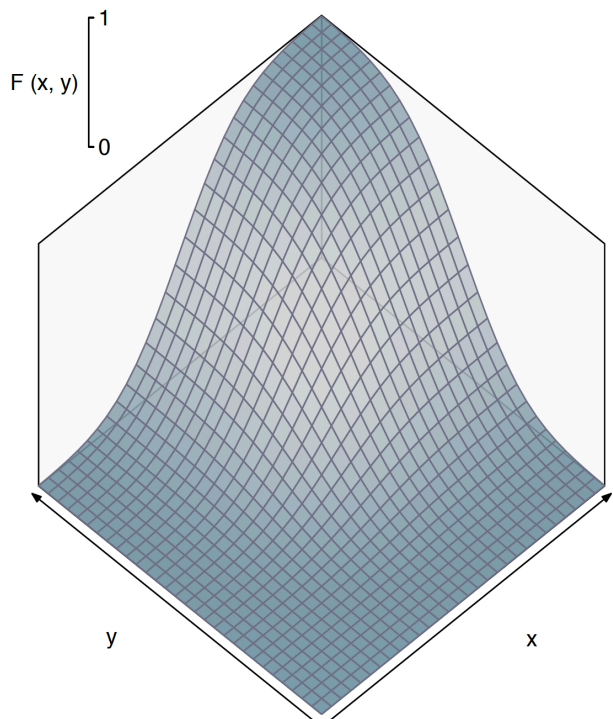
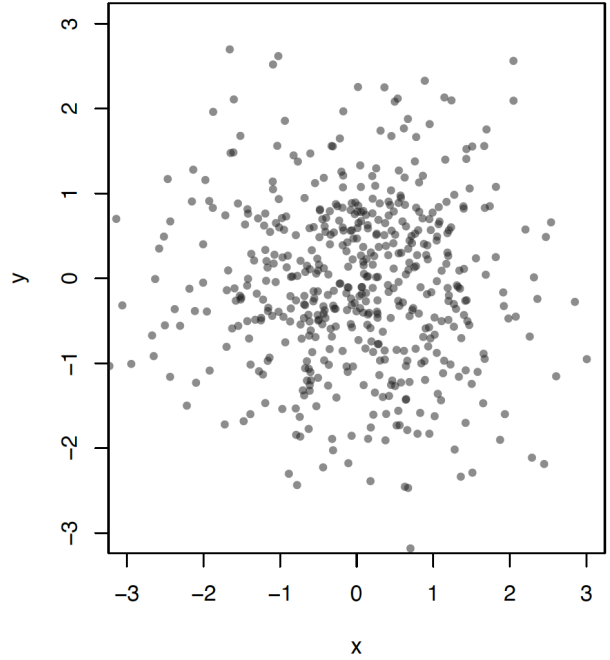
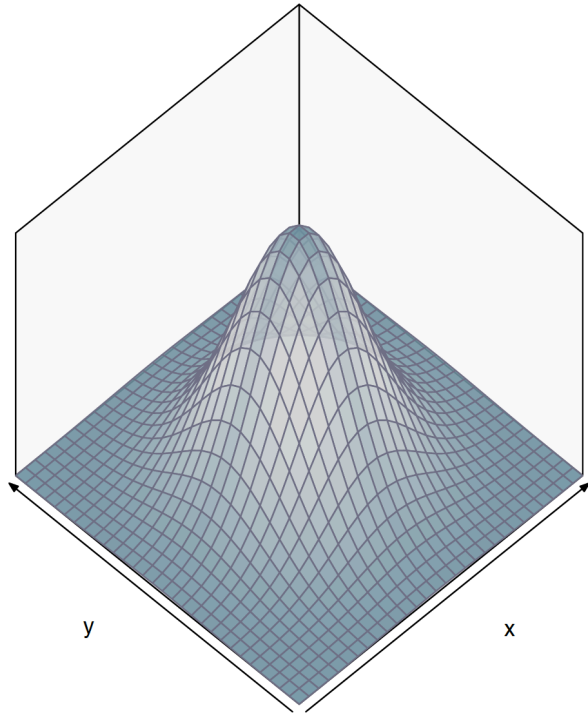
**Note** The notation  $\int \int_A$  means that the limits of integration are set so that the function is integrated over all  $(x, y) \in A$ .

### Properties of a Joint pdf:

1.

2.

**Example (Bivariate Normal Distribution)** Let  $(X,Y) \sim \mathcal{N}\left(\boldsymbol{\mu} = [0,0]', \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$ . A (3-D) plot of the *pdf*,  $f_{XY}(x,y)$ , a (2-D) plot of a random sample of 500  $(x,y)$  pairs, and a (3-D) plot of the *cdf*,  $F_{XY}(x,y)$ , are shown below.



**Recall:** How do we use a univariate *pdf* to find a univariate *cdf*? How do we use a univariate *cdf* to find a univariate *pdf*?

Joint *pdf* and *cdf* relationship: If  $X$  and  $Y$  are jointly distributed (absolutely) continuous random variables, then

$$F_{XY}(x, y) =$$

where  $f_{XY}(x, y)$  is a joint (bivariate) pdf.

