M/STAT 501 Fall 2022: Homework 4

Due *Friday*, Oct 14 by 5:00pm in Gradescope

Directions: For this homework, there are three different types of problems: 1) *Graded for Accuracy Problems*, 2) *Graded for Completion Problems*, and 3) *Extra Practice Problems.* Please turn in your work for the *Graded for Accuracy Problems* and the *Graded for Completion Problems*, as well as any *Sources Used* by the due date via Gradescope. Show work and/or provide justification for credit—neatness counts! *The Extra Practice Problems* do not need to be turned in, and the solutions will be posted for you to check and compare your work.

You are encouraged to use R Markdown for your homework, but it is not required. However, **you must use R Markdown for any problems that require the use of R; include both your R code and the R output on the rendered pdf**.

Graded for Accuracy Problems (turn in)

1. The percentage of alcohol in a certain compound is a random variable X with density function

$$f_X(x) = 20x^3(1-x)I_{[0,1]}(x).$$

Suppose that the compound's selling price depends on its alcohol content. Specifically, if $\frac{1}{3} < x < \frac{2}{3}$, the compound sells for c_1 dollars per gallon; otherwise it sells for c_2 dollars per gallon. The production cost is c_3 dollars per gallon. Let Y be the profit per gallon.

   a. Find the probability function (pdf or pmf) of Y.

   b. Find the expected value of Y. Write a sentence interpreting this value in context of the problem when $c_1 = 40$, $c_2 = 30$, and $c_3 = 15$.

   c. Find the standard deviation of Y. Write a sentence interpreting this value in context of the problem when $c_1 = 40$, $c_2 = 30$, and $c_3 = 15$. .

2. The failure of a circuit board interrupts work that utilizes a computing system until a new board is delivered. Let Y be the delivery time (in days), and assume the moment generating function of Y is

$$M_Y(t) = \frac{e^{5t} - e^t}{4t}$$

for all $t \neq 0$.

   a. Use the table of common distributions in the back of our textbook (p. 621–626) to identify the probability distribution of Y by its mgf. (To completely specify the distribution of a random variable, you must give its name as well as values for its parameters.)

   b. Use the mgf to find $E(Y)$ and $E(Y^2)$. Then verify your answers using the mean and variance given in the table of common distributions. *Hint*: See additional notes on mgfs.

   c. Suppose the cost of a board failure and interruption is represented by $C = 500 + 100Y^2 - 25Y$. Find the expected cost associated with a single failed circuit board.

   d. Now suppose the cost associated with a single failed circuit board is changed and represented instead by $A = 500 + 60Y^2$. Find the variance of this new cost.

3. Suppose X has pdf

$$f_X(x) = \frac{1}{2}\exp(-|x|)$$

for all $-\infty < x < \infty$.

   a. Find $E(X)$.

   b. Find the pdf of $Y = |X|$.

   c. Find $E(Y)$ and compare it to your answer in part a.

Graded for Completion Problems (turn in)

Suppose the random variable X has pdf

$$f_X(x) = \begin{cases} \dfrac{x-1}{2} & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

1. Find a monotone function $g(x)$ such that $Y = g(X)$ has a Uniform(0, 1) distribution.

2. In R, use a probability integral transformation to generate a random sample from the distribution of X. Create a plot to visualize how the distribution of these simulated values compares to $f_X(x)$. Interpret this plot, being sure to comment on how these two distributions compare to one another.

Sources Used (turn in)

At the end of your assignment, list and cite all sources used when working on this assignment, including individuals with whom you discussed the problems. If you did not use any sources besides our textbook or the course notes, please note this.

Extra Practice Problems (do not turn in, solutions posted in D2L)

1. Suppose X has pdf

$$f_X(x) = \begin{cases} 7e^{-7x} & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

   Find the pdf of $Y = 4X + 3$.

2. If X is a nonnegative integer-valued random variable, show

$$E(X) = \sum_{x=1}^{\infty} P(X \geq x).$$

3. Show that the expected value of an indicator variable for event A is just the probability of A. That is, $A \subset S$, and let $I_A = 1$ if A occurs, 0 otherwise. Prove

$$E(I_A) = P(A).$$