From the bivariate Fundamental Theorem of Calculus, this implies that

$$f_{XY}(x, y) =$$

### Example

Let  $X$  and  $Y$  have a joint density function given by  $f_{XY}(x, y) = \begin{cases} kxy & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$

- Find  $k$ .

- Find  $P(X + Y \leq 1)$ .

- Find the joint cdf of  $X$  and  $Y$ . Use this cdf to obtain  $f_{XY}(x, y)$  given above.

## Marginal Distributions

**THM 4.1.6 (Discrete Marginal Distributions)** Let  $X$  and  $Y$  be jointly distributed discrete RVs with joint pmf  $f_{XY}(x, y)$ . Then, the **marginal pmfs** of  $X$  and  $Y$  are given by:

$$f_X(x) =$$

$$f_Y(y) =$$

**THM 4.1.6 (Continuous Marginal Distributions)** Let  $X$  and  $Y$  be jointly distributed continuous RVs with joint pdf  $f_{XY}(x, y)$ . Then, the **marginal pdfs** of  $X$  and  $Y$  are given by:

$$f_X(x) =$$

$$f_Y(y) =$$

### Notes:

- If we sum/integrate one of the RVs out of the bivariate joint *pmf/pdf*, we are left with the *pmf/pdf* of the other variable. Visually, we could “smash down onto a single axis or margin,” which is where the term **marginal probability density function** comes from.

- The marginal pmf/pdf of  $X$  or  $Y$  can be used to compute probabilities or expectations that involve only  $X$  or  $Y$ . However, to compute a probability or an expectation that simultaneously involves both  $X$  and  $Y$ , we must use the joint pmf/pdf of  $X$  and  $Y$ . In general, a joint distribution often tells us additional information about the distribution of  $X$  and  $Y$  that is not found in the marginal distributions. Therefore, we can find a marginal distribution from a joint distribution, but the converse may not be true!



**DEF (Bivariate Expectations)** Suppose  $g(x, y)$  is a real-valued function. If  $X$  and  $Y$  are random variables with joint pmf/pdf  $f_{XY}(x, y)$ ,

$$E[g(X, Y)] =$$

**Example (cont)**

Let  $X$  and  $Y$  have a joint density function given by  $f_{XY}(x, y) = \begin{cases} 4xy & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$ .

- Find the marginal density functions (pdfs) of  $X$  and  $Y$ .
- Find  $E(X^2)$  using both  $f_{XY}(x, y)$  and  $f_X(x)$  to verify you get the same result!

### Example

Let  $X$  and  $Y$  have a joint density function given by  $f_{XY}(x, y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{else} \end{cases}$

- Find  $E(X^2Y)$ .

- Find  $P(X + Y \geq 1)$ .

### Example

Let  $X$  and  $Y$  have a joint density function given by

$$f_{XY}(x, y) = \begin{cases} \binom{y}{x} p^x (1-p)^{y-x} \frac{e^{-\lambda} \lambda^y}{y!} & y = 0, 1, 2, \dots; x = 0, \dots, y \\ 0 & \text{else} \end{cases}$$

- Use the joint *pmf* to show that  $Y \sim \text{Poisson}(\lambda)$  and  $X \sim \text{Poisson}(p\lambda)$ .

**Example (cont)**

- Find  $E(XY)$ .

## 4.2 Conditional Distributions and Independence

Often, when two random variables are observed, the values of the two variables are related. For example, a randomly chosen person's height is typically related to that person's weight—we would think it more likely that a randomly selected person's weight is more than 200 pounds if we were told the person is 73 inches tall than if we were told the person is 41 inches tall. Knowledge about the value of a randomly chosen person's height,  $X$ , gives us some information about the value of that person's weight,  $Y$ , even if it doesn't tell us the exact value. In such cases, we are often interested in the conditional probability of  $Y$  given knowledge that  $X = x$ .

**DEF 4.2.1 (Conditional pmf)** Let  $(X, Y)$  be a discrete bivariate random vector with joint pmf  $f_{XY}(x, y)$  and marginal pmfs  $f_X(x)$  and  $f_Y(y)$ . For any  $x$  such that  $P(X = x) = f_X(x) > 0$ , the *conditional pmf* of  $Y$  given that  $X = x$  is the function of  $y$  denoted by  $f_{Y|X}(y|x)$  and defined by

$$f_{Y|X}(y|x) =$$

For any  $y$  such that  $P(Y = y) > 0$ , the *conditional pmf* of  $X$  given that  $Y = y$  is the function of  $x$  denoted by  $f_{X|Y}(x|y)$  and defined by

$$f_{X|Y}(x|y) =$$

**DEF 4.2.2 (Conditional pdf)** Let  $(X, Y)$  be a continuous bivariate random vector with joint pmf  $f_{XY}(x, y)$  and marginal pmfs  $f_X(x)$  and  $f_Y(y)$ . For any  $x$  such that  $f_X(x) > 0$ , the *conditional pdf* of  $Y$  given that  $X = x$  is the function of  $y$  denoted by  $f_{Y|X}(y|x)$  and defined by

$$f_{Y|X}(y|x) =$$

For any  $y$  such that  $f_Y(y) > 0$ , the *conditional pdf* of  $X$  given that  $Y = y$  is the function of  $x$  denoted by  $f_{X|Y}(x|y)$  and defined by

$$f_{X|Y}(x|y) =$$

**DEF p. 150 (Conditional Expectation)** Let  $X$  and  $Y$  be two random variables. If  $g(Y)$  is a function of  $Y$ , then the conditional expected value of  $g(Y)$  given  $X = x$  is,

$$E(g(Y)|X = x) =$$

**Example**

Consider the joint pdf  $f_{XY}(x, y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{else} \end{cases}$ . We can show the marginal pdfs of  $X$

and  $Y$  are  $f_X(x) = \begin{cases} e^{-x} & 0 < x < \infty \\ 0 & \text{else} \end{cases}$  and  $f_Y(y) = \begin{cases} ye^{-y} & 0 < y < \infty \\ 0 & \text{else} \end{cases}$ , respectively. (*This is left as extra practice for you.*)

a. For  $x > 0$ , find the conditional pdf of  $Y|X = x$ .

b. For  $y > 0$ , find the conditional pdf of  $X|Y = y$ .

c. Find  $E(X|Y = y)$  and  $Var(X|Y = y)$ .

Sometimes the conditional distribution of  $Y$  given that  $X = x$  is different for different values of  $x$  (i.e.,  $Y|X = x$  depends on  $x$ ). However, in some situations, the knowledge that  $X = x$  doesn't give us any more information about  $Y$  than what we already had. This important relationship between  $X$  and  $Y$  is called ***independence***.

**DEF 4.2.5 (Independence)** Let  $(X, Y)$  be a bivariate random vector with joint pdf or pmf  $f_{XY}(x, y)$  and marginal pdfs or pmfs  $f_X(x)$  and  $f_Y(y)$ . Then  $X$  and  $Y$  are independent random variables if, for every  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$

$$f_{XY}(x, y) =$$

Note: If  $X$  and  $Y$  are independent random variables, then

$$f_{X|Y}(x|y) =$$

regardless of the value of  $y$ . Similarly,  $f_{Y|X}(y|x) =$

### Example

Let  $X$  and  $Y$  be jointly distributed random variables with joint pdf

$$f_{XY}(x, y) = \begin{cases} 2x & 0 < x < 1; x \leq y \leq x + 1 \\ 0 & \text{else} \end{cases}$$

a. Are  $X$  and  $Y$  independent? Explain why or why not.

b. Find  $E(e^Y|X = x)$ .

c. For what values of  $X$  does the expectation in part b. hold?

## Independence—Revisited

**Lemma 4.2.7 (AKA "Factorization THM")** Let  $(X, Y)$  be a bivariate random vector with joint pdf or pmf  $f_{XY}(x, y)$ . Then  $X$  and  $Y$  are independent random variables if and only if there exist two functions  $g(x)$  and  $h(y)$  such that, for every  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$ ,

Note:  $g$  and  $h$  do not necessarily need to be pdfs or pmfs

**Proof:** On your own (see p. 153).

### Examples

For each of the following joint distributions, determine whether or not  $X$  and  $Y$  are independent. Explain why or why not.

1. 
$$f_{XY}(x, y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{else} \end{cases}.$$

2. 
$$f_{XY}(x, y) = \begin{cases} 4xy & 0 \leq x \leq 1; 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

**THM 4.2.10 (Using Independence)** Let  $X$  and  $Y$  be independent random variables.

- For any  $A \subset \mathbb{R}$  and  $B \subset \mathbb{R}$ ,  $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ .
- Let  $g(x)$  be a function only of  $x$  and  $h(y)$  a function only of  $y$ . Then

$$E(g(X)h(Y)) =$$

Proof: On your own (see p. 155).



**THM 4.2.12 (Using Independence in MGFs)** Let  $X$  and  $Y$  be independent random variables with moment generating functions  $M_X(t)$  and  $M_Y(t)$ . Then the moment generating function of the random variable  $Z = X + Y$  is given by

$$M_Z(t) =$$

Proof: On your own (see p. 155).

Theorems 4.2.10 and 4.2.12 extend to  $n$  independent random variables (see Theorems 4.6.6 and 4.6.7). That is, if  $X_1, \dots, X_n$  are mutually independent random variables with mgfs  $M_{X_1}(t), \dots, M_{X_n}(t)$ , and  $g_1, \dots, g_n$  are real-valued functions such that  $g_i(x_i)$  is a function only of  $x_i$ ,  $i = 1, \dots, n$ , then

- $E[\prod_{i=1}^n g_i(X_i)] = \prod_{i=1}^n E(g_i(X_i))$ , and
- $M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n M_{X_i}(t)$ .

### Example

Let  $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Bernoulli}(p)$  (“iid” = independent and identically distributed), and define  $Y = \sum_{i=1}^n X_i$ . What distribution does  $Y$  have? Find the mgf of  $Y$ .

## 4.4 Hierarchical Models and Mixture Distributions

Through the use of conditional distributions, we can often model complicated processes by a sequence of relatively simple models placed in a hierarchy.

**DEF 4.4.4 (Mixture Distribution)** A random variable  $X$  is said to have a *mixture distribution* if the distribution of  $X$  depends on a quantity that also has a distribution.

**Example (Random Sum)** An insect lays a large number of eggs, each surviving with probability  $p$ . Suppose the number of eggs laid by the insect, denoted by  $Y$ , follows a Poisson distribution with mean  $\lambda$  and assume that each egg's survival is independent of the others. Let  $X$  be a random variable denoting the number of survivors. Why is this a mixture distribution?

How could we find  $E(X)$  and  $Var(X)$ ?

At times, finding the mean and variance of a random variable in this manner may be difficult and/or tedious. Sometimes, such calculations can be greatly simplified using the following theorems:

**THM 4.4.3 (Conditional Mean Identity)** For any two random variables  $X$  and  $Y$ ,

$$E(X) = \qquad \qquad \qquad \text{and} \qquad E(Y) =$$

provided the expectations exist.

**Proof:**

**THM 4.4.7 (Conditional Variance Identity)** For any two random variables  $X$  and  $Y$ ,

$$\text{Var}(X) =$$

provided the expectations exist.

**Proof:**

**Example (Revisited)** An insect lays a large number of eggs, each surviving with probability  $p$ . Suppose the number of eggs laid by the insect, denoted by  $Y$  follows a Poisson distribution with mean  $\lambda$  and assume the each egg's survival is independent of the others. Let  $X$  be a random variable denoting the number of survivors. Earlier, we stated  $X|Y \sim \text{Binomial}(Y, p)$ , where  $Y \sim \text{Poisson}(\lambda)$ . Use this information to find  $E(X)$  and  $\text{Var}(X)$ .

Recall: Earlier in the notes (pg. 11), we used the following joint pmf

$$f_{XY}(x, y) = \begin{cases} \binom{y}{x} p^x (1-p)^{y-x} \frac{e^{-\lambda} \lambda^y}{y!} & y = 0, 1, 2, \dots; x = 0, \dots, y \\ 0 & \text{else} \end{cases}$$

to show that  $Y \sim \text{Poisson}(\lambda)$  and  $X \sim \text{Poisson}(\lambda p)$ . We could use this to also show that  $X|Y \sim \text{Binomial}(Y, p)$ . This turns out to be the same mixture distribution that we used above! Use the marginal pmf of  $X$  to find  $E(X)$  and  $\text{Var}(X)$  and compare to the results above.

**Example (Revisited)** Consider a generalization of the previous example, where instead of one mother insect, there are a large number of mothers and one mother is chosen at random. We are still interested in the number of survivors, but the number of eggs laid does not follow the same Poisson distribution for each mother, i.e., assume the rate at which eggs are laid for a randomly selected insect,  $\Lambda$ , follows an exponential distribution with mean  $\beta > 0$ . Let  $X$  be a random variable denoting the number of survivors, and let  $Y$  be a random variable denoting the number of eggs laid.

a. Use this information to find  $E(X)$ ,  $E(Y)$ , and  $Var(Y)$ .

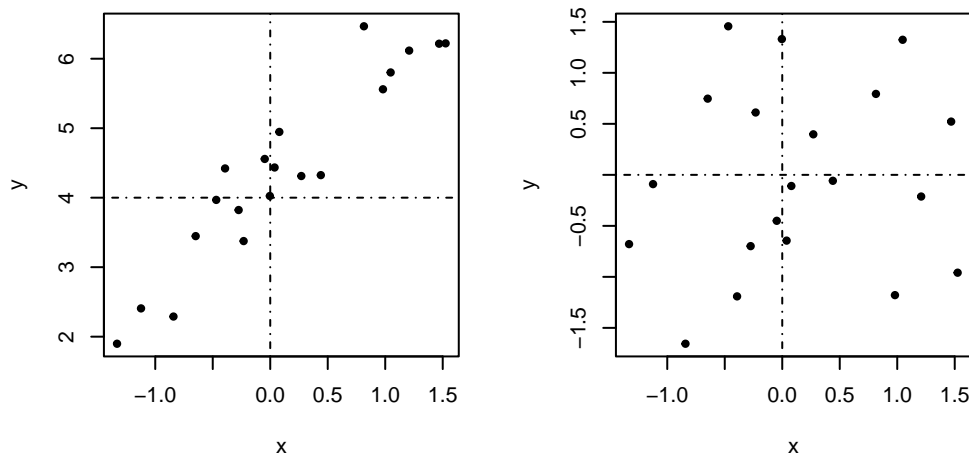
b. Show that  $Y \sim Geometric^* \left( \frac{1}{1+\beta} \right)^1$  and use this result to find  $E(Y)$  and  $Var(Y)$ .

---

<sup>1</sup>The asterisk denotes that we are modeling the number of “failures” until the first success rather than the number of trials.

## 4.5 Covariance and Correlation

How can we measure the *linear* association between two RVs? Consider the following figures, dashed lines represent the true mean values ( $\mu_X$  and  $\mu_Y$ ) for  $X$  and  $Y$ , respectively.



**DEF 4.5.1 and THM 4.5.3 (Covariance)** Let  $X$  and  $Y$  be bivariate random variables with  $E(X) = \mu_X$ ,  $E(Y) = \mu_Y$ ,  $Var(X) = \sigma_X^2$  and  $Var(Y) = \sigma_Y^2$ . The *covariance* between  $X$  and  $Y$  is

$$Cov(X, Y) =$$

Note:  $Cov(X, Y) = Cov(Y, X)$ . And  $Cov(X, X) =$

Proof: On own, (see pg 170 in text).

- What does a large covariance imply?
- What does a small covariance imply?
- How about a covariance of  $\approx 0$ ?
- Are covariances easy to compare? (i.e,  $Cov(X, Y)$  vs.  $Cov(U, V)$ )

**DEF 4.5.2 (Correlation)** The correlation between  $X$  and  $Y$  is given by,

$$\rho_{XY} =$$

Where  $\rho_{XY}$  is also called the correlation coefficient.

- **Covariance** is a measure of direction of linear association between two quantitative RVs, whereas **correlation** measures the direction *and* strength of the linear association between two quantitative random variables.

**Example** Consider the joint pdf  $f_{XY} = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{else} \end{cases}$ . The marginal pdfs of  $X$  and  $Y$  are  $f_X(x) = \begin{cases} e^{-x} & 0 < x < \infty \\ 0 & \text{else} \end{cases}$  and  $f_Y(y) = \begin{cases} ye^{-y} & 0 < y < \infty \\ 0 & \text{else} \end{cases}$ , respectively. Find  $Corr(X, Y)$ .

**Facts about Correlation,  $\rho_{X,Y}$ , (THM 4.5.7)** For any random variables  $X$  and  $Y$ ,

- 
- $|\rho_{XY}| = 1$  if and only if there exists numbers  $a \neq 0$  and  $b$  such that  $P(Y = aX + b) = 1$ . If  $\rho_{XY} = 1$ , then  $a > 0$  and if  $\rho_{XY} = -1$ , then  $a < 0$ .

The second property states that if there is a line  $Y = aX + b$  with  $a \neq 0$  such that the values of  $(X, Y)$  have a high probability of being near this line, then the correlation between  $X$  and  $Y$  will be near 1 or -1. But, if no such line exists, the correlation will be near 0.

Proof: See text 172–173 (on own).

**THM 4.5.5 (Independence)** If  $X$  and  $Y$  are independent random variables,  $Cov(X, Y) = 0 \implies Corr(X, Y) = 0$ .

Proof:

- Is the converse true? That is does  $Cov(X, Y) = \rho_{XY} = 0 \stackrel{?}{\implies} X$  independent of  $Y$ ?

**THM 4.5.6 (Variance of Linear Functions of Random Variables)** If  $X$  and  $Y$  are any two RVs and  $a$  and  $b$  are any two constants, then

$$Var(aX + bY) =$$

If  $X$  and  $Y$  are independent RVs, then

$$Var(aX + bY) =$$

Note: In addition, if  $X$  and  $Y$  are any two random variables and  $a$  and  $b$  are any two constants, then

$$Cov(aX, bY) =$$

$$Cov(X + a, Y + b) =$$

$$Cov(X, aX + b) =$$

Proofs: On own, see pg. 172 in text.

**Example (Revisited)** Consider the joint pdf  $f_{XY} = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{else} \end{cases}$ . The marginal pdfs of  $X$  and  $Y$  are  $f_X(x) = \begin{cases} e^{-x} & 0 < x < \infty \\ 0 & \text{else} \end{cases}$  and  $f_Y(y) = \begin{cases} ye^{-y} & 0 < y < \infty \\ 0 & \text{else} \end{cases}$ , respectively. Find  $Var(3X - 4Y + 2)$ .



## 4.3 Bivariate Transformations

Often, we are interested in finding the probability distribution of a function of two or more random variables and/or the joint distribution of functions of multiple random variables. For example, suppose two random variables  $X$  and  $Y$  have the joint distribution  $f_{XY}(x, y)$ . We are often interested in a new bivariate random vector  $(U, V)$  defined by  $U = g_1(X, Y)$  and  $V = g_2(X, Y)$ , where we want to find either the cdf/pdf/pmf of  $U$  or  $V$  or the joint distribution of  $(U, V)$ . In such instances, we can extend the methods used in the univariate case to the bivariate case.

### Discrete Case

Suppose two discrete random variables  $X$  and  $Y$  have the joint pmf  $f_{XY}(x, y)$ . To find the joint distribution of  $U = g_1(X, Y)$  and  $V = g_2(X, Y)$

1. Find the support for either
  - $U$  and  $V|U = u$  or
  - $V$  and  $U|V = v$ .
2. Rewrite the joint pmf of  $(U, V)$  in terms of  $(X, Y)$ .
  - Recall:  $f_{XY}(x, y) = P(X = x, Y = y)$
  - No Jacobians!

*Note:* Generally (in this class),  $g_1$  and  $g_2$  will be 1-1 and given  $(u, v) \in \{(u, v) : u = g_1(x, y), v = g_2(x, y) \text{ for some } (x, y) \text{ such that } f_{XY}(x, y) > 0\}$ , the set  $A_{UV} = \{(x, y) : u = g_1(x, y), v = g_2(x, y)\}$  will be a singleton set. In this case,

$$\begin{aligned} f_{UV}(u, v) &= P(U = u, V = v) = P(g_1(X, Y) = u, g_2(X, Y) = v) \\ &= P(X = h_1(u, v), Y = h_2(u, v)) = f_{XY}(h_1(u, v), h_2(u, v)). \end{aligned}$$

Where  $h_1$  and  $h_2$  are the single-valued inverse mappings of  $u = g_1(x, y)$  and  $g_2(x, y)$ .

**Example:** Let  $X \sim \text{Poisson}(\theta)$  and  $Y \sim \text{Poisson}(\lambda)$  where  $X$  and  $Y$  are independent random variables. Let  $U = X + Y$  and  $V = Y$ . Find the distribution (pmf) of  $U = X + Y$ .

**Example ctd.**

**Continuous Case** Now, suppose  $X$  and  $Y$  are (absolutely) continuous random variables. If there is a one-to-one transformation from the support of  $(X, Y)$  to the support of  $(U, V)$ , the Jacobian method discussed in the univariate case can be extended to find the joint distribution,  $f_{UV}(u, v)$ , in the bivariate case.

Suppose  $X$  and  $Y$  are (absolutely) continuous random variables with joint pdf  $f_{XY}(x, y)$ . Let  $U = g_1(X, Y)$  and  $V = g_2(X, Y)$ . Also suppose that for all  $(x, y)$  such that  $f_{XY}(x, y) > 0$ , the transformation set  $u = g_1(x, y)$  and  $v = g_2(x, y)$  is one-to-one, and for each  $(u, v)$  in the support of  $(U, V)$ ,  $x = h_1(u, v)$  and  $y = h_2(u, v)$ , where  $h_1$  and  $h_2$  are single-valued inverse mappings. If  $x$  and  $y$  have continuous partial derivatives with respect to  $u$  and  $v$  and Jacobian,

$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \neq 0$ , then the joint pdf of  $U$  and  $V$  is

**Example** Let  $X$  and  $Y$  be independent exponential random variables, both with mean  $\beta$ . Let  $U = \frac{X}{X+Y}$  and  $V = X + Y$ . Find the marginal pdf of  $U$ .

**Example** Let the random variables  $X$  and  $Y$  respectively denote the premium and claims of a randomly selected insurance policy. Assume  $X$  and  $Y$  are independent random variables, where  $X \sim \text{Gamma}(2, 5)$  and  $Y \sim \text{Exp}(5)$ .

- a. What is the probability a randomly chosen claim will be more than the premium?

### Example ctd

- b. Find the pdf  $Z = X/Y$

### What if the transformations are not one-to-one?

- Example 4.3.6: If the transformations are not one-to-one, we can proceed similar to how we did with univariate transformations. That is, we can divide the support of  $(X, Y)$  into regions on which the transformations are one-to-one and sum those which have common support for  $(U, V)$ . For an example of how to use this approach, please see Example 4.3.6 on p. 162 of the text.

In this situation, another option is to find the distribution of  $(U, V)$  by rewriting the cdf  $F_{UV}(u, v) = P(U \leq u, V \leq v)$  in terms of  $X$  and  $Y$ .

## 4.6 Multivariate Distributions

Recall: A (univariate) random variable is a function from a sample space  $S$  into the real numbers,  $\mathbb{R}$ . In the bivariate case, we are interested in a two-dimensional random vector  $(X, Y)$  that is, a function from a sample space  $S$  into  $\mathbb{R}^2$ . However, we can also consider  $n$ -dimensional random vectors, or random vectors with more than two random variables. In this case, many of the concepts discussed earlier generalize from the bivariate to the multivariate setting.

### DEF 4.1.1 ( $n$ -dimensional Random Vector)

An  $n$ -dimensional random vector is a function from a sample space  $S$  into  $\mathbb{R}^n$ ,  $n$ -dimensional Euclidean space.

As before, the concepts of marginal and conditional distributions, independence, and so on that were extended to bivariate random vectors, can also be extended to  $n$ -dimensional random vectors.

**Multivariate cdf Definition** The multivariate cumulative distribution function of  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  on  $\mathbb{R}^n$  is defined as,

$$F_{\mathbf{X}}(x_1, \dots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n)$$

**Multivariate pmf (Eqn 4.6.1)** If  $(X_1, X_2, \dots, X_n)$  is a discrete random vector (the sample space is countable), then the **joint pmf** is a function

$$f_{\mathbf{X}}(\mathbf{x}) = P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) \text{ for each } (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Then for any  $A \in \mathbb{R}^n$ ,  $P(\mathbf{X} \in A) = \sum_A f_{\mathbf{X}}(\mathbf{x})$ .

**Multivariate pdf (Eqn 4.6.2)** If  $(X_1, X_2, \dots, X_n)$  is an (absolutely) continuous random vector, then the joint pdf is a function  $f_{\mathbf{X}}(\mathbf{x})$  that satisfies

$$P(\mathbf{X} \in A) = \int \int \dots \int_A f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = \int \int \dots \int_A f(x_1, x_2, \dots, x_n) dx_1 \dots dx_n \text{ for any } (x_1, \dots, x_n) \in \mathbb{R}^n.$$

*Note:* These integrals are  $n$ -fold integrals with limits of integration set so that the integration is over all points  $\mathbf{x} \in A$ .

**Marginal Distributions (Equations 4.64 & 4.65)** The marginal *pdf* or *pmf* of any subset of the coordinates of  $\mathbf{X}$  can be computed by integrating or summing the joint pdf or pmf over all possible values of the other coordinates.

- Continuous Case: If  $\mathbf{X} = (X_1, X_2, \dots, X_k, X_{k+1}, \dots, X_n)$  is an (absolutely) continuous random vector, then the *marginal pdf* of  $(X_1, X_2, \dots, X_k)$  is  $f(x_1, \dots, x_k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x_1, \dots, x_n) dx_{k+1} dx_{k+2} \dots dx_n$  for every  $(x_1, \dots, x_k) \in \mathbb{R}^k$ . For example, the marginal pdf of the random variable  $X_j$  is  $f_{X_j}(x_j) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x_1, \dots, x_n) dx_1 dx_2 \dots dx_{j-1} dx_{j+1} \dots dx_n$  for every  $x_j \in \mathbb{R}$ .

That is, if we integrate  $n - k$  ( $k < n$ ) continuous random variables out of the multivariate *pdf*, we are left with the marginal *pdf* of the other  $k$  variable(s).

Similarly, we can get the *marginal pmf* of a discrete random variable by summing the other random variables out of the multivariate *pmf*.

- Discrete Case: If  $\mathbf{X} = (X_1, X_2, \dots, X_k, X_{k+1}, \dots, X_n)$  is discrete random vector, then the *marginal pmf* of  $(X_1, X_2, \dots, X_k)$  is  $f(x_1, \dots, x_k) = \sum_{(x_{k+1}, x_{k+2}, \dots, x_n) \in \mathbb{R}^{n-k}} f(x_1, \dots, x_n)$  for every  $(x_1, \dots, x_k) \in \mathbb{R}^k$ .

**Multivariate Expectations (Eqn 4.6.3)** Suppose  $g(\mathbf{x}) = g(x_1, \dots, x_n)$  is a real-valued function defined on the sample space of  $(X_1, \dots, X_n)$ . Then the expected value of  $g(\mathbf{X})$  is

- Continuous Case:  $E(g(\mathbf{X})) = E(g(X_1, \dots, X_n)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_1, \dots, dx_n$
- Discrete Case:  $E(g(\mathbf{X})) = E(g(X_1, \dots, X_n)) = \sum_{\mathbf{x} \in \mathbb{R}^n} g(x_1, \dots, x_n) f(x_1, \dots, x_n)$

**Conditional Distributions (Eqn 4.6.6)** The conditional *pdf* or *pmf* of a subset of the coordinates of  $(X_1, \dots, X_n)$  given the values of the remaining coordinates is obtained by dividing the joint *pdf* or *pmf* by the *marginal pdf* or *pmf* of the remaining coordinates. For example, if  $f(x_1, x_2, \dots, x_k) > 0$ , the conditional pdf or pmf of  $(X_{k+1}, \dots, X_n)$  given  $(X_1, \dots, X_k)$  is defined by

$$f(x_{k+1}, \dots, x_n | x_1, \dots, x_k) = \frac{f(x_1, \dots, x_n)}{f(x_1, \dots, x_k)}.$$

**Example:** Let  $X_1, X_2, X_3$  be continuous random variables with joint pdf

$$f(x_1, x_2, x_3) = \begin{cases} c & 0 < x_1 < x_2 < x_3 < 1 \\ 0 & \text{else} \end{cases}$$

- a. Find  $c$
- b. What is the marginal pdf of  $X_3$
- c. Find  $f_{X_1 X_2}(x_1, x_2)$
- d. Find  $E(X_1 X_2 X_3)$
- e. Find the conditional pdf  $f_{X_1 X_2 | X_3}(x_1, x_2 | x_3)$
- f. Find the conditional pdf  $f_{X_3 | X_1 X_2}(x_3 | x_1, x_2)$



The concept of independence can also be extended from the bivariate case to the multivariate case.

### Mutual Independence (DEF 4.6.5 Random Variable Version)

- Let  $(X_1, \dots, X_n)$  be an  $n$ -dimensional random vector with joint pdf or pmf  $f(x_1, \dots, x_n)$  and marginal pdfs or pmfs for each  $X_i$  denoted by  $f_{X_i}(x_i)$ . Then  $X_1, \dots, X_n$  are mutually independent random variables if, for every  $(x_1, \dots, x_n) \in \mathbb{R}^n$ ,

$$f(x_1, x_2, \dots, x_n) = f_{X_1}(x_1) \dots f_{X_n}(x_n) = \prod_{i=1}^n f_{X_i}(x_i).$$

- Note:** *Mutual independence implies that any pair of random variables is pairwise independent. However, the converse is not true – pairwise independence does not imply mutual independence!*

### Mutual Independence – Revisited

- Theorem 4.6.11 (Generalization of Lemma 4.2.7 (Factorization THM); Random Variable Version)** Let  $(X_1, \dots, X_n)$  be an  $n$ -dimensional random vector with joint pdf or pmf  $f(x_1, \dots, x_n)$ . Then,  $X_1, \dots, X_n$  are mutually independent random variables if and only if there exist functions  $g_i(x_i)$  such that, for every  $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  the joint pdf or pmf of  $(X_1, \dots, X_n)$  can be written as  $f(x_1, x_2, \dots, x_n) = g_1(x_1)g_2(x_2) \dots g_n(x_n)$ .
- Note:**  $g_i$  are not necessarily pdfs or pmfs, and the supports must also factor!

### Mutual Independence – Continued: THM 4.6.12. (Generalization of THM 4.3.5 Random Variable Version)

- Let  $X_1, \dots, X_n$  be independent random variables, and let  $g_i(x_i)$  be a function of  $x_i$  only. Then the random variables  $U_i = g_i(X_i)$  are mutually independent.
- Mutually independent random variables have many nice properties!*

### Variance of Linear Functions of Random Variables

- Let  $(X_1, \dots, X_n)$  be an  $n$ -dimensional random vector. In general,  $Var\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i^2 Var(X_i) + 2 \sum_{i < j} a_i a_j Cov(X_i, X_j)$ .
- However, if  $X_1, \dots, X_n$  are mutually independent random variables, then  $Var\left(\sum_{i=1}^n a_i X_i\right) =$

### Multivariate Expectations – Mutually Independent Random Variables

- Theorem 4.6.6 (Generalization of Theorem 4.2.12):** Let  $X_1, \dots, X_n$  be mutually independent random variables, and let  $g_i(x_i)$  be real-valued functions such that  $g_i(x_i)$

is a function of  $x_i$  only. Then  $E(g_1(x_1)g_2(x_2)...g_n(x_n)) = [E(g_1(X_1))][E(g_2(X_2))] \times ... \times [E(g_n(X_n))]$ .

### Multivariate Transformations for Continuous Random Variables (One-to-One Version)

Suppose  $X_1, \dots, X_n$  are (absolutely) continuous random variables with joint pdf  $f_{X_1 \dots X_n}(x_1, \dots, x_n)$ . Let  $A = \{\mathbf{x} : f_{\mathbf{X}}(\mathbf{x}) > 0\}$ . Consider the random vector  $(U_1, \dots, U_n)$ , defined by  $U_i = g_i(X_1, \dots, X_n)$ . Suppose that the transformation  $(U_1, \dots, U_n) = g_1(\mathbf{X}), \dots, g_n(\mathbf{X})$  is a one-to-one transformation from  $A$  to  $B = \{\mathbf{u} : (u_1, \dots, u_n) = (g_1(\mathbf{x}), \dots, g_n(\mathbf{x})) \text{ for some } \mathbf{x}\}$ . Then, for each  $i$ , if the  $i^{th}$  inverse exists,  $x_i = h_i(u_1, \dots, u_n)$ ,  $i = 1, \dots, n$ . If  $x_1, x_2, \dots, x_n$  have continuous partial derivatives with respect to  $u_1, u_2, \dots, u_n$  and Jacobian,

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \dots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \dots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \dots & \frac{\partial x_n}{\partial u_n} \end{vmatrix} \neq 0$$

, then the joint pdf of  $U_1, U_2, \dots$  and  $U_n$  for  $\mathbf{u} \in B$  is,

$$f_{U_1 \dots U_n} = \begin{cases} f_{X_1 \dots X_n}(h_1(u_1, \dots, u_n), \dots, h_n(u_1, \dots, u_n))|J| & (u_1, \dots, u_n) \in B \\ 0 & \text{else} \end{cases}.$$

Although we can use this approach, we are often interested in obtaining the distribution of  $Y = \sum_{i=1}^n X_i$ . In some of these instances, moment generating functions can be very helpful.

**DEF: Multivariate Moment Generating Function** Suppose  $X_1, \dots, X_n$  are random variables with joint pdf  $f_{X_1, \dots, X_n}(x_1, \dots, x_n)$ . The **multivariate moment generating function** of  $X_1, \dots, X_n$  is,

for all  $t_i$  near zero ( $|t_i| < h_i, i = 1, 2, \dots, n$ ) where the expectation exists.

\end{mdframed}

**Multivariate MGFs can be used to...**

- find moments:

$$E(X_i^k) =$$

- Example:

$$\frac{\partial}{\partial t_1} m_{X_1, \dots, X_n}(t_1 = 0, \dots, t_n = 0) =$$

- Example:

$$\frac{\partial^2}{\partial t_1^2} m_{X_1, \dots, X_n}(t_1 = 0, \dots, t_n = 0) =$$

- Example:

$$\frac{\partial^2}{\partial t_1 \partial t_3} m_{X_1, \dots, X_n}(t_1 = 0, \dots, t_n = 0) =$$

- find Univariate MGFs

$$m_{X_1, \dots, X_n}(t_1 = t, t_2 = 0, \dots, t_n = 0) =$$

- conduct transformations: Suppose  $X_1, \dots, X_n$  are random variables. Often, we are interested in the distribution of  $Y = \sum_{i=1}^n X_i = X_1 + X_2 + X_3 + \dots + X_n$ . In this case, the MGF of the transformed RV,  $Y$  is

$$m_Y(t) =$$

**THM 4.6.7:** In the case where  $X_1, \dots, X_n$  are mutually independent,

**THM 4.6.7:** In the very special case where  $X_1, \dots, X_n$  are independent and identically distributed (iid),

**Example** Suppose  $X_1, \dots, X_n$  are independent random variables where  $X_i \sim \text{Poisson}(\lambda_i)$  Use mgfs to find the distribution of  $Y = \sum_{i=1}^n X_i$

**Example** Suppose  $X_1, \dots, X_n$  are independent random variables where  $X_i \sim \text{Normal}(\mu, \sigma^2)$  Use mgfs to find the distribution of  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$